Hopf and Frobenius algebras: generalizations and the Larson-Sweedler theorem

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Riverside Mathematics Workshop for Excellence and Diversity

20 October 2018

Outline

1. Hopf and Frobenius algebras
2. Hopf and Frobenius (enriched) categories
3. Larson-Sweedler theorem

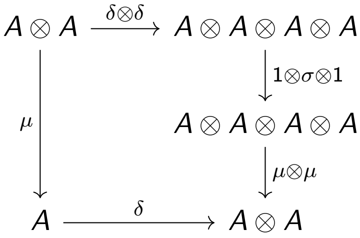
A Hopf k-algebra is Frobenius
if and only if it is finite dimensional.

$\star$ How can generalize this result from a field k to commutative rings or more general structures?

Bialgebras

Let k be a field.

- A k-algebra (M, μ, η) is a monoid in the symmetric monoidal category $(\mathbf{Vect}_k, \otimes, k)$.
- A k-coalgebra (C, δ, ϵ) is a comonoid in $(\mathbf{Vect}_k, \otimes, k)$; Sweedler's sigma notation $\delta(x) = \Sigma_{(x)} x_{(1)} \otimes x_{(2)}$.
- A k-bialgebra $(A, \mu, \eta, \delta, \epsilon)$ is a bimonoid in $(\mathbf{Vect}_k, \otimes, k)$:

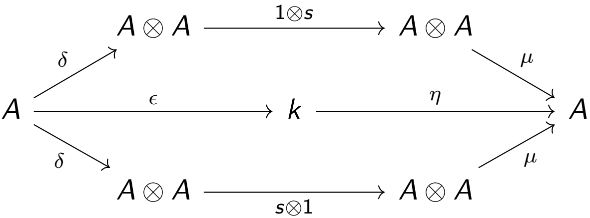


$$\sum_{(xy)} (xy)_{(1)} \otimes (xy)_{(2)} = \sum_{(x),(y)} x_{(1)} y_{(1)} \otimes x_{(2)} y_{(2)}$$

+3 more axioms.

Hopf algebras

▶A k-bialgebra A is called Hopf if there exists $s\colon A \to A$, the *antipode*, such that



It is a Hopf monoid in $(\mathbf{Vect}_k, \otimes, k)$.

Examples

- Group-algebra $k[G] = \{\sum_{r_g \in k} r_g g\}$ of a group, $\delta(x){=}x{\otimes}x$.
- Universal enveloping algebra $U(\mathfrak{g})$ of Lie algebra, $\delta(x){=}x \otimes 1{+}1 \otimes x$.
- Coordinate algebra $\mathcal{O}(G)$ of an algebraic group, $\delta(f)(x, y) = f(xy)$.

Frobenius algebras

▶A k-algebra and k-coalgebra $(A, \mu, \eta, \delta, \epsilon)$ is Frobenius if

$$\begin{array}{ccccc}
A \otimes A & & \xrightarrow{\delta\otimes 1} & & A \otimes A \otimes A \\
\downarrow{\scriptstyle 1\otimes\delta} & \searrow^{\mu} & & & \downarrow{\scriptstyle 1\otimes\mu} \\
 & & A & & \\
 & & & \searrow^{\delta} & \\
A \otimes A \otimes A & & \xrightarrow[\mu\otimes 1]{} & & A \otimes A
\end{array}$$

It is a Frobenius monoid in $(\mathbf{Vect}_k, \otimes, k)$.

$\star$ Every Frobenius k-algebra is finite dimensional.

Examples

- Matrix algebra $M_n(k)$, $\epsilon(X) = \mathrm{tr}(X)$.
- $k[G]$ for finite group, $\delta(g) = \sum_h gh^{-1} \otimes h$.
- Frobenius algebras correspond to 2-dimensional TQFT.

Hopf categories

⋆ If $(\mathcal{V}, \otimes, I, \sigma)$ symmetric monoidal, **Comon**$(\mathcal{V})$ is monoidal:

$$C \otimes D \xrightarrow{\delta\otimes\delta} C \otimes C \otimes D \otimes D \xrightarrow{1\otimes\sigma\otimes1} C \otimes D \otimes C \otimes D$$

Definition

A Hopf $\mathcal{V}$-category $(\mathcal{H}, m, j, d, e, s)$ is

- a **Comon**$(\mathcal{V})$-enriched category (every $H_{x,y}$ is a coalgebra),
- equipped with a collection of maps $s_{x,y} \colon H_{x,y} \to H_{y,x}$ (+properties).

▶ A one-object **Comon**$(\mathcal{V})$-category is a bimonoid in $\mathcal{V}$.

▶ A one-object Hopf $\mathcal{V}$-category is a Hopf monoid in $\mathcal{V}$.

Axioms

- $m_{xyz} \colon H_{x,y} \otimes H_{y,z} \to H_{x,z}$, $\;j_x \colon I \to H_{x,x}$ 'global' multiplication
- $d_{ab} \colon H_{a,b} \to H_{a,b} \otimes H_{a,b}$, $\;e_{ab} \colon H_{a,b} \to I$ 'local' comultiplication

$$
\begin{array}{ccc}
H_{x,y} \otimes H_{y,z} & \xrightarrow{d_{xy} \otimes d_{yz}} & H_{x,y} \otimes H_{x,y} \otimes H_{y,z} \otimes H_{y,z} \\
 & & \big\downarrow{\scriptstyle 1 \otimes \sigma \otimes 1} \\
{\scriptstyle m_{xyz}}\big\downarrow & & H_{x,y} \otimes H_{y,z} \otimes H_{x,y} \otimes H_{y,z} \\
 & & \big\downarrow{\scriptstyle m_{xyz} \otimes m_{xyz}} \\
H_{x,z} & \xrightarrow[d_{xz}]{} & H_{x,z} \otimes H_{x,z}
\end{array}
$$

- $s_{xy} \colon H_{x,y} \to H_{y,x}$ 'global' antipode

$$
\begin{array}{ccccc}
 & H_{x,y} \otimes H_{x,y} & \xrightarrow{1 \otimes s_{xy}} & H_{x,y} \otimes H_{y,x} & \\
{\scriptstyle d_{xy}} \nearrow & & & & \searrow {\scriptstyle m_{xyx}} \\
H_{x,y} & \xrightarrow{e_{xy}} & I & \xrightarrow{j_x} & H_{x,x}
\end{array}
$$

Frobenius categories

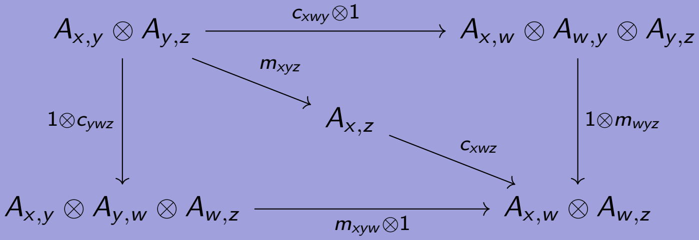
⋆ A $\mathcal{V}$-*opcategory* is a $\mathcal{V}^{\mathrm{op}}$-enriched category

$c_{xyz} : A_{x,z} \to A_{x,y} \otimes A_{y,z}, \quad \varepsilon_x : A_{x,x} \to I$ 'global' comultiplication

Definition

A $\mathcal{V}$-Frobenius category $(\mathcal{A}, m, j, c, \varepsilon)$ is

- $\mathcal{V}$-enriched and $\mathcal{V}^{\mathrm{op}}$-enriched, and
- satisfies 'parameterized' Frobenius properties



▶ A one-object Frobenius $\mathcal{V}$-category is a Frobenius monoid in $\mathcal{V}$.

Larson-Sweedler for Hopf and Frobenius categories

▶ A $\mathcal{V}$-category $\mathcal{A}$ is *locally rigid* when each $A_{x,y}$ has a dual in $\mathcal{V}$.
E.g. every $\mathbf{Vect}_k^f$-category.

▶ For a Hopf $\mathcal{V}$-category $(\mathcal{H}, m, j, d, e, s)$, the *left integral space* is

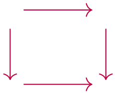
$$\int_{H_y}^{\ell} := \{ \overset{xy}{t} : I \to H_{x,y} \mid h \circ \overset{xy}{t} = e_{zx}(h) \overset{zy}{t}, \forall h \in H_{z,x} \}$$

For any Hopf $\mathcal{V}$-category $\mathcal{H}$, the following are equivalent:

- $\mathcal{H}$ is a Frobenius $\mathcal{V}$-category
- $\mathcal{H}$ is locally rigid and $\int_{H_y}^{\ell} \cong I$ for all y.

⋆ In particular, one-object case gives k-algebras, R-algebras, any Hopf and Frobenius monoids...
⋆ In general, R-linear categories, weak and Turaev Hopf algebras...

Thank you for your attention!



Joint work with Mitch Buckley, Timmy Fieremans and Joost Vercruysse