# Generalized (or Confluent) Vandermonde Determinants

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**Abstract**

We present an explicit computation of some determinants which can be considered as generalizations of the Vandermonde determinant. The result is not new [1]. As an application we compute the Wronskian of the standard solutions of the general linear homogeneous ordinary differential equation with constant coefficients, whose associated characteristic equation has repeated roots.

**Keywords.** (Generalized or confluent) Vandermonde determinant; linear homogeneous ordinary differential equation with constant coefficients; Wronskian.
**2010 AMS Mathematics Classification.** 15A15; 34A30; 34A05.

## 1 The calculation of generalized (or confluent) Vandermonde determinants

It is well known that

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_\nu \\ x_1^2 & x_2^2 & \cdots & x_\nu^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\nu-1} & x_2^{\nu-1} & \cdots & x_\nu^{\nu-1} \end{vmatrix} = \prod_{1 \le j < k \le \nu} (x_k - x_j), \tag{1.1}$$

where the left-hand side of (1.1) is the so-called $\nu \times \nu$ *Vandermonde determinant*. The justification of equation (1.1) is relatively easy. One can use, e.g., induction on $\nu$ or, alternatively, one can first notice that the sides of (1.1) have to be equal up to a constant factor $c_\nu$, since both sides are polynomials in the variables $x_1, \ldots, x_\nu$ of the same degree and having the same one-degree factors. Then, the evaluation of $c_\nu$ can be done by, say, comparing coefficients of some monomial.

**Definition.** Let $A$ and $\alpha$ be integers with $A \geq \alpha \geq 1$. The $A \times \alpha$ (*generalized*) *Vandermonde block* is the matrix

$$B(x; A \times \alpha) = (c_{jk})_{\substack{1 \leq j \leq A \\ 1 \leq k \leq \alpha}}, \qquad \text{where} \quad c_{jk} := \binom{j-1}{k-1} x^{j-k}, \tag{1.2}$$

with the convention that $\binom{j-1}{k-1} = 0$ for $j < k$. Notice that $B(x; A \times \alpha)$ is a square matrix only if $A = \alpha$, and in this case its determinant is 1.

Next, let $\alpha_1, \ldots, \alpha_m$ be strictly positive integers and

$$A = \alpha_1 + \cdots + \alpha_m\,. \tag{1.3}$$

Putting the blocks $B(x_1; A \times \alpha_1), \ldots, B(x_m; A \times \alpha_m)$ side by side we form the $A \times A$ (square) matrix

$$M(x_1, \ldots, x_m; \alpha_1, \ldots, \alpha_m) := [B(x_1; A \times \alpha_1) \cdots B(x_m; A \times \alpha_m)]\,. \tag{1.4}$$

Then, we consider its determinant

$$F(x_1, \ldots, x_m; \alpha_1, \ldots, \alpha_m) := \det M(x_1, \ldots, x_m; \alpha_1, \ldots, \alpha_m), \tag{1.5}$$

namely

$$F(x_1, \ldots, x_m; \alpha_1, \ldots, \alpha_m)$$

$$= \begin{vmatrix} 1 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ x_1 & 1 & \cdots & 0 & \cdots & x_m & \cdots & 0 \\ x_1^2 & 2x_1 & \cdots & 0 & \cdots & x_m^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ x_1^{A-2} & (A-2)x_1^{A-3} & \cdots & \binom{A-2}{\alpha_1-1} x_1^{A-1-\alpha_1} & \cdots & x_m^{A-2} & \cdots & \binom{A-2}{\alpha_m-1} x_m^{A-1-\alpha_m} \\ x_1^{A-1} & (A-1)x_1^{A-2} & \cdots & \binom{A-1}{\alpha_1-1} x_1^{A-\alpha_1} & \cdots & x_m^{A-1} & \cdots & \binom{A-1}{\alpha_m-1} x_m^{A-\alpha_m} \end{vmatrix}. \tag{1.6}$$

Thus, $F(x_1,\dots,x_m;\alpha_1,\dots,\alpha_m)$ is a polynomial in $x_1,\dots,x_m$. For instance, if $m=3$ and $(\alpha_1,\alpha_2,\alpha_3)=(2,3,1)$ we get

$$F(x_1,x_2,x_3;2,3,1)=\begin{vmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ x_1 & 1 & x_2 & 1 & 0 & x_3 \\ x_1^2 & 2x_1 & x_2^2 & 2x_2 & 1 & x_3^2 \\ x_1^3 & 3x_1^2 & x_2^3 & 3x_2^2 & 3x_2 & x_3^3 \\ x_1^4 & 4x_1^3 & x_2^4 & 4x_2^3 & 6x_2^2 & x_3^4 \\ x_1^5 & 5x_1^4 & x_2^5 & 5x_2^4 & 10x_2^3 & x_3^5 \end{vmatrix} = (x_2-x_1)^6(x_3-x_1)^2(x_3-x_2)^3. \tag{1.7}$$

In the case $\alpha_1=\cdots=\alpha_A=1$ (hence $m=A$), $F(x_1,\dots,x_A;1,\dots,1)$ becomes the standard Vandermonde determinant and we have

$$F(x_1,\dots,x_A;1,\dots,1)=\prod_{1\le j<k\le A}(x_k-x_j).$$

On the other hand, in the extreme case $m=1$ we have $\alpha_1=A$ and

$$F(x_1;A)\equiv 1.$$

**Observation.** Assume $\alpha_j\ge 2$ for some $j=1,\dots,m$. Set

$$f(y):=F(x_1,\dots,x_{j-1},x_j,y,x_{j+1},\dots,x_m;\alpha_1,\dots,\alpha_{j-1},(\alpha_j-1),1,\alpha_{j+1},\dots,\alpha_m) \tag{1.8}$$

(thus, $f(y)$ is a polynomial in the $m+1$ variables $x_1,\dots,x_m$ and $y$). Then

$$F(x_1,\dots,x_m;\alpha_1,\dots,\alpha_m)=\frac{f^{(\alpha_j-1)}(x_j)}{(\alpha_j-1)!}. \tag{1.9}$$

For example, if we take $m=3$, $(\alpha_1,\alpha_2,\alpha_3)=(2,3,1)$, and $j=2$ we have

$$f(y)=F(x_1,x_2,y,x_3;2,2,1,1)=\begin{vmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ x_1 & 1 & x_2 & 1 & y & x_3 \\ x_1^2 & 2x_1 & x_2^2 & 2x_2 & y^2 & x_3^2 \\ x_1^3 & 3x_1^2 & x_2^3 & 3x_2^2 & y^3 & x_3^3 \\ x_1^4 & 4x_1^3 & x_2^4 & 4x_2^3 & y^4 & x_3^4 \\ x_1^5 & 5x_1^4 & x_2^5 & 5x_2^4 & y^5 & x_3^5 \end{vmatrix} \tag{1.10}$$

and $f''(x_2)=2!F(x_1,x_2,x_3;2,3,1)$, where $F(x_1,x_2,x_3;2,3,1)$ is the determinant of (1.7).

The following proposition appears as a problem in [1].

**Proposition.** Let $m\ge 2$. Then,

$$F(x_1,\dots,x_m;\alpha_1,\dots,\alpha_m)=\prod_{1\le j<k\le m}(x_k-x_j)^{\alpha_j\alpha_k}. \tag{1.11}$$

*Proof.* We will use induction on $\max\{\alpha_1,\ldots,\alpha_m\}$, i.e. the maximum of the $\alpha_j$'s. If $\alpha_1=\cdots=\alpha_m=1$, the left-hand side of (1.11) becomes the standard Vandermonde determinant and (1.11) holds.
First inductive hypothesis: Assume that (1.11) is true for $\max\{\alpha_1,\ldots,\alpha_m\}<n$, where $n\geq 2$. We need to show that (1.11) also holds for $\max\{\alpha_1,\ldots,\alpha_m\}=n$. We will prove this by induction on $\#\{\alpha_j:\alpha_j=n\}$, namely the number of $\alpha_j$'s that assume the maximum value $n$.
We begin by considering the case where $\alpha_i=n$ for some $i\in\{1,\ldots,m\}$ and $\max_{j\neq i}\alpha_j<n$, namely $\#\{\alpha_j:\alpha_j=n\}=1$. Set

$$f(y):=F(x_1,\ldots,x_{i-1},x_i,y,x_{i+1},\ldots,x_m;\alpha_1,\ldots,\alpha_{i-1},(\alpha_i-1),1,\alpha_{i+1},\ldots,\alpha_m). \tag{1.12}$$

Then, since $\max\{\alpha_1,\ldots,\alpha_{i-1},(\alpha_i-1),1,\alpha_{i+1},\ldots,\alpha_m\}=n-1$, the first inductive hypothesis implies that

$$f(y)=(y-x_i)^{n-1}\prod_{\substack{l=1\\ l\neq i}}^{m}(y-x_l)_i^{\alpha_l}\prod_{\substack{l=1\\ l\neq i}}^{m}(x_i-x_l)_i^{(n-1)\alpha_l}\prod_{\substack{1\leq j<k\leq m\\ j,k\neq i}}(x_k-x_j)^{\alpha_j\alpha_k}, \tag{1.13}$$

where for typographical convenience we have set $(y-x_l)_i:=(y-x_l)\mathrm{sgn}(i-l)$ and $(x_i-x_l)_i:=(x_i-x_l)\mathrm{sgn}(i-l)$. We continue by writing (1.13) in the form

$$f(y)=(y-x_i)^{n-1}f_1(y), \tag{1.14}$$

where

$$f_1(y):=\prod_{\substack{l=1\\ l\neq i}}^{m}(y-x_l)_i^{\alpha_l}\prod_{\substack{l=1\\ l\neq i}}^{m}(x_i-x_l)_i^{(n-1)\alpha_l}\prod_{\substack{1\leq j<k\leq m\\ j,k\neq i}}(x_k-x_j)^{\alpha_j\alpha_k}. \tag{1.15}$$

Now, the observation (1.9) applied to (1.12) gives

$$F(x_1,\ldots,x_m;\alpha_1,\ldots,\alpha_m)=\frac{f^{(n-1)}(x_i)}{(n-1)!}. \tag{1.16}$$

Applying (1.16) to (1.14) yields

$$F(x_1,\ldots,x_m;\alpha_1,\ldots,\alpha_m)=f_1(x_i) \tag{1.17}$$

and hence, in view of (1.15) we get that $F(x_1,\ldots,x_m;\alpha_1,\ldots,\alpha_m)$ satisfies (1.11).
Second inductive hypothesis: Assume now that (1.11) is true for $\max\{\alpha_1,\ldots,\alpha_m\}=n$ and $\#\{\alpha_j:\alpha_j=n\}<p$, where $p\geq 2$. It remains to show that (1.11) is also true for $\max\{\alpha_1,\ldots,\alpha_m\}=n$ and $\#\{\alpha_j:\alpha_j=n\}=p$.

Of course, $p \leq m$ (since it is impossible to have $p > m$) and there are indices $1 \leq i_1 < \cdots < i_p \leq m$ such that $\alpha_{i_1} = \cdots = \alpha_{i_p} = n$ (while $\alpha_j < n$ for any index $j \notin \{i_1, \ldots, i_p\}$).
Let us set

$$g(y) := F(x_1, \ldots, x_{i_p-1}, x_{i_p}, y, x_{i_p+1}, \ldots, x_m; \alpha_1, \ldots, \alpha_{i_p-1}, (\alpha_{i_p}-1), 1, \alpha_{i_p+1}, \ldots, \alpha_m). \tag{1.18}$$

Among the $m+1$ numbers $\alpha_1, \ldots, \alpha_{i_p-1}, (\alpha_{i_p}-1), 1, \alpha_{i_p+1}, \ldots, \alpha_m$, there are exactly $p-1$ which are equal to $n$, hence the second inductive hypothesis implies that

$$g(y) = (y - x_{i_p})^{n-1} \prod_{\substack{l=1 \\ l \neq i_p}}^{m} (y - x_l)_{i_p}^{\alpha_l} \prod_{\substack{l=1 \\ l \neq i_p}}^{m} (x_{i_p} - x_l)_{i_p}^{(n-1)\alpha_l} \prod_{\substack{1 \leq j < k \leq m \\ j,k \neq i_p}} (x_k - x_j)^{\alpha_j \alpha_k}, \tag{1.19}$$

where, as before $(y - x_l)_{i_p} = (y - x_l)\text{sgn}(i_p - l)$ and $(x_{i_p} - x_l)_{i_p} = (x_i - x_l)\text{sgn}(i_p - l)$. We write (1.19) in the form

$$g(y) = (y - x_{i_p})^{n-1} g_1(y), \tag{1.20}$$

where

$$g_1(y) := \prod_{\substack{l=1 \\ l \neq i_p}}^{m} (y - x_l)_{i_p}^{\alpha_l} \prod_{\substack{l=1 \\ l \neq i_p}}^{m} (x_{i_p} - x_l)_{i_p}^{(n-1)\alpha_l} \prod_{\substack{1 \leq j < k \leq m \\ j,k \neq i_p}} (x_k - x_j)^{\alpha_j \alpha_k}. \tag{1.21}$$

Next, the observation (1.9) applied to (1.18) gives

$$F(x_1, \ldots, x_m; \alpha_1, \ldots, \alpha_m) = \frac{g^{(n-1)}(x_{i_p})}{(n-1)!}. \tag{1.22}$$

Applying (1.22) to (1.20) yields

$$F(x_1, \ldots, x_m; \alpha_1, \ldots, \alpha_m) = g_1(x_{i_p}) \tag{1.23}$$

and hence, in view of (1.21) we get that $F(x_1, \ldots, x_m; \alpha_1, \ldots, \alpha_m)$ satisfies (1.11). ∎

# 2 An application

Consider the differential equation

$$\frac{d^A u}{dt^A} + \sum_{k=0}^{A-1} c_k \frac{d^k u}{dt^k} = 0, \tag{2.1}$$

where the $c_k$'s, $k = 0, \dots, A-1$ are complex constants.
The characteristic equation associated to (2.1) is

$$p(r) := r^A + \sum_{k=0}^{A-1} c_k r^k = 0. \tag{2.2}$$

Let us assume that the polynomial $p(r)$ of (2.2) can be factored as

$$p(r) = \prod_{j=1}^{m} (r - x_j)^{\alpha_j}, \tag{2.3}$$

where $x_1, \dots, x_m$ are distinct complex numbers (of course, $\alpha_1 + \cdots + \alpha_m = A$). Then, it is well known that the functions

$$e^{x_1 t}, te^{x_1 t}, \dots, \frac{t^{\alpha_1 - 1} e^{x_1 t}}{(\alpha_1 - 1)!}; \dots; e^{x_m t}, te^{x_m t}, \dots, \frac{t^{\alpha_m - 1} e^{x_m t}}{(\alpha_m - 1)!} \tag{2.4}$$

(a total of $A$ functions) are solutions of (2.1). Their Wronskian $W(t)$ satisfies the Abel's formula, which in our case reads

$$W(t) = W(0) \exp(-c_{A-1} t). \tag{2.5}$$

Using the fact that

$$\frac{d^j}{dt^j} \left[ \frac{t^k e^{xt}}{k!} \right] \Bigg|_{t=0} = \binom{j}{k} x^{j-k}, \qquad j, k = 0, 1, \dots \tag{2.6}$$

(we, again, use the convention that $\binom{j}{k} = 0$, if $j < k$) one obtains that

$$W(0) = F(x_1, \dots, x_m; \alpha_1, \dots, \alpha_m), \tag{2.7}$$

where $F(x_1, \dots, x_m; \alpha_1, \dots, \alpha_m)$ is the generalized (or confluent) Vandermonde determinant introduced in (1.5). Hence, in view of (1.11) we have that (2.7) becomes

$$W(0) = \prod_{1 \le j < k \le m} (x_k - x_j)^{\alpha_j \alpha_k} \tag{2.8}$$

and, furthermore, an immediate corollary of (2.8) is the well-known fact that the functions appearing in (2.4) are linearly independent.

# References


[1] R. Horn and C. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991.