STOCHASTIC ANALYSIS FOR JUMP PROCESSES

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ABSTRACT. Lecture notes from courses at TU Berlin in WS 2009/10, WS 2011/12 and WS 2012/13.

CONTENTS

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1. Introduction

These lecture notes aim at providing a self-contained introduction to Lévy processes. We start by defining Lévy processes and study a simple but very interesting example: a Lévy jump-diffusion.

2. Definition of Lévy processes

2.1. **Notation and auxiliary definitions.**

Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ denote a stochastic basis, or filtered probability space, i.e. a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$.

A filtration is an increasing family of sub-σ-algebras of $\mathcal{F}$, i.e. $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$. By convention $\mathcal{F}_\infty = \mathcal{F}$ and $\mathcal{F}_{\infty-} = \bigvee_{s\geq 0} \mathcal{F}_s$.

A stochastic basis satisfies the *usual conditions* if it is *right-continuous*, i.e. $\mathcal{F}_t = \mathcal{F}_{t+}$, where $\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$, and is *complete*, i.e. the σ-algebra $\mathcal{F}$ is $\mathbb{P}$-complete and every $\mathcal{F}_t$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$.

Definition 2.1. A stochastic process $X = (X_t)_{t\geq 0}$ has *independent increments* if, for any $n \geq 1$ and $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.

Alternatively, we say that X has independent increments if, for any $0 \leq s < t$, $X_t - X_s$ is independent of $\mathcal{F}_s$.

Definition 2.2. A stochastic process $X = (X_t)_{t\geq 0}$ has *stationary increments* if, for any $s, t \geq 0$, the distribution of $X_{t+s} - X_s$ does not depend on s. Alternatively, we say that X has stationary increments if, for any $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.

Definition 2.3. A stochastic process $X = (X_t)_{t\geq 0}$ is *stochastically continuous* if, for every $t \geq 0$ and $\epsilon > 0$

$$\lim_{s\to t} \mathbb{P}(|X_s - X_t| > \epsilon) = 0.$$

2.2. **Definition of Lévy processes.**

Definition 2.4 (Lévy process)**.** An adapted, $\mathbb{R}^d$-valued stochastic process $X = (X_t)_{t\geq 0}$ with $X_0 = 0$ a.s. is called a *Lévy process* if:

(L1) X has independent increments,
(L2) X has stationary increments,
(L3) X is stochastically continuous.

In the sequel, we will always assume that X has *càdlàg paths*. The next two results provide the justification.

Lemma 2.5. *If X is a Lévy process and Y is a modification of X (i.e. $\mathbb{P}(X_t \neq Y_t) = 0$ a.s. for each $t \geq 0$), then Y is a Lévy process and has the same characteristics as X.*

Proof. [App09, Lemma 1.4.8]. □

Theorem 2.6. *Every Lévy process has a unique càdlàg modification that is itself a Lévy process.*

Proof. [App09, Theorem 2.1.8] or [Pro04, Theorem I.30]. □

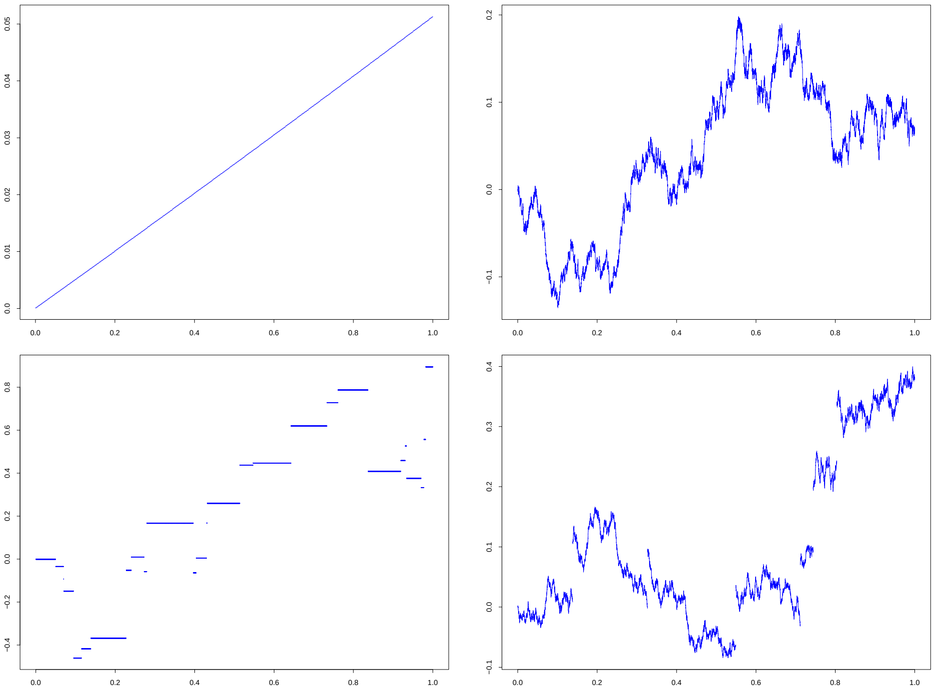

Figure 2.1. Sample paths of a linear drift processs (top-left), a Brownian motion (top-right), a compound Poisson process (bottom-left) and a Lévy jump-diffusion.

2.3. **Examples.** The following are some well-known examples of Lévy processes:

- The *linear drift* is the simplest Lévy process, a deterministic process; see Figure 5.3 for a sample path.
- The *Brownian motion* is the only non-deterministic Lévy process with continuous sample paths; see Figure 5.3 for a sample path.
- The *Poisson*, the *compound Poisson* and the *compensated (compound) Poisson* processes are also examples of Lévy processes; see Figure 5.3 for a sample path of a compound Poisson process.

The sum of a linear drift, a Brownian motion and a (compound or compensated) Poisson process is again a Lévy process. It is often called a "jump-diffusion" process. We shall call it a *Lévy jump-diffusion* process, since there exist jump-diffusion processes which are not Lévy processes. See Figure 5.3 for a sample path of a Lévy jump-diffusion process.

3. Toy example: a Lévy jump-diffusion

Let us study the Lévy jump-diffusion process more closely; it is the simplest Lévy process we have encountered so far that contains both a diffusive part and a jump part. We will calculate the characteristic function of the Lévy jump-diffusion, since it offers significant insight into the structure of the characteristic function of general Lévy processes.

Assume that the process $X = (X_t)_{t\geq 0}$ is a Lévy jump-diffusion, i.e. a linear deterministic process, plus a Brownian motion, plus a compensated compound Poisson process. The paths of this process are described by

$$X_t = bt + \sigma W_t + \Big(\sum_{k=1}^{N_t} J_k - t\lambda\beta\Big), \tag{3.1}$$

where $b \in \mathbb{R}$, $\sigma \in \mathbb{R}_{\geqslant 0}$, $W = (W_t)_{t\geq 0}$ is a standard Brownian motion, $N = (N_t)_{t\geq 0}$ is a Poisson process with intensity $\lambda \in \mathbb{R}_{\geqslant 0}$ (i.e. $\mathbb{E}[N_t] = \lambda t$), and $J = (J_k)_{k\geq 1}$ is an i.i.d. sequence of random variables with probability distribution F and $\mathbb{E}[J_k] = \beta < \infty$. Here F describes the distribution of the jumps, which arrive according to the Poisson process N. All sources of randomness are assumed *mutually independent*.

The characteristic function of X_t, taking into account that all sources of randomness are independent, is

$$\begin{aligned}
\mathbb{E}\big[\mathrm{e}^{iuX_t}\big] &= \mathbb{E}\Big[\exp\big(iu\big(bt + \sigma W_t + \sum_{k=1}^{N_t} J_k - t\lambda\kappa\big)\big)\Big] \\
&= \exp\big[iubt\big]\mathbb{E}\Big[\exp\big(iu\sigma W_t\big)\Big]\mathbb{E}\Big[\exp\big(iu\sum_{k=1}^{N_t} J_k - iut\lambda\kappa\big)\Big];
\end{aligned}$$

recalling that the characteristic functions of the normal and the compound Poisson distributions are

$$\mathbb{E}[\mathrm{e}^{iu\sigma W_t}] = \mathrm{e}^{-\frac{1}{2}\sigma^2u^2t}, \quad W_t \sim \mathcal{N}(0,t)$$

$$\mathbb{E}[\mathrm{e}^{iu\sum_{k=1}^{N_t} J_k}] = \mathrm{e}^{\lambda t(\mathbb{E}[\mathrm{e}^{iuJ_k}-1])}, \quad N_t \sim \mathrm{Poi}(\lambda t)$$

(cf. Example 4.14 and Exercise 1), we get

$$\begin{aligned}
&= \exp\big[iubt\big]\exp\Big[-\frac{1}{2}u^2\sigma^2t\Big]\exp\Big[\lambda t\big(\mathbb{E}[\mathrm{e}^{iuJ_k} - 1] - iu\mathbb{E}[J_k]\big)\Big] \\
&= \exp\big[iubt\big]\exp\Big[-\frac{1}{2}u^2\sigma^2t\Big]\exp\Big[\lambda t\big(\mathbb{E}[\mathrm{e}^{iuJ_k} - 1 - iuJ_k]\big)\Big];
\end{aligned}$$

and since the distribution of J_k is F we have

$$= \exp\big[iubt\big]\exp\Big[-\frac{1}{2}u^2\sigma^2t\Big]\exp\Big[\lambda t\int_{\mathbb{R}}\big(\mathrm{e}^{iux} - 1 - iux\big)F(\mathrm{d}x)\Big].$$

Finally, since t is a common factor, we can rewrite the above equation as

$$\mathbb{E}\big[\mathrm{e}^{iuX_t}\big] = \exp\Big[t\Big(iub - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}}(\mathrm{e}^{iux} - 1 - iux)\lambda F(\mathrm{d}x)\Big)\Big]. \tag{3.2}$$

We can make the following observations based on the structure of the characteristic function of the random variable X_t from the Lévy jump-diffusion:

(O1) time and space *factorize*;
(O2) the drift, the diffusion and the jump parts are *separated*;
(O3) the jump part *decomposes* to $\lambda \times F$, where λ is the expected number of jumps and F is the distribution of jump size.

One would naturally ask if these observations are true for *any* Lévy process. The answer for (O1) and (O2) is *yes*, because Lévy processes have stationary and independent increments. The answer for (O3) is *no*, because there exist

Lévy processes with *infinitely* many jumps (on any compact time interval), thus their expected number of jumps is also infinite.

Since the characteristic function of a random variable determines its distribution, (3.2) provides a characterization of the distribution of the random variables X_t from the Lévy jump-diffusion X. We will soon see that this distribution belongs to the class of *infinitely divisible distributions* and that equation (3.2) is a special case of the celebrated *Lévy-Khintchine formula.*

3.1. **The basic connections.** The next sections will be devoted to establishing the connection between the following mathematical objects:

- Lévy processes $X = (X_t)_{t\geq 0}$
- infinitely divisible distributions $\rho = \mathcal{L}(X_1)$
- Lévy triplets (b, c, ν).

The following commutative diagram displays how these connections can be proved, where LK stands for the Lévy–Khintchine formula, LI for the Lévy–Itô decomposition, CFE for the Cauchy functional equation and SII for stationary and independent increments.

$$
\begin{tikzcd}
(X_t)_{t\geq 0} \arrow[rr, shift left, "SII"] \arrow[dr, shift left, "CFE"] & & \mathcal{L}(X_1) = \rho \arrow[ll, shift left, "LI"] \arrow[dl, Leftrightarrow, "LK"] \\
& (b, c, \nu) \arrow[ul, shift left, "LI"] &
\end{tikzcd}
$$

FIGURE 3.2. The basic connections between Lévy processes, infinitely divisible distributions and Lévy triplets.

Exercise 1. Let $X = (X_t)_{t\geq 0}$ be a compound Poisson process with intensity $\lambda > 0$ and jump distribution F, i.e.

$$X_t = \sum_{k=1}^{N_t} J_k,$$

where $N = (N_t)_{t\geq 0}$ is a Poisson process with $\mathbb{E}[N_t] = \lambda t$ and $J = (J_k)_{k\geq 0}$ is an i.i.d. sequence of random variables with distribution F. Show that

$$\mathbb{E}\left[\mathrm{e}^{iuX_t}\right] = \exp\left(\lambda t \int_{\mathbb{R}} (\mathrm{e}^{iux} - 1) F(\mathrm{d}x)\right).$$

Exercise 2. Consider the setting of the previous exercise and assume that $\mathbb{E}[J_k] = \beta < \infty$. Show that the compensated compound Poisson process $\overline{X} = (\overline{X}_t)_{t\geq 0}$ is a martingale, where

$$\overline{X}_t = X_t - \lambda\beta t.$$

4. Infinitely Divisible distributions

4.1. **Notation and auxiliary results.** Let X be a random variable and denote by $\mathbb{P}_X$ its law, by φ_X its characteristic function, and by M_X its moment generating function. They are related as follows:

$$\varphi_X(u) = \mathbb{E}\big[\mathrm{e}^{i\langle u,X\rangle}\big] = \int_\Omega \mathrm{e}^{i\langle u,X\rangle}\mathrm{d}\mathbb{P} = \int_{\mathbb{R}^d} \mathrm{e}^{i\langle u,x\rangle}\mathbb{P}_X(\mathrm{d}x) = M_X(iu), \quad (4.1)$$

for all $u \in \mathbb{R}^d$.

Let ρ be a probability measure; we will denote by $\widehat{\rho}$ its characteristic function (or Fourier transform), i.e.

$$\widehat{\rho}(u) = \int_{\mathbb{R}^d} \mathrm{e}^{i\langle u,x\rangle}\rho(\mathrm{d}x). \quad (4.2)$$

Let $S \subseteq \mathbb{R}^d$, we will denote by $\mathcal{B}(S)$ the Borel σ-algebra of S and by $B_b(S)$ the space of bounded, Borel measurable functions from S to $\mathbb{R}$. We will also denote convergence in law by $\xrightarrow{d}$, weak convergence by $\xrightarrow{w}$ and uniform convergence on compact sets by $\xrightarrow{\mathrm{uc}}$.

We also recall some results from probability theory and complex analysis.

Proposition 4.1. *Let ρ, ρ_n, $n \in \mathbb{N}$, be probability measures on $\mathbb{R}^d$.*

(1) If $\rho_n \xrightarrow{w} \rho$ then $\widehat{\rho}_n(u) \xrightarrow{uc} \widehat{\rho}(u)$.
(2) If $\widehat{\rho}_n(u) \longrightarrow \widehat{\rho}(u)$ for every u, then $\rho_n \xrightarrow{w} \rho$.
(3) Let $f, f_n : \mathbb{R}^d \to \mathbb{C}$, $n = 1, 2, \dots,$ be continuous functions such that $f(0) = f_n(0) = 1$ and $f(u) \neq 0$ and $f_n(u) \neq 0$, for any u and any n. If $f_n(u) \xrightarrow{uc} f(u)$, then also $\log f_n(u) \xrightarrow{uc} \log f(u)$.

Proof. For (1) and (2) see [Shi96, p. 325], for (3) see [Sat99, Lemma 7.7]. □

Theorem 4.2 (Lévy continuity theorem)**.** *Let $(\rho_n)_{n\in\mathbb{N}}$ be probability measures on $\mathbb{R}^d$ whose characteristic functions $\widehat{\rho}_n(u)$ converge to some function $\widehat{\rho}_n(u)$, for all u, where $\widehat{\rho}$ is continuous at 0. Then, $\widehat{\rho}$ is the characteristic function of a probability distribution ρ and $\rho_n \xrightarrow{d} \rho$.*

Proof. [Dud02, Theorem 9.8.2] □

4.2. **Convolution.** Let μ, ρ be two probability measures on $\mathbb{R}^d$. We define the *convolution* of μ and ρ as

$$(\mu * \rho)(A) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} 1_A(x+y)\mu(\mathrm{d}x)\rho(\mathrm{d}y), \quad (4.3)$$

for each $A \in \mathcal{B}(\mathbb{R}^d)$.

Denote by $A - x := \{y - x : y \in A\}$, then we have that $1_A(x+y) = 1_{A-x}(y) = 1_{A-y}(x)$, and Fubini's theorem yields

$$(\mu * \rho)(A) = \int_{\mathbb{R}^d} \mu(A-x)\rho(\mathrm{d}x) = \int_{\mathbb{R}^d} \rho(A-y)\mu(\mathrm{d}y). \quad (4.4)$$

Proposition 4.3. *The convolution of two probability measures is again a probability measure.*

Proof. [App09, Proposition 1.2.1]. □

Definition 4.4. We define the n-fold convolution of a measure ρ as

$$\rho^{*n} = \underbrace{\rho * \cdots * \rho}_{n \text{ times}}. \tag{4.5}$$

We say that the measure ρ has a *convolution n-th root* if there exists a measure ρ_n such that

$$\rho = (\rho_n)^{*n}. \tag{4.6}$$

In the sequel we will make use of the following results.

Proposition 4.5. *Let ρ_1, ρ_2, be Borel probability measures on $\mathbb{R}^d$ and let $f \in B_b(\mathbb{R}^d)$, then*

$$\int\limits_{\mathbb{R}^d} f(y)(\rho_1 * \rho_2)(\mathrm{d}y) = \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(x+y)\rho_1(\mathrm{d}x)\rho_2(\mathrm{d}y). \tag{4.7}$$

Proof. [App09, Proposition 1.2.2(1)]. □

Corollary 4.6. *Let X_1, X_2 be independent random variables with marginals ρ_1, ρ_2. Then, for any $f \in B_b(\mathbb{R}^d)$*

$$\mathbb{E}[f(X_1 + X_2)] = \int\limits_{\mathbb{R}^d} f(x)(\rho_1 * \rho_2)(\mathrm{d}x). \tag{4.8}$$

In particular, for the indicator function we get

$$\mathbb{P}(X_1 + X_2 \in A) = \mathbb{E}[1_A(X_1 + X_2)] = (\rho_1 * \rho_2)(A), \tag{4.9}$$

where $A \in \mathcal{B}(\mathbb{R}^d)$.

Proof. Direct consequences of independence and Proposition 4.5. □

4.3. **Infinite divisibility.** We start by defining infinitely divisible random variables and then provide some properties of their probability distributions and characteristic functions.

Definition 4.7. A random variable X is *infinitely divisible* if, for all $n \in \mathbb{N}$, there exist i.i.d. random variables $X_1^{(n)}, \ldots, X_n^{(n)}$ such that

$$X \overset{\mathrm{d}}{=} X_1^{(n)} + \cdots + X_n^{(n)}. \tag{4.10}$$

The next result provides some insight into the structure of infinitely divisible distributions.

Proposition 4.8. *The following are equivalent:*

(1) X is infinitely divisible;
(2) $\mathbb{P}_X$ has a convolution n-th root that is itself the law of a random variable, for all $n \in \mathbb{N}$;
(3) φ_X has an n-th root that is itself the characteristic function of a random variable, for all $n \in \mathbb{N}$.

Proof. (1) $\Rightarrow$ (2) Since X is infinitely divisible, we have for any $A \in \mathcal{B}(\mathbb{R}^d)$

$$\begin{aligned}
\mathbb{P}_X(A) &= \mathbb{P}(X \in A) = \mathbb{P}(X_1^{(n)} + \cdots + X_n^{(n)} \in A) \\
&= (\mathbb{P}_{X_1^{(n)}} * \cdots * \mathbb{P}_{X_n^{(n)}})(A) \qquad \text{(by independence and (4.9))} \\
&= (\mathbb{P}_{X^{(n)}} * \cdots * \mathbb{P}_{X^{(n)}})(A) \qquad \text{(by identical laws)} \\
&= (\mathbb{P}_{X^{(n)}})^{*n}(A).
\end{aligned} \tag{4.11}$$

(2) $\Rightarrow$ (3) Since $\mathbb{P}_X$ has a convolution n-th root, we have

$$\begin{aligned}
\varphi_X(u) &= \int_{\mathbb{R}^d} \mathrm{e}^{i\langle u,x\rangle} \mathbb{P}_X(\mathrm{d}x) = \int_{\mathbb{R}^d} \mathrm{e}^{i\langle u,x\rangle} (\mathbb{P}_{X^{(n)}} * \cdots * \mathbb{P}_{X^{(n)}})(\mathrm{d}x) \\
&= \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \mathrm{e}^{i\langle u,x_1+\cdots+x_n\rangle} \mathbb{P}_{X^{(n)}}(\mathrm{d}x_1) \dots \mathbb{P}_{X^{(n)}}(\mathrm{d}x_n) \qquad \text{(Prop. 4.5)} \\
&= \prod_{i=1}^n \int_{\mathbb{R}^d} \mathrm{e}^{i\langle u,x_i\rangle} \mathbb{P}_{X^{(n)}}(\mathrm{d}x_i) \qquad \text{(by independence)} \\
&= \prod_{i=1}^n \varphi_{X^{(n)}}(u) = \Big(\varphi_{X^{(n)}}(u)\Big)^n.
\end{aligned} \tag{4.12}$$

(3) $\Rightarrow$ (1) Choose $X_1^{(n)}, \dots, X_n^{(n)}$ to be independent copies of a given r.v. $X^{(n)}$. Since the characteristic function has an n-th root, we get

$$\begin{aligned}
\mathbb{E}\big[\mathrm{e}^{i\langle u,X\rangle}\big] &= \varphi_X(u) \\
&= \Big(\varphi_{X^{(n)}}(u)\Big)^n = \prod_{i=1}^n \varphi_{X_i^{(n)}}(u) \\
&= \mathbb{E}\big[\mathrm{e}^{i\left\langle u, X_1^{(n)}+\cdots+X_n^{(n)}\right\rangle}\big] \qquad \text{(by independence)},
\end{aligned} \tag{4.13}$$

and the result follows, since the characteristic function deteremines the law of a random variable. $\square$

These results motivate us to give the following more general definition of infinite divisibility.

Definition 4.9. A probability measure ρ is *infinitely divisible* if, for all $n \in \mathbb{N}$, there exists another probability measure ρ_n such that

$$\rho = \underbrace{\rho_n * \cdots * \rho_n}_{n \text{ times}}. \tag{4.14}$$

Proposition 4.10. *A probability measure ρ is infinitely divisible if and only if, for all $n \in \mathbb{N}$, there exists another probability measure ρ_n such that*

$$\widehat{\rho}(u) = \Big(\widehat{\rho_n}(u)\Big)^n. \tag{4.15}$$

Proof. Similar to the proof of Proposition 4.8, thus left as an exercise. $\square$

Next, we will discuss some properties of infinitely divisible distributions, in particular that they are closed under convolutions and weak limits.

Lemma 4.11. *If μ, ρ are infinitely divisible probability measures then $\mu * \rho$ is also infinitely divisible.*

Proof. Since μ and ρ are infinitely divisible, we know that for any $n \in \mathbb{N}$ it holds

$$\mu = (\mu_n)^{*n} \quad \text{and} \quad \rho = (\rho_n)^{*n}. \tag{4.16}$$

Hence, from the commutativity of the convolution we get that

$$\mu * \rho = (\mu_n)^{*n} * (\rho_n)^{*n} = (\mu_n * \rho_n)^{*n}. \qquad \square$$

Lemma 4.12. *If ρ is infinitely divisible then $\widehat{\rho}(u) \neq 0$ for any $u \in \mathbb{R}^d$.*

Proof. Since ρ is infinitely divisible, we know that for every $n \in \mathbb{N}$ there exists a measure ρ_n such that $\widehat{\rho} = (\widehat{\rho}_n)^n$. Using [Sat99, Prop. 2.5(v)] we have that $|\widehat{\rho}_n(u)|^2 = |\widehat{\rho}(u)|^{2/n}$ is a characteristic function. Define the function

$$\varphi(u) = \lim_{n\to\infty} |\widehat{\rho}_n(u)|^2 = \lim_{n\to\infty} |\widehat{\rho}(u)|^{2/n} = \begin{cases} 1, & \text{if } \widehat{\rho}(u) \neq 0 \\ 0, & \text{if } \widehat{\rho}(u) = 0. \end{cases}$$

Since $\widehat{\rho}(0) = 1$ and $\widehat{\rho}$ is a continuous function, we get that $\varphi(u) = 1$ in a neighborhood of 0. Now, using Lévy's continuity theorem we get that $\varphi(u)$ is a continuous function, thus $\varphi(u) = 1$ for all $u \in \mathbb{R}^d$. Hence $\widehat{\rho}(u) \neq 0$ for any $u \in \mathbb{R}^d$. $\square$

Lemma 4.13. *If $(\rho_k)_{k\geq 0}$ is a sequence of infinitely divisible distributions and $\rho_k \xrightarrow{w} \rho$, then ρ is also infinitely divisible.*

Proof. Since $\rho_k \xrightarrow{w} \rho$ as $k \to \infty$ we get from Proposition 4.1(1) that $\widehat{\rho}_k(z) \xrightarrow{uc} \widehat{\rho}(z)$ and $\widehat{\rho}$ is the characteristic function of the probability measure ρ. In order to prove the claim, it suffices to show that $\widehat{\rho}^{1/n}$ is well-defined and the characteristic function of a probability measure. Then, the trivial equality $\widehat{\rho}(z) = (\widehat{\rho}(z)^n)^{1/n}$ yields that ρ is infinitely divisible.

Since $\widehat{\rho}_k$ and $\widehat{\rho}$ are characteristic functions, we know that they are continuous and $\widehat{\rho}_k(0) = \widehat{\rho}(0) = 1$ for every k. Moreover, $\widehat{\rho}_k$ is the characteristic function of an infinitely divisible distribution, thus from Lemma 4.12 we get that $\widehat{\rho}_k(u) \neq 0$ for any k, u. One can also show that $\widehat{\rho}(u) \neq 0$ for every u, see [Sat99, Lemma 7.8]. Therefore, we can apply Proposition 4.1(3) and we get that $\log \widehat{\rho}_k(u) \xrightarrow{uc} \log \widehat{\rho}(u)$, hence also $\widehat{\rho}_k(u)^{1/n} \xrightarrow{uc} \widehat{\rho}(u)^{1/n}$, for every n, as $k \to \infty$. We have that $\widehat{\rho}_k^{1/n}$ is a continuous function, and using the uniform convergence to $\widehat{\rho}^{1/n}$, we can conclude that this is also continuous (around zero). Now, an application of Lévy's continuity theorem yields that $\widehat{\rho}^{1/n}$ is the characteristic function of a probability distribution. $\square$

4.4. **Examples.** Below we present some examples of infinitely divisible distributions. In particular, using Proposition 4.8 we can easily show that the normal, the Poisson and the exponential distributions are infinitely divisible.

Example 4.14 (Normal distribution)**.** Let $X \sim \mathcal{N}(\mu, \sigma^2)$, then we have

$$\begin{aligned}
\varphi_X(u) &= \exp\left(iu\mu - \frac{1}{2}u^2\sigma^2\right) = \exp\left(n\Big[iu\frac{\mu}{n} - \frac{1}{2}u^2\frac{\sigma^2}{n}\Big]\right) \qquad (4.17)\\
&= \left(\exp\Big[iu\frac{\mu}{n} - \frac{1}{2}u^2\frac{\sigma^2}{n}\Big]\right)^n\\
&= \Big(\varphi_{X^{(n)}}(u)\Big)^n,
\end{aligned}$$

where $X^{(n)} \sim \mathcal{N}(\frac{\mu}{n}, \frac{\sigma^2}{n})$.

Example 4.15 (Poisson distribution)**.** Let $X \sim \mathrm{Poi}(\lambda)$, then we have

$$\begin{aligned}
\varphi_X(u) &= \exp\Big(\lambda(\mathrm{e}^{iu} - 1)\Big) = \left(\exp\Big[\frac{\lambda}{n}(\mathrm{e}^{iu} - 1)\Big]\right)^n \qquad (4.18)\\
&= \Big(\varphi_{X^{(n)}}(u)\Big)^n,
\end{aligned}$$

where $X^{(n)} \sim \mathrm{Poi}(\frac{\lambda}{n})$.

Example 4.16 (Exponential distribution)**.** Let $X \sim \mathrm{Exp}(\lambda)$, then we have

$$\begin{aligned}
\varphi_X(u) &= \left(1 - \frac{iu}{\lambda}\right)^{-1} = \left[\left(1 - \frac{iu}{\lambda}\right)^{-\frac{1}{n}}\right]^n \qquad (4.19)\\
&= \Big(\varphi_{X^{(n)}}(u)\Big)^n,
\end{aligned}$$

where $X^{(n)} \sim \Gamma(\frac{1}{n}, \lambda)$.

Remark 4.17. Other examples of infinitely divisible distributions are the geometric, the negative binomial, the Cauchy and the strictly stable distributions. Counter-examples are the uniform and the binomial distributions.

Exercise 3. Show that the law of the random variable

$$X_t = bt + \sigma W_t + \sum_{k=1}^{N_t} J_k, \quad (t \geq 0, \quad \text{fixed}) \qquad (4.20)$$

is infinitely divisible, *without* using Proposition 4.8.

4.5. **Lévy processes have infinitely divisible laws.** We close this section by taking a glimpse of the deep connections between infinitely divisible distributions and Lévy processes. In particular, we will show that if $X = (X_t)_{t\geq 0}$ is a Lévy process then X_t is an infinitely divisible random variable (for all $t \geq 0$).

Lemma 4.18. *Let $X = (X_t)_{t\geq 0}$ be a Lévy process. Then the random variables X_t, $t \geq 0$, are infinitely divisible.*

Proof. Let $X = (X_t)_{t\geq 0}$ be a Lévy process; for any $n \in \mathbb{N}$ and any $t > 0$ we trivially have that

$$X_t = X_{\frac{t}{n}} + (X_{\frac{2t}{n}} - X_{\frac{t}{n}}) + \ldots + (X_t - X_{\frac{(n-1)t}{n}}). \qquad (4.21)$$

The stationarity of the increments of the Lévy process yields that

$$X_{\frac{tk}{n}} - X_{\frac{t(k-1)}{n}} \overset{\mathrm{d}}{=} X_{\frac{t}{n}},$$

for any $k \geq 1$, while the independence of the increments yields that the random variables $X_{\frac{tk}{n}} - X_{\frac{t(k-1)}{n}}$, $k \geq 1$, are indepedent of each other. Thus, $(X_{\frac{tk}{n}} - X_{\frac{t(k-1)}{n}})_{k\geq 1}$ is an i.i.d. sequence of random variables, and from Definition 4.7 we conclude that the random variable X_t is infinitely divisible. $\square$

5. The Lévy–Khintchine representation

The next result provides a complete characterization of infinitely divisible distributions in terms of their characteristic functions. This is the celebrated *Lévy-Khintchine formula.* B. de Finetti and A. Kolmogorov were the first to prove versions of this representation under certain assumptions. P. Lévy and A. Khintchine indepedently proved it in the general case, the former by analyzing the sample paths of the process and the latter by a direct analytic method.

5.1. **Statement, "if part".**

We first define a Lévy measure and then state the Lévy–Khintchine representation and prove the "if part" of the theorem.

Definition 5.1 (Lévy measure)**.** Let ν be a Borel measure on $\mathbb{R}^d$. We say that ν is a *Lévy measure* if it satisfies

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(\mathrm{d}x) < \infty. \tag{5.1}$$

Remark 5.2. Since $|x|^2 \wedge \varepsilon \leq |x|^2 \wedge 1$ for all $0 < \varepsilon \leq 1$, it follows that $\nu\big((-\varepsilon, \varepsilon)^c\big) < \infty$ for all $\varepsilon > 0$. In other words, any Lévy measure becomes a probability measure once restricted to the complement of an ε-neighborhood of the origin (after an appropriate normalization).

Theorem 5.3 (Lévy–Khintchine)**.** *A measure ρ is* infinitely divisible *if and only if there exists a triplet (b, c, ν) with $b \in \mathbb{R}^d$, c a symmetric, non-negative definite, $d \times d$ matrix, and ν a Lévy measure, such that*

$$\widehat{\rho}(u) = \exp\left(i\,\langle u, b\rangle - \frac{\langle u, cu\rangle}{2} + \int_{\mathbb{R}^d} \big(\mathrm{e}^{i\langle u,x\rangle} - 1 - i\,\langle u, x\rangle\, 1_D\big)\nu(\mathrm{d}x) \right). \tag{5.2}$$

Here D denotes the closed unit ball in $\mathbb{R}^d$, i.e. $D := \{|x| \leq 1\}$.

Definition 5.4. We will call (b, c, ν) the *Lévy* or *characteristic triplet* of the infinitely divisible measure ρ. We call b the *drift* characteristic and c the *Gaussian* or *diffusion* characteristic.

Example 5.5. An immediate consequence of Definitions 5.1 and 5.4 and Theorem 5.3 is that the distribution of the r.v. X_1 from the Lévy jump-diffusion is infinitely divisible with Lévy triplet

$$\left(b - \int_{D^c} x\lambda F(\mathrm{d}x), \sigma^2, \lambda \times F \right).$$

Proof of Theorem 5.3, "If" part. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$, monotonic and decreasing to zero (e.g. $\varepsilon_n = \frac{1}{n}$). Define for all $u \in \mathbb{R}^d$ and $n \in \mathbb{N}$

$$\widehat{\rho}_n(u) = \exp\left(i\Big\langle u, b - \int\limits_{\varepsilon_n<|x|\le 1} x\nu(\mathrm{d}x)\Big\rangle - \frac{\langle u, cu\rangle}{2} + \int\limits_{|x|>\varepsilon_n} (\mathrm{e}^{i\langle u,x\rangle} - 1)\nu(\mathrm{d}x)\right).$$

Each $\widehat{\rho}_n$ is the characteristic function of the convolution of a normal and a compound Poisson distribution, hence $\widehat{\rho}_n$ is the characteristic function of an infinitely divisible probability measure ρ_n. We clearly have that

$$\lim_{n\to\infty} \widehat{\rho}_n(u) = \widehat{\rho}(u).$$

Then, by Lévy's continuity theorem and Lemma 4.13, $\widehat{\rho}$ is the characteristic function of an infinitely divisible law, provided that $\widehat{\rho}$ is continuous at 0.

Now, continuity of $\widehat{\rho}$ at 0 boils down to the continuity of the integral term in (5.2), i.e.

$$\begin{aligned}\psi_\nu(u) &:= \int\limits_{\mathbb{R}^d} (\mathrm{e}^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle 1_D(x))\nu(\mathrm{d}x)\\ &= \int\limits_D (\mathrm{e}^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle)\nu(\mathrm{d}x) + \int\limits_{D^c} (\mathrm{e}^{i\langle u,x\rangle} - 1)\nu(\mathrm{d}x).\end{aligned}$$

Using Taylor's expansion, the Cauchy–Schwarz inequality, the definition of the Lévy measure and dominated convergence, we get

$$\begin{aligned}|\psi_\nu(u)| &\le \frac{1}{2}\int\limits_D |\langle u, x\rangle|^2\nu(\mathrm{d}x) + \int\limits_{D^c} |\mathrm{e}^{i\langle u,x\rangle} - 1|\nu(\mathrm{d}x)\\ &\le \frac{|u|^2}{2}\int\limits_D |x^2|\nu(\mathrm{d}x) + \int\limits_{D^c} |\mathrm{e}^{i\langle u,x\rangle} - 1|\nu(\mathrm{d}x)\\ &\longrightarrow 0 \quad \text{as} \quad u \to 0.\end{aligned}$$

□

Exercise 4 (Frullani integral)**.** (i) Consider a function f such that f' exists and is continuous, and $f(0), f(\infty)$ are finite. Show that

$$\int\limits_0^\infty \frac{f(ax) - f(bx)}{x}\mathrm{d}x = (f(0) - f(\infty))\log\left(\frac{b}{a}\right),$$

for $b > a > 0$.

(ii) Consider the function $f(x) = \mathrm{e}^{-x}$ and set $a = \alpha > 0$ and $b = \beta = \alpha - z$ with $z < 0$. Show that

$$\exp\Big(\int\limits_0^\infty (\mathrm{e}^{zx} - 1)\frac{\beta}{x}\mathrm{e}^{-\alpha x}\mathrm{d}x\Big) = \frac{1}{(1 - z/\alpha)^\beta}.$$

Explain why this equality remains true for $z \in \mathbb{C}$ with $\Re z \le 0$.

Exercise 5. Consider the $\Gamma(\alpha, \beta)$ distribution, described by the density

$$f_{\alpha,\beta}(x) = \frac{\alpha^\beta}{\Gamma(\beta)}x^{\beta-1}\mathrm{e}^{-\alpha x},$$

concentrated on $(0, \infty)$.

(i) Compute the characteristic function of the $\Gamma(\alpha, \beta)$ distribution and show it is infinitely divisible.

(ii) Show that the Lévy triplet of the $\Gamma(\alpha, \beta)$ distribution is

$$b = \int_0^1 x\nu(\mathrm{d}x), \quad c = 0, \quad \nu(\mathrm{d}x) = \beta x^{-1}\mathrm{e}^{-\alpha x}\mathrm{d}x.$$

5.2. **Truncation functions and uniqueness.** We will now introduce truncation functions and discuss about the uniqueness of the representation (5.2).

The integrand in (5.2) is integrable with respect to the Lévy measure ν because it is bounded outside any neighborhoud of zero and

$$\mathrm{e}^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle 1_D(x) = O(|x|^2) \quad \text{as } |x| \to 0, \tag{5.3}$$

for any fixed u. There are many other ways to construct an integrable integrand, and we will be particularly interested in *continuous* ones because they are suitable for limit arguments. This leads to the notion of a *truncation function*. The following definitions are taken from [JS03] and [Sat99] respectively.

Definition 5.6. A truncation function is a bounded function $h : \mathbb{R}^d \to \mathbb{R}^d$ that satisfies $h(x) = x$ in a neighborhood of zero.

Definition 5.7. A truncation function $h' : \mathbb{R}^d \to \mathbb{R}$ is a bounded and measurable function, satisfying

$$h'(x) = 1 + o(|x|), \quad \text{as } |x| \to 0, \tag{5.4}$$

$$h'(x) = O(1/|x|), \quad \text{as } |x| \to \infty. \tag{5.5}$$

Remark 5.8. The two definitions are related via $h(x) = x \cdot h'(x)$.

Example 5.9. The following are some well-known examples of truncation functions:

(i) $h(x) = x1_D(x)$, typically called the *canonical* truncation function;
(ii) $h(x) \equiv 0$ and $h(x) \equiv x$, are also commonly used truncation functions; note that contrary to the other two examples, these are not always permissible choices;
(iii) $h(x) = \frac{x}{1+|x|^2}$, a continuous truncation function.

The Lévy–Khintchine representation of $\widehat{\rho}$ in (5.2) depends on the choice of the truncation function. Indeed, if we use another truncation function h instead of the canonical one, then (5.2) can be rewritten as

$$\widehat{\rho}(u) = \exp\left(i\langle u, b_h\rangle - \frac{\langle u, cu\rangle}{2} + \int_{\mathbb{R}^d} \big(\mathrm{e}^{i\langle u,x\rangle} - 1 - i\langle u, h(x)\rangle\big)\nu(\mathrm{d}x)\right), \tag{5.6}$$

with b_h defined as follows:

$$b_h = b + \int_{\mathbb{R}^d} \big(h(x) - x1_D(x)\big)\nu(\mathrm{d}x). \tag{5.7}$$

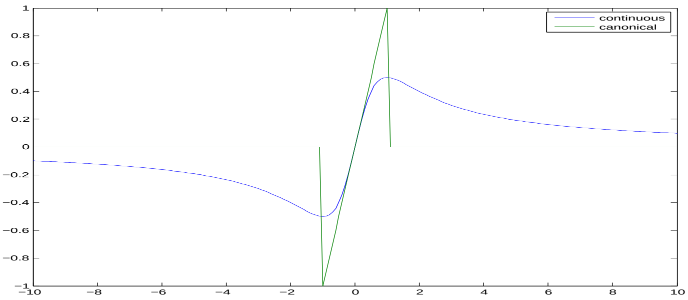

FIGURE 5.3. Illustration of the canonical and the continuous truncation functions from Example 5.9.

If we want to stress the dependence of the Lévy triplet on the truncation function, we will denote it by

$$(b_h, c, \nu)_h \qquad \text{or} \qquad (b, c, \nu)_h.$$

Note that diffusion characteristic c and the Lévy measure ν are invariant with respect to the choice of the truncation function.

Remark 5.10. There is no rule about which truncation function to use, among the permissible choices. One simply has to be *consistent* with ones choice of a truncation function. That is, the same choice should be made for the Lévy–Khintchine representation of the characteristic function, the Lévy triplet and the path decomposition of the Lévy process.

Example 5.11. Let us revisit the Lévy jump-diffusion process (3.1). In this example, since the Lévy measure is finite and we have assumed that $\mathbb{E}[J_k] < \infty$, all the truncation functions of Example 5.9 are permissible. The Lévy triplet of this process with respect to the canonical truncation function was presented in Example 5.5. The triplets with respect to the zero and the linear truncation functions are

$$\left(b - \int_{\mathbb{R}} x\lambda F(\mathrm{d}x), \sigma^2, \lambda \times F\right)_0 \qquad \text{and} \qquad \left(b, \sigma^2, \lambda \times F\right)_{\mathrm{id}}.$$

Although the Lévy–Khintchine representation depends on the choice of the truncation function, the Lévy triplet determines the law of the distribution uniquely (once the truncation function has been fixed).

Proposition 5.12. *The representation of $\widehat{\rho}$ by (b, c, ν) in* (5.2) *is unique.*

Sketch of Proof. We will outline the argument for the diffusion coefficient c; the complete proof can be found in [Sat99, Theorem 8.1(ii)].

Let $\widehat{\rho}$ be expressed by (b, c, ν)according to (5.2). By Taylor's theorem we get that

$$|\mathrm{e}^{i\langle u,x\rangle} - 1 - i\,\langle u, x\rangle\, 1_D(x)| \leq \frac{1}{2}|u|^2|x|^2 1_D(x) + 2 1_{D^c}(x). \tag{5.8}$$

Since the exponent in (5.2) is continuous in u, we have

$$\log \widehat{\rho}(su) = -s^2 \frac{\langle u, cu\rangle}{2} + is\,\langle u, b\rangle + \int\limits_{\mathbb{R}^d} \left(\mathrm{e}^{is\langle u,x\rangle} - 1 - is\,\langle u, x\rangle\, 1_D(x)\right)\nu(\mathrm{d}x),$$

for $s \in \mathbb{R}$. Now, by (5.8) and dominated convergence we get

$$s^{-2} \log \widehat{\rho}(su) \longrightarrow -\frac{\langle u, cu \rangle}{2}, \quad \text{as} \quad s \to \infty. \tag{5.9}$$

Therefore, the diffusion coefficient c is uniquely identified by ρ. The proof for ν is analogous, while once c, ν are uniquely determined, then b is identified as well. □

5.3. **Proof, "only if" part.** The next theorem contains an essential step in the proof of the "only if" part of the Lévy–Khintchine representation (Theorem 5.3). We denote by $C_\sharp$ the space of bounded continuous functions $f : \mathbb{R}^d \to \mathbb{R}$, vanishing in a neighborhood of 0.

Theorem 5.13. *Let $h' : \mathbb{R}^d \to \mathbb{R}$ be a continuous truncation function, i.e. satisfying* (5.4) *and* (5.5). *Suppose that ρ_n, $n \in \mathbb{N}$, are infinitely divisible distributions on $\mathbb{R}^d$ and that each $\widehat{\rho}_n$ has the Lévy–Khintchine representation with triplet $(\beta_n, c_n, \nu_n)_h$. Let ρ be a probability distribution on $\mathbb{R}^d$. Then* $\rho_n \xrightarrow{w} \rho$ if and only if *(i) ρ is infinitely divisible and (ii) $\widehat{\rho}$ has the Lévy–Khintchine representation with triplet $(\beta, c, \nu)_h$, where β, c and ν satisfy the following conditions:*

(1) If $f \in C_\sharp$ then

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x)\nu_n(\mathrm{d}x) = \int_{\mathbb{R}^d} f(x)\nu(\mathrm{d}x). \tag{5.10}$$

(2) Define the symmetric, non-negative definite matrices $c_{n,\varepsilon}$ via

$$\langle u, c_{n,\varepsilon} u \rangle = \langle u, c_n u \rangle + \int_{|x|\le\varepsilon} \langle u, x \rangle^2 \nu_n(\mathrm{d}x). \tag{5.11}$$

Then

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \big| \langle u, c_{n,\varepsilon} u \rangle - \langle u, cu \rangle \big| = 0 \quad \textit{for } u \in \mathbb{R}^d. \tag{5.12}$$

(3) $\beta_n \to \beta$.

Proof. "Only If" part. Assume that $\rho_n \to \rho$. Then ρ is infinitely divisible (by Lemma ...) and $\widehat{\rho}(u) \neq 0$ for all u (by Lemma ...). It follows from Lemma ... and Proposition ... that

$$\log \widehat{\rho}_n(u) \to \log \widehat{\rho}(u) \tag{5.13}$$

uniformly on any compact set.

Define the measure $\phi_n(\mathrm{d}x) = (|x|^2 \wedge 1)\nu_n(\mathrm{d}x)$, and note that $\phi_n(\mathbb{R}^d) = \int_{\mathbb{R}^d} \phi_n(\mathrm{d}x) < \infty$ by the definition of the Lévy measure. We claim that (ϕ_n) is *tight*, i.e. that

$$\sup_n \phi_n(\mathbb{R}^d) < \infty \quad \text{and} \quad \lim_{l\to\infty} \sup_n \int_{|x|>l} \phi_n(\mathrm{d}x) = 0; \tag{5.14}$$

for a proof of the tightness of (ϕ_n) we refer to [Sat99, pp.. . .]. Then, by Prokhorov's selection theorem [. . .] there exists a subsequence (ϕ_{n_k}) that converges to some finite measure ϕ.

Now, define ν via: $\nu(\{0\}) = 0$ and $\nu(\mathrm{d}x) = (|x|^2 \wedge 1)^{-1}\phi(\mathrm{d}x)$ on the set $\{|x| > 0\}$. The measure ϕ might have a point mass at 0, but this is ignored when defining ν. Let

$$g(u,x) = \mathrm{e}^{i\langle u,x\rangle} - 1 - i\langle u, h(x)\rangle, \tag{5.15}$$

which is bounded and continuous in x, for fixed u, due to the choice of a continuous truncation function h. We have that

$$\begin{aligned}\log\widehat{\rho}_n(u) &= -\frac{1}{2}\langle u, A_n u\rangle + i\langle u, \beta_n\rangle + \int\limits_{\mathbb{R}^d} g(u,x)\nu_n(\mathrm{d}x)\\ &= -\frac{1}{2}\langle u, A_{n,\epsilon} u\rangle + i\langle u, \beta_n\rangle + I_{n,\epsilon} + J_{n,\epsilon},\end{aligned} \tag{5.16}$$

where

$$I_{n,\epsilon} = \int\limits_{|x|\leq\epsilon} \left(g(u,x) + \frac{1}{2}\langle u,x\rangle^2\right)(|x|^2\wedge 1)^{-1}\rho_n(\mathrm{d}x) \tag{5.17}$$

and

$$J_{n,\epsilon} = \int\limits_{|x|>\epsilon} g(u,x)(|x|^2\wedge 1)^{-1}\rho_n(\mathrm{d}x). \tag{5.18}$$

Let $E := \{\epsilon > 0 : \int_{|x|=\epsilon}\rho(\mathrm{d}x) = 0\}$, then

$$\lim_{k\to\infty} J_{n_k,\epsilon} = \int\limits_{|x|>\epsilon} g(u,x)(|x|^2\wedge 1)^{-1}\rho(\mathrm{d}x) \tag{5.19}$$

hence

$$\lim_{E\ni\epsilon\downarrow 0}\lim_{k\to\infty} J_{n_k,\epsilon} = \int\limits_{\mathbb{R}^d} g(u,x)\nu(\mathrm{d}x), \tag{5.20}$$

because $g \in C_\sharp$. Furthermore,

$$\lim_{\epsilon\downarrow 0}\sup_n |I_{n,\epsilon}| = 0, \tag{5.21}$$

since

$$\left(g(u,x) + \frac{1}{2}\langle u,x\rangle^2\right)(|x|^2\wedge 1)^{-1} \longrightarrow_{x\to 0} 0, \tag{5.22}$$

by the definition of the truncation function h.

Consider the real and imaginary part in (5.16) separately, then:

$$\limsup_{k\to\infty}\langle u, \beta_{n_k}\rangle = \liminf_{k\to\infty}\langle u, \beta_{n_k}\rangle \Longrightarrow \beta_{n_k} \longrightarrow \beta, \tag{5.23}$$

$$\ldots \tag{5.24}$$

It follows that $\widehat{\rho}(u)$ has the Lévy–Khintchine representation with triplet $(\beta, c, \nu)_h$ and that (1), (2), (3) hold via the subsequence (ρ_{n_k}). The β, c and ν in the triplet are unique, hence the results hold for any subsequence and thus for the whole sequence. □

Finally, using the "only if" part of Theorem 5.13 we are ready to complete the proof of Theorem 5.3.

Proof of Theorem 5.3, "Only If" part. Let ρ be an infinitely divisible distribution. Choose a sequence $t_n \downarrow 0$ arbitrarily, and define ρ_n via

$$\widehat{\rho}_n(u) = \exp\Big(t_n^{-1}\left(\widehat{\rho}(u)^{t_n} - 1\right)\Big) = \exp\left(t_n^{-1}\int\limits_{\mathbb{R}^d\setminus\{0\}}\Big(\mathrm{e}^{i\langle u,x\rangle} - 1\Big)\rho^{t_n}(\mathrm{d}x)\right). \tag{5.25}$$

Clearly, the distribution ρ_n is compound Poisson and thus also infinitely divisible. Moreover, Taylor's expansion yields

$$\begin{aligned}\widehat{\rho}_n(u) &= \exp\Big(t_n^{-1}\Big(\mathrm{e}^{t_n\log\widehat{\rho}(u)} - 1\Big)\Big) = \exp\left(t_n^{-1}\left(t_n\log\widehat{\rho}(u) + O(t_n^2)\right)\right)\\ &= \exp\Big(\log\widehat{\rho}(u) + O(t_n)\Big),\end{aligned} \tag{5.26}$$

for fixed u, as $n\to\infty$. Hence $\widehat{\rho}_n(u)\to\widehat{\rho}(u)$ as $n\to\infty$.

Since ρ_n is infinitely divisible it has the Lévy–Khintchine representation (5.2) for some triplet $(b_n, c_n, \nu_n)_h$ (in this case with $h\equiv 0$). However, $\widehat{\rho}_n(u) \longrightarrow \widehat{\rho}(u)$ implies that $\rho_n \stackrel{w}{\longrightarrow} \rho$, by Proposition 4.1. Hence, using Theorem 5.13 yields that $\widehat{\rho}$ has the Lévy–Khintchine representation with some triplet $(b,c,\nu)_h$. Now, we can rewrite this as (5.1) and the result is proved. □

Corollary 5.14. *Every infinitely divisible distribution is the limit of a sequence of compound Poisson distributions.*

5.4. **The Lévy–Khintchine formula for Lévy processes.** In section 4.5 we showed that for any Lévy process $X = (X_t)_{t\geq 0}$, the random variables X_t are infinitely divisible. Next, we would like to compute the characteristic function of X_t. Since X_t is infinitely divisible for any $t\geq 0$, we know that X_1 is also infinitely divisible and has the Lévy–Khintchine representation in terms of some triplet (b,c,ν).

Definition 5.15. We define the *Lévy exponent* ψ of X by

$$\psi(u) = i\langle u,b\rangle - \frac{\langle u,cu\rangle}{2} + \int\limits_{\mathbb{R}}\left(\mathrm{e}^{i\langle u,x\rangle} - 1 - i\langle u,x\rangle 1_D(x)\right)\nu(\mathrm{d}x), \tag{5.27}$$

where

$$\mathbb{E}\left[\mathrm{e}^{i\langle u,X_1\rangle}\right] = \mathrm{e}^{\psi(u)}. \tag{5.28}$$

Theorem 5.16. *Let $X = (X_t)_{t\geq 0}$ be a Lévy process, then*

$$\mathbb{E}\left[\mathrm{e}^{i\langle u,X_t\rangle}\right] = \mathrm{e}^{t\psi(u)}, \tag{5.29}$$

where ψ is the Lévy exponent of X.

Proof. Define the function $\phi_u(t) = \mathbb{E}[\mathrm{e}^{i\langle u,X_t\rangle}]$. Using the independence and stationarity of the increments we have that

$$\begin{aligned}\phi_u(t+s) &= \mathbb{E}[\mathrm{e}^{i\langle u,X_{t+s}\rangle}] = \mathbb{E}[\mathrm{e}^{i\langle u,X_{t+s}-X_s\rangle}\mathrm{e}^{\langle iu,X_s\rangle}]\\ &= \mathbb{E}[\mathrm{e}^{i\langle u,X_{t+s}-X_s\rangle}]\mathbb{E}[\mathrm{e}^{\langle iu,X_s\rangle}] = \phi_u(t)\phi_u(s).\end{aligned} \tag{5.30}$$

Moreover, $\phi_u(0) = \mathbb{E}[\mathrm{e}^{i\langle u,X_0\rangle}] = 1$ by definition. Since X is stochastically continuous we can show that $t\mapsto\phi_u(t)$ is continuous (cf. Exercise 6).

Notice that (5.30) is Cauchy's second functional equation. The unique continuous solution to this equation has the form

$$\phi_u(t) = \mathrm{e}^{t\vartheta(u)}, \qquad \text{where} \qquad \vartheta : \mathbb{R}^d \to \mathbb{C}.$$

Now the result follows since X_1 is infinitely divisible, which yields

$$\phi_u(1) = \mathbb{E}[\mathrm{e}^{i\langle u, X_1\rangle}] = \mathrm{e}^{\psi(u)}.$$

□

Corollary 5.17. *The infinitely divisible random variable X_t has the Lévy triplet $(bt, ct, \nu t)$.*

Exercise 6. Let $X = (X_t)_{t\geq 0}$ be a stochastically continuous process. Show that the map $t \mapsto \varphi_{X_t}(u)$ is continuous for every $u \in \mathbb{R}^d$.

Exercise 7. Let X be a Lévy process with triplet (b, c, ν). Show that $-X$ is also a Lévy process and determine its triplet.

6. The Lévy–Itô decomposition

In the previous sections, we showed that for any Lévy process $X = (X_t)_{t\geq 0}$ the random variables X_t, $t \geq 0$ have an infinitely divisible distribution and determined this distribution using the Lévy–Khintchine representation. The aim of this section is to prove an "inverse" result: starting from an infinitely divisible distribution ρ, or equivalently from a Lévy triplet (b, c, ν), we want to construct a Lévy process $X = (X_t)_{t\geq 0}$ such that $\mathbb{P}_{X_1} = \rho$.

Theorem 6.1. *Let ρ be an infinitely divisible distribution with Lévy triplet (b, c, ν), where $b \in \mathbb{R}^d$, $c \in \mathbb{S}^d_{\geqslant 0}$ and ν is a Lévy measure. Then, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which four independent Lévy processes exist, $X^{(1)}, \dots, X^{(4)}$, where: $X^{(1)}$ is a constant drift, $X^{(2)}$ is a Brownian motion, $X^{(3)}$ is a compound Poisson process and $X^{(4)}$ is a square integrable, pure jump martingale with an a.s. countable number of jumps of magnitude less than 1 on each finite time interval. Setting $X = X^{(1)} + \dots + X^{(4)}$, we have that there exists a probability space on which a Lévy process $X = (X_t)_{t\geq 0}$ is defined, with Lévy exponent*

$$\psi(u) = i\langle u, b\rangle - \frac{\langle u, cu\rangle}{2} + \int_{\mathbb{R}^d} \big(\mathrm{e}^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle 1_D(x)\big)\nu(\mathrm{d}x) \tag{6.1}$$

for all $u \in \mathbb{R}^d$, and path, or Lévy–Itô, decomposition

$$X_t = bt + \sqrt{c}W_t + \int_0^t \int_{D^c} x\mu^X(\mathrm{d}s, \mathrm{d}x) + \int_0^t \int_D x(\mu^X - \nu^X)(\mathrm{d}s, \mathrm{d}x), \tag{6.2}$$

where $\nu^X = \mathrm{Leb} \otimes \nu$.

6.1. **Roadmap of the Proof.** We first provide an informal description of the proof, in order to motivate the mathematical tools required. Consider the exponent in the Lévy–Khintchine formula and rewrite it as follows:

$$\begin{aligned}\psi(u) &= \psi^{(1)}(u) + \psi^{(2)}(u) + \psi^{(3)}(u) + \psi^{(4)}(u)\\ &= i\,\langle u, b\rangle - \frac{\langle u, cu\rangle}{2} + \nu(D^c)\int_{D^c}\left(\mathrm{e}^{i\langle u,x\rangle} - 1\right)\frac{\nu(\mathrm{d}x)}{\nu(D^c)}\\ &\quad + \int_D \left(\mathrm{e}^{i\langle u,x\rangle} - 1 - i\,\langle u, x\rangle\right)\nu(\mathrm{d}x). \end{aligned} \tag{6.3}$$

Clearly $\psi^{(1)}$ corresponds to the characteristic exponent of a linear drift process with rate b, $\psi^{(2)}$ to a Brownian motion with covariance matrix c, and $\psi^{(3)}$ to a compound Poisson process with intensity $\lambda := \nu(D^c)$ and jump distribution $F(\mathrm{d}x) := \frac{\nu(\mathrm{d}x)}{\nu(D^c)} 1_{D^c}(\mathrm{d}x)$.

The most difficult part is to handle the process with characteristic exponent $\psi^{(4)}$. We can express this as follows:

$$\begin{aligned}\psi^{(4)}(u) &= \int_D \left(\mathrm{e}^{i\langle u,x\rangle} - 1 - i\,\langle u, x\rangle\right)\nu(\mathrm{d}x)\\ &= \sum_{n\geq 0}\left(\lambda_n \int_{D_n}\left(\mathrm{e}^{i\langle u,x\rangle} - 1\right)\nu_n(\mathrm{d}x) - i\Big\langle u, \lambda_n \int_{D_n} x\nu_n(\mathrm{d}x)\Big\rangle\right),\end{aligned}$$

where we define the discs $D_n := \{2^{-(n+1)} \leq |x| < 2^{-n}\}$, the intensities $\lambda_n := \nu(D_n)$ and the probability measures $\nu_n(\mathrm{d}x) := \frac{\nu(\mathrm{d}x)}{\lambda_n} 1_{D_n}(\mathrm{d}x)$ (see again Remark 5.2). We can intuitively understand this as the Lévy exponent of a superposition of compound Poisson processes with arrival rates λ_n and jump distributions ν_n, and an additional drift term that turns these processes into martingales. In order to convert this intuition into precise mathematical statements, we will need results on Poisson random measures and square integrable martingales.

6.2. **Poisson random measures.** Let us first consider a compound Poisson process with drift $X = (X_t)_{t\geq 0}$, with

$$X_t = bt + \sum_{k=1}^{N_t} J_k,$$

where $b \in \mathbb{R}$, N is a Poisson process with intensity λ and $J = (J_k)_{k\geq 0}$ is an i.i.d. sequence of random variables with distribution F. This process has a finite number of jumps in any finite time interval, and the time between consecutive jumps is exponentially distributed with parameter λ, the rate of the Poisson process. Denote the jump times of X by $(T_k)_{k\geq 1}$, and for a set $A \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}\backslash\{0\})$ define the random variable $\mu(A)$ via

$$\mu(A) := \#\{k \geq 1 : (T_k, J_k) \in A\} = \sum_{k\geq 1} 1_{\{(T_k,J_k)\in A\}}.$$

The random variable $\mu(A)$ takes values in $\mathbb{N}$ and counts the total number of jumps that belong to the time-space set A. The following lemma provides some important properties of μ.

Lemma 6.2. *Suppose that $A_1, \dots, A_k$, $k \geq 1$ are disjoint subsets of $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}\backslash\{0\})$. Then $\mu(A_1), \dots, \mu(A_k)$ are mutually independent random variables, and for each $i \in \{1, \dots, k\}$ the random variable $\mu(A_i)$ has a Poisson distribution with intensity*

$$\lambda_i = \lambda \int_{A_i} \mathrm{d}t \times F(\mathrm{d}x).$$

Moreover, for $\mathbb{P}$-a.e. realization of X, $\mu : \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}\backslash\{0\}) \to \mathbb{N} \cup \{\infty\}$ is a measure.

Exercise 8. Prove Lemma 6.2. Steps and Hints:

(i) Recall that the law of $\{T_1, \dots, T_n\}$ conditioned on the event $\{N_t = n\}$ has the same law as the ordered independent sample from n uniformly distributed r.v. on $[0, t]$.

(ii) Use (i) and the independence of J_k to show that the law of $\{(T_k, J_k), k = 1, \dots, n\}$ conditioned on $\{N_t = n\}$ equals the law of n independent bivariate r.v. with common distribution $t^{-1}\mathrm{d}s \times F(\mathrm{d}x)$ on $[0, t] \times \mathbb{R}$, ordered in time.

(iii) Show that, for $A \in \mathcal{B}([0, t]) \times \mathcal{B}(\mathbb{R})$, $\mu(A)$ conditioned on $\{N_t = n\}$ is a Binomial r.v. with probability of success $\int_A t^{-1}\mathrm{d}s \times F(\mathrm{d}x)$.

(iv) Show that

$$\mathbb{P}(\mu(A_1) = n_1, \dots, \mu(A_k) = n_k | N_t = n) = \frac{n!}{n_0! n_1! \dots n_k!} \prod_{i=0}^{k} \left(\frac{\lambda_i}{\lambda t}\right)^{n_i},$$

where $n_0 = n - \sum_{i=1}^k n_i$ and $\lambda_0 = \lambda t - \sum_{i=1}^k \lambda_i$.

(v) Finally, integrate out the conditioning to show that

$$\mathbb{P}(\mu(A_1) = n_1, \dots, \mu(A_k) = n_k) = \prod_{i=1}^{k} \mathrm{e}^{-\lambda_i} \frac{(\lambda_i)^{n_i}}{n_i!}.$$

The random measure introduced above is a special case of the more general notion of a Poisson random measure, defined as follows.

Definition 6.3 (Poisson random measure)**.** Let $(E, \mathcal{E}, \nu)$ be a σ-finite measure space. Consider a mapping $\mu : \mathcal{E} \to \mathbb{N} \cup \{\infty\}$ such that $\{\mu(A) : A \in \mathcal{E}\}$ is a family of random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then μ is called a *Poisson random measure with intensity* ν if

(1) μ is $\mathbb{P}$-a.s. a measure on $(E, \mathcal{E})$;

(2) for each $A \in \mathcal{E}$, $\mu(A)$ is Poisson distributed with parameter $\nu(A)$, where $\nu(A) \in [0, \infty]$;

(3) for mutually disjoint sets $A_1, \dots, A_n$ in $\mathcal{E}$, the random variables $\mu(A_1), \dots, \mu(A_n)$ are independent.

Remark 6.4. Note that if $\nu(A) = 0$ then we get that $\mathbb{P}(\mu(A) = 0) = 1$, while if $\nu(A) = \infty$ then we have that $\mathbb{P}(\mu(A) = \infty) = 1$.

Exercise 9. Show that every Lévy measure is a σ-finite measure on $\mathbb{R}^d\backslash\{0\}$, i.e. there exist sets $(A_i)_{i\in\mathbb{N}}$ such that $\cup_i A_i = \mathbb{R}^d\backslash\{0\}$ and $\nu(A_i) < \infty$, for all $i \in \mathbb{N}$.

Theorem 6.5. *Let $(E, \mathcal{E}, \nu)$ be a σ-finite measure space. Then, a Poisson random measure μ as defined above always exists.*

Proof. Step 1. Assume that $\nu(E) < \infty$. There is a standard construction of an infinite product space $(\Omega, \mathcal{F}, \mathbb{P})$ on which the following independent random variables are defined:

$$N \quad \text{and} \quad \{v_1, v_2, \dots\},$$

such that N is Poisson distributed with intensity $\nu(E)$ and each v_i has the probability distribution $\frac{\nu(\mathrm{d}x)}{\nu(E)}$. Define, for every $A \in \mathcal{E}$

$$\mu(A) = \sum_{i=1}^{N} 1_{\{v_i \in A\}}, \tag{6.4}$$

such that $N = \mu(E)$. For each $A \in \mathcal{E}$ and $i \geq 1$, the random variables $1_{\{v_i \in A\}}$ are $\mathcal{F}$-measurable, hence $\mu(A)$ is also $\mathcal{F}$-measurable. Let $A_1, \dots, A_k$ be mutually disjoint sets, then we can show that

$$\mathbb{P}(\mu(A_1) = n_1, \dots, \mu(A_k) = n_k) = \prod_{i=1}^{k} \mathrm{e}^{-\nu(A_i)} \frac{(\nu(A_i))^{n_i}}{n_i!}; \tag{6.5}$$

the derivation is similar to the proof of Lemma 6.2. Now, we can directly deduce that conditions (1)–(3) in the definition of a Poisson random measure are satisfied.

Step 2. Let ν be a σ-finite measure on $(E, \mathcal{E})$. Then, there exist subsets $(A_i)_{i\geq 1}$ of E such that $\cup_i A_i = E$ and $\nu(A_i) < \infty$. Define the measures

$$\nu_i(\cdot) := \nu(\cdot \cap A_i), \quad i \geq 1.$$

The first step yields that for each $i \geq 1$ there exists a probability space $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$ such that a Poisson random measure μ_i can be defined on $(A_i, \mathcal{E}_i, \nu_i)$, where $\mathcal{E}_i := \{B \cap A_i, B \in \mathcal{E}\}$. Now, we just have to show that

$$\mu(\cdot) := \sum_{i\geq 1} \mu(\cdot \cap A_i),$$

is a Poisson random measure on E with intensity ν, defined on the product space

$$(\Omega, \mathcal{F}, \mathbb{P}) := \bigotimes_{i\geq 1} (\Omega_i, \mathcal{F}_i, \mathbb{P}_i).$$

... □

The construction of the Poisson random measure leads immediately to the following corollaries.

Corollary 6.6. *Let μ be a Poisson random measure on $(E, \mathcal{E}, \nu)$. Then, for every $A \in \mathcal{E}$, we have that $\mu(\cdot \cap A)$ is a Poisson random measure on $(E \cap A, \mathcal{E} \cap A, \nu(\cdot \cap A))$. Moreover, if $A, B \in \mathcal{E}$ are disjoint, then the random variables $\mu(\cdot \cap A)$ and $\mu(\cdot \cap B)$ are independent.*

Corollary 6.7. *Let μ be a Poisson random measure on $(E, \mathcal{E}, \nu)$. Then, the support of μ is $\mathbb{P}$-a.s. countable. If, in addition, ν is a finite measure, then the support of μ is $\mathbb{P}$-a.s. finite.*

Corollary 6.8. *Assume that the measure ν has an atom, say at the point $\varepsilon \in E$. Then, it follows from the construction of the Poisson random measure μ that $\mathbb{P}(\mu(\{\varepsilon\}) \geq 1) > 0$. Conversely, if ν has no atoms then $\mathbb{P}(\mu(\{\varepsilon\}) = 0) = 1$ for all singletons $\varepsilon \in E$.*

6.3. **Integrals wrt Poisson random measures.** Let μ be a Poisson random measure defined on the space $(E, \mathcal{E}, \nu)$. The fact that μ is $\mathbb{P}$-a.s. a measure, allows us to use Lebesgue's theory of integration and consider, for a measurable function $f : E \to [0, \infty)$,

$$\int_E f(x)\mu(\mathrm{d}x),$$

which is then a well-defined, $[0, \infty]$-valued random variable. The same holds true for a signed function f, which yields a $[-\infty, \infty]$-valued random variable, provided that either f^+ or f^- are finite. This integral can be understood as follows:

$$\int_E f(x)\mu(\mathrm{d}x) = \sum_{v \in \mathrm{supp}(\mu)} f(v) \cdot m_v,$$

where m_v denotes the multiplicity of points at v (e.g., if μ has no atoms then $m_v = 0$ for every v). Convergence of integrals with respect to Poisson random measures and related properties are provided by the following result.

Theorem 6.9. *Let μ be a Poisson random measure on $(E, \mathcal{E}, \nu)$ and $f : E \to \mathbb{R}^d$ be a measurable function. Then:*

(i) $X = \int_E f(x)\mu(\mathrm{d}x)$ *is almost surely absolutely convergent if and only if*

$$\int_E (1 \wedge |f(x)|)\nu(\mathrm{d}x) < \infty. \tag{6.6}$$

(ii) If (6.6) *holds then*

$$\mathbb{E}\left[\mathrm{e}^{i\langle u,X\rangle}\right] = \exp\left(\int_E \left(\mathrm{e}^{i\langle u,f(x)\rangle} - 1\right)\nu(\mathrm{d}x)\right). \tag{6.7}$$

(iii) Moreover, if $f \in L^1(\nu)$ then

$$\mathbb{E}[X] = \int_E f(x)\nu(\mathrm{d}x), \tag{6.8}$$

while if $f \in L^2(\nu)$ then

$$\mathrm{Var}[X] = \int_E f(x)^2\nu(\mathrm{d}x). \tag{6.9}$$

Proof. Theorem 2.7 in [Kyp06]. The proof follows the "usual" recipe of first showing the result for a simple function f, then for a positive function using monotone convergence and finally for a general function by writing it as a difference of two positive functions. □

6.4. **Poisson random measures and stochastic processes.** In the sequel, we want to make the connection between Poisson random measures and stochastic processes. We will work in the following σ-finite space:

$$\big(E, \mathcal{E}, \nu^X\big) = \big(\mathbb{R}_{\geqslant 0} \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}_{\geqslant 0}) \times \mathcal{B}(\mathbb{R}^d), \mathrm{Leb} \otimes \nu\big)$$

where ν is a Lévy measure; see again Definition 5.1. We will denote the Poisson random measure on this space by μ^X. If we consider a time-space interval of the form $[s,t] \times A$, $s \leq t$, where $A \subset \mathbb{R}^d$ such that $0 \notin \overline{A}$, then the integral with respect to μ^X, denoted by

$$\int\limits_{[s,t]} \int\limits_{A} x \mu^X(\mathrm{d}s, \mathrm{d}x) =: X, \tag{6.10}$$

is a compound Poisson random variable with intensity $(t-s) \cdot \nu(A)$. This follows directly from Theorem 6.9, while we also get that

$$\mathbb{E}\big[\mathrm{e}^{i\langle u, X\rangle}\big] = \exp\bigg((t-s) \int_A x \nu(\mathrm{d}x)\bigg). \tag{6.11}$$

Let us consider the collection of random variables

$$\Bigg(\int\limits_0^t \int\limits_A x \mu^X(\mathrm{d}s, \mathrm{d}x)\Bigg)_{t \geq 0} \tag{6.12}$$

then one would naturally expect that this is a compound Poisson *stochastic process*.

Lemma 6.10. *Let μ^X be a Poisson random measure with intensity* $\mathrm{Leb} \otimes \nu$ *and assume that $A \subset \mathcal{B}(\mathbb{R}^d)$ such that $\nu(A) < \infty$. Then*

$$X_t = \int\limits_0^t \int\limits_A x \mu^X(\mathrm{d}s, \mathrm{d}x), \quad t \geq 0$$

is a compound Poisson process with arrival rate $\nu(A)$ and jump distribution $\frac{\nu(\mathrm{d}x)}{\nu(A)}|_A$.

Proof. Since $\nu(A) < \infty$, we have from Corollary 6.7 that the support of μ^X is finite. Hence, we can write X_t as follows

$$X_t = \sum_{0 \leq s \leq t} x \mu^X(\{s\} \times A) = \sum_{0 \leq s \leq t} \Delta X_s 1_{\{\Delta X_s \in A\}},$$

which shows that $t \mapsto X_t$ is a càdlàg function. Let $0 \leq s \leq t$, then the random variable

$$X_t - X_s = \int\limits_{(s,t]} \int\limits_A x \mu^X(\mathrm{d}s, \mathrm{d}x)$$

is independent from $\{X_u : u \leq s\}$, since Poisson random measures over disjoint sets are independent. From Theorem 6.9 we know that

$$\mathbb{E}\big[\mathrm{e}^{i\langle u,X_t\rangle}\big] = \exp\left(t\int_A(\mathrm{e}^{i\langle u,x\rangle}-1)\nu(\mathrm{d}x)\right). \tag{6.13}$$

The independence of increments allows us to deduce that

$$\begin{aligned}\mathbb{E}\big[\mathrm{e}^{i\langle u,X_t-X_s\rangle}\big] &= \frac{\mathbb{E}\big[\mathrm{e}^{i\langle u,X_t\rangle}\big]}{\mathbb{E}\big[\mathrm{e}^{i\langle u,X_s\rangle}\big]}\\ &= \exp\left((t-s)\int_A(\mathrm{e}^{i\langle u,x\rangle}-1)\nu(\mathrm{d}x)\right)\\ &= \mathbb{E}\big[\mathrm{e}^{i\langle u,X_{t-s}\rangle}\big],\end{aligned}$$

which yields that the increments are also stationary. Moreover, from (6.13) we have that X_t is compound Poisson distributed with arrival rate $t\cdot\nu(A)$ and jump distribution $\frac{\nu(\mathrm{d}x)}{\nu(A)}|_A$. Finally, we have that $X=(X_t)_{t\geq 0}$ is a compound Poisson process since it is a process with stationary and independent increments, whose increment distributions are compound Poisson. □

Lemma 6.11. *Consider the setting of the previous lemma and assume that* $\int_A |x|\nu(\mathrm{d}x)<\infty$. *Then*

$$M_t = \int_0^t\int_A x\mu^X(\mathrm{d}s,\mathrm{d}x) - t\int_A x\nu(\mathrm{d}x), \quad t\geq 0 \tag{6.14}$$

is a $\mathbb{P}$*-martingale relative to the filtration generated by the Poisson random measure* μ^X

$$\mathcal{F}_t := \sigma\left(\mu^X(G) : G\in\mathcal{B}([0,t])\times\mathcal{B}(\mathbb{R}^d)\right), \quad t\geq 0. \tag{6.15}$$

If, in addition, $\int_A|x|^2\nu(\mathrm{d}x)<\infty$ *then* M *is a square-integrable martingale.*

Proof. The process $M=(M_t)_{t\geq 0}$ is clearly adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by μ^X. Moreover, Theorem 6.9 together with the assumption $\int_A|x|\nu(\mathrm{d}x)<\infty$ immediately yield that

$$\mathbb{E}|M_t| \leq \mathbb{E}\left(\int_0^t\int_A |x|\mu^X(\mathrm{d}s,\mathrm{d}x)\right) - t\int_A|x|\nu(\mathrm{d}x) < \infty.$$

Using that M has stationary and independent increments, which follows directly from Lemma 6.10, we get that, for $0\leq s<t$,

$$\begin{aligned}\mathbb{E}[M_t-M_s|\mathcal{F}_s] &= \mathbb{E}[M_{t-s}]\\ &= \mathbb{E}\left(\int_s^t\int_A x\mu^X(\mathrm{d}s,\mathrm{d}x)\right) - (t-s)\int_A x\nu(\mathrm{d}x) = 0,\end{aligned}$$

again using Theorem 6.9.

Next, we just have to show that M is a square integrable. We have, using the martingale property of M, the properties of the variance and Theorem 6.9 once more, that

$$\begin{aligned}\mathbb{E}\big[M_t^2\big] = \operatorname{Var}\big[M_t\big] &= \operatorname{Var}\left(\int_s^t \int_A x\mu^X(\mathrm{d}s, \mathrm{d}x)\right)\\ &= t\int_A |x|^2 \nu(\mathrm{d}x) < \infty,\end{aligned}$$

which concludes the proof. □

The results of this section allow us to construct compound Poisson processes with jumps taking values in discs of the form $D_\varepsilon := \{\varepsilon < |x| \leq 1\}$, for any $\varepsilon \in (0, 1)$. However, we cannot consider the ball $D = \{|x| \leq 1\}$, i.e. set $\varepsilon = 0$, since there exist Lévy measures such that $\int_D |x|\nu(\mathrm{d}x) = \infty$. We will thus study the limit of the martingale M in Lemma 6.11 when the jumps belong to D_ε for $\varepsilon \downarrow 0$.

Exercise 10. Consider the measure on $\mathbb{R}^d \setminus \{0\}$ provided by

$$\nu(\mathrm{d}x) = |x|^{-(1+\alpha)} 1_{\{x<0\}} \mathrm{d}x + x^{-(1+\alpha)} 1_{\{x>0\}} \mathrm{d}x,$$

for $\alpha \in (1, 2)$. Show that it is a Lévy measure, such that $\int_D |x|\nu(\mathrm{d}x) = \infty$.

6.5. **Square integrable martingales.** Denote by $\mathcal{M}_T^2$ the space of right-continuous, zero mean, square integrable martingales. This is a Hilbert space with inner product defined by

$$\langle M, N\rangle := \mathbb{E}[M_T N_T].$$

Therefore, for any Cauchy sequence M^n in $\mathcal{M}_T^2$ there exists an element $M \in \mathcal{M}_T^2$ such that $\|M^n - M\| \to 0$ as $n \to \infty$, where $\|\cdot\| = \langle \cdot, \cdot \rangle$. A proof of this result can be found in Section 2.4 of [Kyp06]. In the sequel, we will make use of *Doob's martingale inequality* which states that for any $M \in \mathcal{M}_T^2$ it holds that

$$\mathbb{E}\big[\sup_{0\leq s\leq T} M_s^2\big] \leq 4\mathbb{E}\big[M_T^2\big].$$

The following result is crucial for the proof of the Lévy–Itô decomposition.

Theorem 6.12. *Consider the setting of Lemma 6.10 and recall that for any Lévy measure $\int_{|x|\leq 1} |x|^2\nu(\mathrm{d}x) < \infty$. For each $\varepsilon \in (0, 1)$ define the martingale*

$$M_t^\varepsilon = \int_0^t \int_{D_\varepsilon} x\mu^X(\mathrm{d}s, \mathrm{d}x) - t\int_{D_\varepsilon} x\nu(\mathrm{d}x), \tag{6.16}$$

where $D_\varepsilon = \{\varepsilon < |x| \leq 1\}$. Let $\overline{\mathcal{F}}_t$ denote the completion of $\bigcap_{s>t} \mathcal{F}_s$ by all the $\mathbb{P}$-null sets. Then, there exists a square integrable martingale $M = (M_t)_{t\geq 0}$ that satisfies:

(i) for each $T > 0$, there exists a deterministic subsequence $(\varepsilon_n^T)_{n\in\mathbb{N}}$ with $\varepsilon_n^T \downarrow 0$, along which

$$\mathbb{P}\left(\lim_{n\to\infty}\sup_{0\le s\le T}\left(M_s^{\varepsilon_n^T} - M_s\right)^2 = 0\right) = 1,$$

(ii) it is adapted to the filtration $(\overline{\mathcal{F}}_t)_{t\ge 0}$,
(iii) it has a.s. càdlàg paths,
(iv) it has stationary and independent increments,
(v) it has an a.s. countable number of jumps on each compact time interval.

Henceforth, there exists a Lévy process $M = (M_t)_{t\ge 0}$, which is a square integrable martingale, with an a.s. countable number of jumps such that, for each fixed $T > 0$, the sequence of martingales $(M_t^\varepsilon)_{0\le t\le T}$ converges uniformly to M on $[0,T]$ a.s. along a subsequence in ε.

Proof. (i) Consider a fixed $T > 0$ and set $0 < \eta < \varepsilon < 1$, then

$$\begin{aligned}\|M^\eta - M^\varepsilon\| &= \mathbb{E}\big[(M_T^\eta - M_T^\varepsilon)^2\big]\\ &= \mathbb{E}\left(\int_0^T\int_{\eta<|x|\le\varepsilon} x\mu^X(\mathrm{d}s,\mathrm{d}x) - T\int_{\eta<|x|\le\varepsilon} x\nu(\mathrm{d}x)\right)^2\\ &= T\int_{\eta<|x|\le\varepsilon} x^2\nu(\mathrm{d}x), \end{aligned}\tag{6.17}$$

see also Exercise 13. Since $\int_D |x|^2\nu(\mathrm{d}x) < \infty$, we have that

$$\|M^\eta - M^\varepsilon\| \longrightarrow 0, \text{ as } \varepsilon\downarrow 0,\tag{6.18}$$

hence (M^ε) is a Cauchy sequence on $\mathcal{M}_T^2$. Moreover, since $\mathcal{M}_T^2$ is a Hilbert space, there exists a martingale $M = (M_t)_{0\le t\le T}$ in $\mathcal{M}_T^2$ such that

$$\lim_{\varepsilon\downarrow 0}\|M - M^\varepsilon\| = 0.\tag{6.19}$$

Using Doob's maximal inequality, we get that

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\left[\sup_{0\le s\le T}(M_s^\varepsilon - M_s)^2\right] \le 4\lim_{\varepsilon\downarrow 0}\|M - M^\varepsilon\| = 0.\tag{6.20}$$

This allows us to conclude that the limit does not depend on T, thus we have a well-defined martingale limit $M = (M_t)_{t\ge 0}$. In addition, (6.20) yields that there exists a deterministic subsequence $(\varepsilon_n^T)_{n\ge 0}$, possibly depending on T, such that

$$\lim_{\varepsilon_n^T\downarrow 0}\sup_{0\le s\le T}(M_s^{\varepsilon_n^T} - M_s)^2 = 0, \qquad \mathbb{P}\text{-a.s.}\tag{6.21}$$

(ii) Follows directly from the definition of the filtration.
(iii) We can use the following facts:

- M^ε has càdlàg paths,
- M^ε converges uniformly to M, $\mathbb{P}$-a.s.;
- the space of càdlàg functions is closed under the supremum metric;

which yield immediately that M has càdlàg paths.
(iv) We have that a.s. uniform convergence along a subsequence implies also convergence in distribution along the same subsequence. Let $0 \leq q < r < s < t \leq T$ and $u, v \in \mathbb{R}^d$, then using dominated convergence we get

$$
\begin{aligned}
\mathbb{E}\big[&\exp\left(i\langle u, M_t - M_s\rangle + i\langle v, M_r - M_q\rangle\right)\big] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\exp\left(i\left\langle u, M_t^{\varepsilon_n^T} - M_s^{\varepsilon_n^T}\right\rangle + i\left\langle v, M_r^{\varepsilon_n^T} - M_q^{\varepsilon_n^T}\right\rangle\right)\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\exp\left(i\left\langle u, M_{t-s}^{\varepsilon_n^T}\right\rangle\right)\right] \mathbb{E}\left[\exp\left(i\left\langle v, M_{r-q}^{\varepsilon_n^T}\right\rangle\right)\right] \\
&= \mathbb{E}\big[\exp\left(i\langle u, M_{t-s}\rangle\right)\big]\mathbb{E}\big[\exp\left(i\langle v, M_{r-q}\rangle\right)\big],
\end{aligned}
$$

which yields that M has stationary and independent increments.
(v) According to Corollary 6.7, there exist, at most, an a.s. countable number of points in the support of the Poisson random measure μ^X. Moreover, since $\mathrm{Leb} \otimes \nu$ has no atoms, we get that μ^X takes values in $\{0, 1\}$ at singletons. Hence, every discontinuity of $M = (M_t)_{t\geq 0}$ corresponds to a single point in the support of μ^X, which yields that M has an a.s. countable number of jumps in every compact time interval. $\square$

6.6. **Proof of the Lévy–Itô decomposition.** Now, we are ready to complete the proof of the Lévy–Itô decomposition.

Proof of Theorem 6.1. Step 1. We first consider the processes $X^{(1)}$ and $X^{(2)}$ with characteristic exponents

$$
\psi^{(1)}(u) = i\langle u, b\rangle \quad \text{and} \quad \psi^{(2)}(u) = \frac{\langle u, cu\rangle}{2}, \tag{6.22}
$$

which correspond to a linear drift and a Brownian motion, i.e.

$$
X_t^{(1)} = bt \quad \text{and} \quad X_t^{(2)} = \sqrt{c}W_t, \tag{6.23}
$$

defined on some probability space $(\Omega^\natural, \mathcal{F}^\natural, \mathbb{P}^\natural)$.

Step 2. Given a Lévy measure ν, we know from Theorem 6.5 that there exists a probability space, denoted by $(\Omega^\sharp, \mathcal{F}^\sharp, \mathbb{P}^\sharp)$, such that we can construct a Poisson random measure μ^X on $(\mathbb{R}_{\geqslant 0} \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}_{\geqslant 0}) \times \mathcal{B}(\mathbb{R}^d), \mathrm{Leb} \otimes \nu)$. Let us define the process $X^{(3)} = (X_t^{(3)})_{t\geq 0}$ with

$$
X^{(3)} = \int_0^t \int_{D^c} x\mu^X(\mathrm{d}s, \mathrm{d}x). \tag{6.24}
$$

Using Lemma 6.10 we can deduce that $X^{(3)}$ is a compound Poisson process with intensity $\lambda := \nu(D^c)$ and jump distribution $F(\mathrm{d}x) := \frac{\nu(\mathrm{d}x)}{\nu(D^c)} 1_{D^c}(\mathrm{d}x)$.

Step 3. Next, from the Lévy measure ν we construct a process having only jumps less than 1. For each $0 < \varepsilon \leq 1$, define the compensated compound Poisson process $X^{(4,\varepsilon)} = (X_t^{(4,\varepsilon)})_{t\geq 0}$ with

$$
X^{(4,\varepsilon)} = \int_0^t \int_{\varepsilon<|x|\leq 1} x\mu^X(\mathrm{d}s, \mathrm{d}x) - t\int_{\varepsilon<|x|\leq 1} x\nu(\mathrm{d}x). \tag{6.25}
$$

Using Theorem 6.9 we know that $X^{(4,\varepsilon)}$ has the characteristic exponent

$$\psi^{(4,\varepsilon)}(u) = \int\limits_{\varepsilon<|x|\leq 1} \big(\mathrm{e}^{i\langle u,x\rangle} - 1 - i\,\langle u,x\rangle\,\big)\nu(\mathrm{d}x). \tag{6.26}$$

Now, according to Theorem 6.12 there exists a Lévy process, denoted by $X^{(4)}$, which is a square integrable, pure jump martingale defined on $(\Omega^\sharp, \mathcal{F}^\sharp, \mathbb{P}^\sharp)$, such that $X^{(4,\epsilon)}$ converges to $X^{(4)}$ uniformly on $[0,T]$ along an appropriate subsequence as $\varepsilon \downarrow 0$. Obviously, the characteristic exponent of the latter Lévy process is

$$\psi^{(4)}(u) = \int\limits_{|x|\leq 1} \mathrm{e}^{i\langle u,x\rangle} - 1 - i\,\langle u,x\rangle\,\big)\nu(\mathrm{d}x). \tag{6.27}$$

Since the sets $\{|x|>1\}$ and $\{|x|\leq 1\}$ are obviously disjoint, the processes $X^{(3)}$ and $X^{(4)}$ are independent. Moreover, they are both independent of $X^{(1)}$ and $X^{(2)}$, which are defined on a different probability space.

Step 4. In order to conclude the proof, we consider the product space

$$(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega^\natural, \mathcal{F}^\natural, \mathbb{P}^\natural) \times (\Omega^\sharp, \mathcal{F}^\sharp, \mathbb{P}^\sharp). \tag{6.28}$$

The process $X = (X_t)_{t\geq 0}$ with

$$\begin{aligned} X_t &= X_t^{(1)} + X_t^{(2)} + X_t^{(3)} + X_t^{(4)} \\ &= bt + \sqrt{c}W_t + \int_0^t\int_{D^c} x\mu^X(\mathrm{d}s,\mathrm{d}x) + \int_0^t\int_D x(\mu^X - \nu^X)(\mathrm{d}s,\mathrm{d}x), \end{aligned} \tag{6.29}$$

is defined on this space, has stationary and independent increments, càdlàg paths, and the characteristic exponent is

$$\begin{aligned} \psi(u) &= \psi^{(1)}(u) + \psi^{(2)}(u) + \psi^{(3)}(u) + \psi^{(4)}(u) \\ &= i\,\langle u,b\rangle - \frac{\langle u,cu\rangle}{2} + \int\limits_{\mathbb{R}^d} \big(\mathrm{e}^{i\langle u,x\rangle} - 1 - i\,\langle u,x\rangle\, 1_D(x)\big)\nu(\mathrm{d}x). \end{aligned}$$

□

Remark 6.13 (Truncation function)**.** Assume that the infinitely divisible distribution ρ has the Lévy triplet $(b_h, c, \nu)_h$ relative to the truncation function h, that is, assume that the Fourier transform of ρ is given by (5.6)–(5.7) instead of (5.2). Then, the Lévy–Itô decomposition takes the form

$$X_t = b_h t + \sqrt{c}W_t + \int_0^t\int_{\mathbb{R}^d} h^c(x)\mu^X(\mathrm{d}s,\mathrm{d}x) + \int_0^t\int_{\mathbb{R}^d} h(x)(\mu^X - \nu^X)(\mathrm{d}s,\mathrm{d}x), \tag{6.30}$$

where $h^c(x) = x - h(x)$. This form of the Lévy–Itô decomposition is *consistent* with the choice of the truncation function h; see also Remark 5.10.

Example 6.14. Revisiting the Lévy jump-diffusion process, we can easily see that (3.1) is the Lévy–Itô decomposition of this Lévy process for the truncation function $h(x) = x$, while

$$X_t = b_0 t + \sigma W_t + \sum_{k=1}^{N_t} J_k, \tag{6.31}$$

where $b_0 = b - \lambda\beta$ is the Lévy–Itô decomposition of X relative to the truncation function $h(x) \equiv 0$. See also Example 5.11.

Exercise 11. Suppose X, Y are two independent Lévy processes (on the same probability space). Show that $X + Y$ and $X - Y$ are again Lévy processes. Can $X - Y$ be a Lévy process in case X and Y are not independent?

Exercise 12. Let ν be a measure on the space $(E, \mathcal{E})$ and $f : E \to [0, \infty)$ be a measurable function. Then, for all $u > 0$, show that

$$\int\limits_E (\mathrm{e}^{uf(x)} - 1)\nu(\mathrm{d}x) < \infty \Longleftrightarrow \int\limits_E (1 \wedge f(x))\nu(\mathrm{d}x) < \infty. \tag{6.32}$$

Exercise 13. Consider the space $(\mathbb{R}_{\geqslant 0} \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}_{\geqslant 0}) \times \mathcal{B}(\mathbb{R}^d), \mathrm{Leb} \otimes \nu)$ and denote by μ^X the Poisson random measure with intensity $\mathrm{Leb} \otimes \nu$. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ such that $\int_{\mathbb{R}^d} |f(x)|^2 \nu(\mathrm{d}x) < \infty$. Show that the process $I = (I_t)_{t \geq 0}$ with

$$I_t = \int\limits_0^t \int\limits_{\mathbb{R}^d} f(x)\mu^X(\mathrm{d}s, \mathrm{d}x) - t \int\limits_{\mathbb{R}^d} f(x)\nu(\mathrm{d}x) \tag{6.33}$$

is a square integrable martingale and prove the following simplified version of the Itô isometry

$$\mathbb{E}\big[|I_t|^2\big] = t \int\limits_{\mathbb{R}^d} |f(x)|^2 \nu(\mathrm{d}x). \tag{6.34}$$

Exercise 14. Consider the setting of the previous exercise.
(i) Show that, for each $n \geq 2$ and each $t > 0$,

$$\int\limits_0^t \int\limits_{\mathbb{R}^d} x^n \mu^X(\mathrm{d}s, \mathrm{d}x) < \infty \quad \text{a.s.} \iff \int\limits_{|x|>1} |x|^n \nu(\mathrm{d}x) < \infty. \tag{6.35}$$

(ii) Assuming that the previous condition holds, show that

$$\left(\int\limits_0^t \int\limits_{\mathbb{R}^d} x^n \mu^X(\mathrm{d}s, \mathrm{d}x) - t \int\limits_{\mathbb{R}^d} x^n \nu(\mathrm{d}x) \right)_{t \geq 0} \tag{6.36}$$

is a martingale.

6.7. **Another approach to the basic connections.** We have now proved the basic connections between Lévy processes, infinitely divisible distributions and Lévy triplets, as announced in §3.1. The line of these proofs is diagrammatically represented in Figure 3.2. These relations are useful for the construction of new classes of Lévy processes and for the simulation of Lévy processes.

Naturally, there are other ways to prove these connections. Another approach is diagrammatically represented in Figure 6.4. The steps in these

$$\begin{tikzcd}
(X_t)_{t\geq 0} \arrow[rr, shift left, "SII"] \arrow[dr, "LI"'] & & \mathcal{L}(X_1)=\rho \arrow[ll, shift left, "KET"] \\
& (b,c,\nu) \arrow[ur, "LK"'] &
\end{tikzcd}$$

FIGURE 6.4. Another approach to the basic connections between Lévy processes, infinitely divisible distributions and Lévy triplets.

proofs can be summarized as follows:

(i) show that the law of X_t is infinitely divisible using the stationarity and independence of the increments (cf. Lemma 4.18);
(ii) show that for every Lévy triplet (b, c, ν) that satisfies (5.2) the measure ρ is infinitely divisible (cf. Theorem 5.3, "If" part);
(iii) use Kolmogorov's extension theorem to show that for every infinitely divisible distribution ρ, there exists a Lévy process $X = (X_t)_{t\geq0}$ such that $\mathbb{P}_{X_1} = \rho$;
(iv) prove the following version of the Lévy–Itô decomposition: every Lévy process admits the path decomposition (6.2). A corollary of the last result is the Lévy–Khintchine formula, cf. (5.27)-(5.29).

This line of proofs is based on the analysis of the jumps of Lévy process and follows in spirit the analysis of the jumps of the compound Poisson process in §6.2. We refer the interested reader to [App09] and [Pro04].

7. THE LÉVY MEASURE AND PATH PROPERTIES

The properties of the path of a Lévy process can be completely characterized on the basis of it's Lévy triplet, and in particular on the properties of the Lévy measure and the presence or absence of a Brownian component.

Throughout this section we assume that $X = (X_t)_{t\geq0}$ is a Lévy process with triplet (b, c, ν).

Proposition 7.1. *The paths of $X = (X_t)_{t\geq0}$ are a.s. continuous if and only if $\nu \equiv 0$.*

Exercise 15. Let X be a Lévy process with Lévy measure ν.

(i) Show that for $a > 0$

$$\mathbb{P}\Big(\sup_{0<s\leq t} |X_s - X_{s-}| > a\Big) = 1 - \mathrm{e}^{-t\nu(\mathbb{R}\setminus(-a,a))}.$$

(ii) Prove Proposition 7.1.

Proposition 7.2. *The paths of $X = (X_t)_{t\geq 0}$ are a.s. piecewise constant if and only if X is a compound Poisson process without drift.*

Exercise 16. Prove Proposition 7.2

Definition 7.3. We say that a Lévy process X has *infinite activity* if the sample paths of X have an a.s. countably infinite number of jumps on every compact time interval $[0, T]$. Otherwise we say that X has *finite activity.*

Proposition 7.4. *(1) If $\nu(\mathbb{R}^d) = \infty$ then X has infinite activity.*
(2) If $\nu(\mathbb{R}^d) < \infty$ then X has finite activity.

Exercise 17. Prove Proposition 7.4

Remark 7.5. By the definition of a Lévy measure, cf. Definition 5.1, we get immediately the following equivalences:

$$\nu(\mathbb{R}^d) = \infty \quad \Longleftrightarrow \quad \nu(D) = \infty \tag{7.1}$$

$$\nu(\mathbb{R}^d) < \infty \quad \Longleftrightarrow \quad \nu(D) < \infty. \tag{7.2}$$

7.1. **Variation of the paths.** We want to analyze the variation of the paths of a Lévy process. We will consider a real-valued Lévy process for simplicity, although the main result, Proposition 7.9, is valid for $\mathbb{R}^d$-valued Lévy processes.

Definition 7.6. Consider a function $f : [a, b] \to \mathbb{R}$. The *total variation* of f over $[a, b]$ is

$$\mathrm{TV}(f) = \sup_{\pi} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| \tag{7.3}$$

where $\pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of the interval $[a, b]$.

Lemma 7.7. *If $f : [a, b] \to \mathbb{R}$ is càdlàg and has finite variation on $[a, b]$, then*

$$\mathrm{TV}(f) \geq \sum_{t\in[a,b]} |\Delta f(t)|. \tag{7.4}$$

Proof. [App09, Theorem 2.3.14]. □

Definition 7.8. A stochastic process $X = (X_t)_{t\geq 0}$ has *finite variation* if the paths $(X_t(\omega))_{t\geq 0}$ have finite variation for almost all $\omega \in \Omega$. Otherwise, the process has *infinite variation.*

Proposition 7.9. *A Lévy process $X = (X_t)_{t\geq 0}$ with triplet (b, c, ν) has finite variation if and only if*

$$c = 0 \quad \textit{and} \quad \int_{|x|\leq 1} |x|\nu(\mathrm{d}x) < \infty. \tag{7.5}$$

Proof. Assume that $c = 0$, then the Lévy–Itô decomposition of the Lévy process takes the form

$$X_t = bt + \int_0^t \int_{|x|>1} x\mu^X(\mathrm{d}s, \mathrm{d}x) + \underbrace{\int_0^t \int_{|x|\leq 1} x(\mu^X - \nu^X)(\mathrm{d}s, \mathrm{d}x)}_{=X_t^{(4)}}. \tag{7.6}$$

We know that the first and second processses have finite variation, hence we will concentrate on the last part. Using the definition we have

$$\begin{aligned}
\mathrm{TV}\big(X_t^{(4)}\big) &= \sup_\pi \sum_{i=1}^n \big|X_{t_i}^{(4)} - X_{t_{i-1}}^{(4)}\big| \\
&= \sup_\pi \sum_{i=1}^n \Big| \int_{t_{i-1}}^{t_i} \int_{|x|\leq 1} x(\mu^X - \nu^X)(\mathrm{d}s, \mathrm{d}x) \Big| \\
&\leq \sup_\pi \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_{|x|\leq 1} |x|(\mu^X - \nu^X)(\mathrm{d}s, \mathrm{d}x) \\
&= \int_0^t \int_{|x|\leq 1} |x|(\mu^X - \nu^X)(\mathrm{d}s, \mathrm{d}x) \\
&= \int_0^t \int_{|x|\leq 1} |x|\mu^X(\mathrm{d}s, \mathrm{d}x) - t \int_{|x|\leq 1} |x|\nu^X(\mathrm{d}x) < \infty \text{ a.s.,}
\end{aligned} \tag{7.7}$$

since condition (7.5) for the Lévy measure implies that the integral with respect to the Poisson random measure μ^X is well defined and a.s. finite, cf. Theorem 6.9. Hence, we can split the integral wrt to the compensated random measure $\mu^X - \nu^X$ in two a.s. finite parts.

Conversely, assume that X has finite variation; then, we can use estimation (7.4), which yields

$$\infty > \mathrm{TV}(X_t) \geq \sum_{0\leq s\leq t} |\Delta X_s| \geq \sum_{0\leq s\leq t} |\Delta X_s| 1_{\{|\Delta X_s|\leq 1\}} = \int_0^t \int_{|x|\leq 1} |x|\mu^X(\mathrm{d}s, \mathrm{d}x).$$

Using again Theorem 6.9, finiteness of the RHS implies that

$$\int_0^t \int_{|x|\leq 1} |x|\nu^X(\mathrm{d}s, \mathrm{d}x) < \infty \quad \Longrightarrow \quad \int_{|x|\leq 1} |x|\nu(\mathrm{d}x) < \infty, \tag{7.8}$$

which yields the second condition. The Lévy–Itô decomposition of this Lévy process – where the jumps have finite variation – takes the form

$$X_t = b't + \sqrt{c}W_t + \sum_{s\leq t} \Delta X_s. \tag{7.9}$$

However, the paths of a Brownian motion have infinite variation, see e.g. [RY99], hence X will have paths of finite variation if and only if $c = 0$. □

Remark 7.10. Assume that the *jump part* of the Lévy process X has finite variation, i.e.

$$\int_{|x|\le 1} |x|\nu(\mathrm{d}x) < \infty, \tag{7.10}$$

then the Lévy–Itô decomposition of X takes the form

$$X_t = b_0 t + \sqrt{c}W_t + \int_0^t \int_{\mathbb{R}^d} x\mu^X(\mathrm{d}s, \mathrm{d}x), \tag{7.11}$$

and the Lévy–Khintchine formula can be written as

$$\mathbb{E}\big[\mathrm{e}^{i\langle u, X_1\rangle}\big] = \exp\left(i\left\langle u, b_0\right\rangle - \frac{\langle u, cu\rangle}{2} + \int_{\mathbb{R}^d} \big(\mathrm{e}^{i\langle u,x\rangle} - 1\big)\nu(\mathrm{d}x) \right). \tag{7.12}$$

In other words, we can use the truncation function $h(x) = 0$ and the drift term relative to this truncation function (denoted by b_0) is related to the drift term b in (5.2) via

$$b_0 = b - \int_{|x|\le 1} x\nu(\mathrm{d}x). \tag{7.13}$$

Note that this process is not necessarily a compound Poisson process, as the activity of the process might be infinite (i.e. $\nu(D) = \infty$).

7.2. **Subordinators.** Subordinators are Lévy processes with increasing paths.

Proposition 7.11. *Let $X = (X_t)_{t\ge 0}$ be a real-valued Lévy process with triplet (b, c, ν). The following are equivalent:*

(1) $X_t \ge 0$ a.s. for some $t > 0$;
(2) $X_t \ge 0$ a.s. for all $t > 0$;
(3) The sample paths of X are a.s. non-decreasing, that is
$t \ge s \Longrightarrow X_t \ge X_s$;
(4) The triplet (b, c, ν) satisfies: $b \ge 0$, $c = 0$, $\nu(-\infty, 0]) = 0$ and $\int_0^1 x\nu(\mathrm{d}x) < \infty$. In other words, X has a positive drift, no diffusion component, and jumps are only positive and have finite variation.

Proof. . . . □

8. Elementary operations

We will study some elementary operations on Lévy processes, such as linear transformations, projections and subordination. The resulting processes will be expressed again in terms of the corresponding Lévy exponents or Lévy triplets.

8.1. **Linear transformations of Lévy processes.** We are interested in linear transformations and projections of Lévy processes. The following result provides a complete characterization of linear transformations of Lévy processes in terms of their characteristic exponent and Lévy triplet.

Proposition 8.1. *Let $X = (X_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued Lévy process with triplet $(b, c, \nu)_h$. Let U be an $n \times d$ matrix with real entries ($U \in M_{nd}(\mathbb{R})$). Then, $X^U = (X_t^U)_{t\geq 0}$ with $X_t^U := UX_t$ is an $\mathbb{R}^n$-valued Lévy process with Lévy triplet $(b^U, c^U, \nu^U)_{h'}$, where*

$$\begin{aligned} b^U &= Ub + \int_{\mathbb{R}^d} (h'(Ux) - Uh(x))\nu(\mathrm{d}x) \\ c^U &= UcU^\top \\ \nu^U(E) &= \nu(\{x \in \mathbb{R}^d : Ux \in E\}), \quad E \in \mathcal{B}(\mathbb{R}^n \backslash \{0\}). \end{aligned} \tag{8.1}$$

Here $h'(x)$ denotes a truncation function on $\mathbb{R}^n$.

Proof. Since U defines a linear mapping from $\mathbb{R}^d$ to $\mathbb{R}^n$, it is clear that X^U has independent and stationary increments, and is stochastically continuous; moreover, $X_0^U = 0$ a.s. In other words, X^U is an $\mathbb{R}^n$-valued Lévy process.

We will show that ν^U is a Lévy measure and the integral on the RHS of b^U is finite; hence, the triplet (b^U, c^U, ν^U) in (8.1) is indeed a Lévy triplet. Then we will derive the characteristic function of X_t^U.

Clearly ν^U has no mass at the origin; in addition we have that

$$\begin{aligned} \int_{\mathbb{R}^n} (|y|^2 \wedge 1)\nu^U(\mathrm{d}y) &= \int_{\mathbb{R}^d} (|Ux|^2 \wedge 1)\nu(\mathrm{d}x) \\ &\leq (\|U\|^2 \vee 1) \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(\mathrm{d}x) < \infty, \end{aligned}$$

because the induced norm satisfies $|Ux| \leq \|U\||x|$ for all $U \in M_{nd}(\mathbb{R})$ and $x \in \mathbb{R}^d$.

Next, we restrict ourselves to the canonical truncation function for simplicity, i.e. $h(x) = x1_{\{|x|\leq 1\}}$, and derive the following result for the integral on the RHS of b^U:

$$
\begin{aligned}
&\int_{\mathbb{R}^d} |h'(Ux) - Uh(x)|\nu(\mathrm{d}x) \\
&\qquad \le \int_{\mathbb{R}^d} |Ux||1_{\{|Ux|\le 1\}} - 1_{\{|x|\le 1\}}|\nu(\mathrm{d}x) \\
&\qquad = \int_{\mathbb{R}^d} |Ux||1_{\{|Ux|\le 1<|x|\}} - 1_{\{|x|\le 1<|Ux|\}}|\nu(\mathrm{d}x) \\
&\qquad \le \int_{\{|Ux|\le 1<|x|\}} |Ux|\nu(\mathrm{d}x) + \int_{\{|x|\le 1<|Ux|\}} |Ux|\nu(\mathrm{d}x) \\
&\qquad \le \int_{\{|x|>1\}} \nu(\mathrm{d}x) + \|U\| \int_{\{|x|\le 1<\|U\||x|\}} |x|\nu(\mathrm{d}x) \\
&\qquad \le \int_{\{|x|>1\}} \nu(\mathrm{d}x) + \|U\|^2 \int_{\{\frac{1}{\|U\|}<|x|\le 1\}} |x|^2\nu(\mathrm{d}x) < \infty.
\end{aligned}
$$

Finally, regarding the characteristic function we have for any $z \in \mathbb{R}^n$

$$
\begin{aligned}
\mathbb{E}\left[\mathrm{e}^{i\langle z, X_1^U\rangle}\right] &= \mathbb{E}\left[\mathrm{e}^{i\langle z, UX_1\rangle}\right] = \mathbb{E}\left[\mathrm{e}^{i\langle U^\top z, X_1\rangle}\right] \\
&= \exp\Bigg(i\langle U^\top z, b\rangle - \frac{1}{2}\langle U^\top z, cU^\top z\rangle \\
&\qquad + \int_{\mathbb{R}^d} (\mathrm{e}^{i\langle U^\top z, x\rangle} - 1 - i\langle U^\top z, h(x)\rangle)\nu(\mathrm{d}x)\Bigg) \\
&= \exp\Bigg(i\langle z, Ub\rangle - \frac{1}{2}\langle z, UcU^\top z\rangle \\
&\qquad + \int_{\mathbb{R}^d} (\mathrm{e}^{i\langle z, Ux\rangle} - 1 - i\langle z, Uh(x)\rangle)\nu(\mathrm{d}x)\Bigg) \\
&= \exp\Bigg(i\langle z, b^U\rangle - \frac{1}{2}\langle z, c^U z\rangle \\
&\qquad + \int_{\mathbb{R}^n} (\mathrm{e}^{i\langle z, y\rangle} - 1 - i\langle z, h'(y)\rangle)\nu^U(\mathrm{d}y)\Bigg),
\end{aligned}
$$

where b^U is given by (8.1). Thus, (b^U, c^U, ν^U) is indeed the triplet of the Lévy process X^U. □

8.2. **Subordination.** Subordinators are Lévy processes with a.s. non-decreasing paths; see . . . for a complete characterization. Subordinators can be thought of a stochastic model for the evolution of time. *Subordination* is the tranformation of one stochastic process to a new one through a random time-change by an indepedent subordinator. This idea was introduced by Bochner. Note that one can also subordinate a semigroup of linear operators to create a new semigroup.

In mathematical finance, subordination plays a prominent role. Many popular Lévy modes can be constructed by subordinating Brownian motion, e.g. VG and NIG. In that setting, one often speaks about "calendar" time and "business" time. Subordination is also used to create multidimensional models with dependence structure via a common time-change.

Let $Y = (Y_t)_{t\geq 0}$ be a suborinator, i.e. a Lévy process with a.s. increasing paths. Let ψ_Y denote the characteristic exponent of Y; using ... we know that it has the form

$$\psi_Y(u) = ib_Y u + \int\limits_{(0,\infty)} (\mathrm{e}^{iux} - 1)\nu_Y(\mathrm{d}y). \tag{8.2}$$

Note that $\mathbb{E}[\mathrm{e}^{uY_t}] < \infty$ for all $u \leq 0$ since Y takes only non-negative values; therefore, $\int_{x>1} \mathrm{e}^{uy}\nu_Y(\mathrm{d}y) < \infty$ for all $u \leq 0$. Therefore, the characteristic exponent of Y can be extended to an analytic function for $u \leq 0$, and the moment generating function of Y_t is

$$\mathbb{E}[\mathrm{e}^{\langle u, Y_t\rangle}] = \mathrm{e}^{t\phi_Y(u)} \tag{8.3}$$

where

$$\phi_Y(u) = b_Y u + \int\limits_{(0,\infty)} (\mathrm{e}^{ux} - 1)\nu_Y(\mathrm{d}y). \tag{8.4}$$

Theorem 8.2. *Let X be an $\mathbb{R}^d$-valued Lévy process with characteristic exponent ψ_X. Let Y be a subordinator with cumulant generating function ϕ_Y, where Y is independent of X. Define the process $Z = (Z_t)_{t\geq 0}$ for each $\omega \in \Omega$ via*

$$Z_t(\omega) = X_{Y_t(\omega)}(\omega). \tag{8.5}$$

Then, Z is a Lévy process with characteristic exponent

$$\psi_Z(u) = \phi_Y(\psi_X(u)). \tag{8.6}$$

Proof. ... □

Exercise 18. Show that any Lévy process with finite variation can be written as the difference of two independent subordinators.

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