Wiring Diagrams, a categorical formalism for systems modeling

Christina Vasilakopoulou

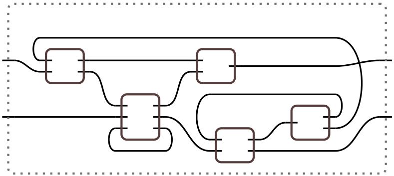
University of Patras, Greece

Department of Mathematics and Statistics, Macquarie University

27 February 2020

Goal: categorical framework for modeling and analysis of systems

systems as boxes
inhabitants



channels of info
flow as wires

Analyse the composite system using the analyses of the particular system components and their specific wired interconnection.

▶ System architecture and behavior in single model!

Outline

1. A few categorical concepts
2. The category of wiring diagrams **WD**
3. Systems as **WD**-algebras
4. Further directions

Categories

A *category* $\mathcal{C}$ consists of

- a collection of objects $X, Y, Z, \ldots$
- a collection of morphisms $f\colon X \to Y$
- an identity morphism $1_X\colon X \to X$ for all $X \in \mathcal{C}$
- a composition rule $X \xrightarrow{f} Y \xrightarrow{g} Z := g \circ f$

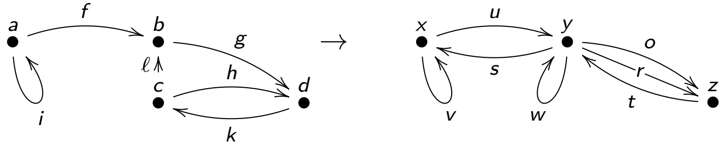
such that $h \circ (g \circ f) = (h \circ g) \circ f$ and $1_Y \circ f = f = f \circ 1_X$

Examples

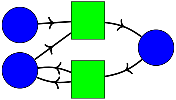
- **Set** the category of sets and functions (usual composition and ids)
- Numerous mathematical structures: **Mon**, **Grp**, **Rng**, $\mathbf{Vect}_k$, $\mathbf{Mod}_R$, **Mat**, **Top**, **Man**, **Hilb**, **Aff**, ...

Examples

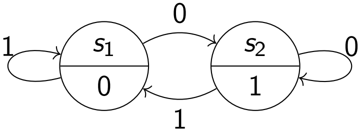
- **Graph** the category of (directed, multi) graphs and homomorphisms



- **Petri** the category of Petri nets and homomorphisms



- **DDS** the category of discrete dynamical systems (Moore machines)



$S = \{s_1, s_2\}, S \times \mathbf{B} \xrightarrow{\text{upd}} S, S \xrightarrow{\text{rdt}} \mathbf{B}$
is the NOT machine

Functors

A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ between two categories consists of

- a mapping $X \mapsto FX$ on objects
- a mapping $(f\colon X \to Y) \mapsto (Ff\colon FX \to FY)$ on morphisms

such that $F(g \circ f) = Fg \circ Ff$ and $F(1_X) = 1_{FX}$

Examples

- The 'forgetful' functor $U\colon \mathbf{Grp} \to \mathbf{Set}$ discards the group (& group homomorphism) structure, keeping the underlying set (& function)
- $\mathrm{List}\colon \mathbf{Set} \to \mathbf{Set}$ maps set S to $\mathrm{List}(S)=\{(x_1,\ldots,x_n)|n\in\mathbb{N}, x_i\in S\}$ & function $f\colon S \to T$ to $\mathrm{List}(f)$ by $(x_1,\ldots,x_n)\mapsto(f(x_1),\ldots,f(x_n))$
- Given a 'theory' $\mathcal{T}$, a *model* or *algebra* is a functor $A\colon \mathcal{T} \to \mathbf{Set}$ that materializes abstract n-ary operations (groups, rings..)

Categories and functors themselves form a category **Cat**

Monoidal structure

A *monoidal category* $\mathcal{V}$ has a *unit* I and a *tensor product* functor

$$\otimes \colon \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$$
$$(X, Y) \longmapsto X \otimes Y$$

with $X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$ and $I \otimes X \cong X \cong X \otimes I$.

Examples

- **Set** with the cartesian product $X \times Y = \{(x, y) | x \in X, y \in Y\}$ and the singleton set $\{*\}$ and similarly $(\mathbf{Cat}, \times, \mathbf{1})$
- $(\mathbf{Vect}_k, \otimes, k)$ with the tensor product of vector spaces

A *monoidal functor* $F \colon \mathcal{V} \to \mathcal{W}$ comes with 'comparison' morphisms

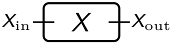
$$FX \otimes_{\mathcal{W}} FY \to F(X \otimes_{\mathcal{V}} Y) \text{ and } I_{\mathcal{W}} \to F(I_{\mathcal{V}})$$

▶ Monoidal functors $F \colon \mathcal{V} \to \mathbf{Cat}$ are sometimes called $\mathcal{V}$-*algebras*.

The monoidal category of wiring diagrams

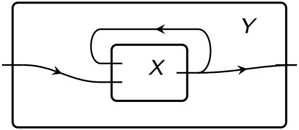
There is a category **WD** where

- objects are pairs of sets $X = (X_{\rm in}, X_{\rm out})$



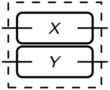
think of X as a placeholder for systems, with input&output info values in $X_{\rm in}, X_{\rm out}$

- morphisms are functions $(X_{\rm out} \times Y_{\rm in} \xrightarrow{\phi_{\rm in}} X_{\rm in}, X_{\rm out} \xrightarrow{\phi_{\rm out}} Y_{\rm out})$



think of $\phi_{\rm in/out}$ expressing the flow of info through the ports

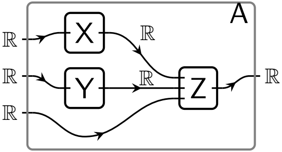
- tensor product $X \otimes Y = (X_{\rm in} \times Y_{\rm in}, X_{\rm out} \times Y_{\rm out})$

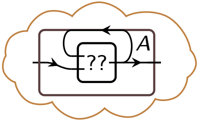


think of parallel execution of processes

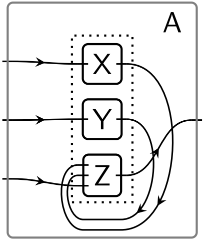
Worked-out Example

Start with three boxes $\mathbb{R}\text{-}\boxed{X}\text{-}\mathbb{R}$, $\mathbb{R}\text{-}\boxed{Y}\text{-}\mathbb{R}$ and $\mathbb{R}\,\mathbb{R}\,\mathbb{R}\text{-}\boxed{Z}\text{-}\mathbb{R}$ where all input and output data of possible processes are real numbers , interconnected as in





This is a morphism $X \otimes Y \otimes Z \to A$ in the category **WD** expressed as



$$\begin{cases} \phi_{\text{in}}: \overbrace{\mathbb{R}\times\mathbb{R}\times\mathbb{R}}^{(X\otimes Y\otimes Z)_{\text{out}}} \times \overbrace{\mathbb{R}\times\mathbb{R}\times\mathbb{R}}^{A_{\text{in}}} \xrightarrow{\pi_{12456}} \overbrace{\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}}^{(X\otimes Y\otimes Z)_{\text{in}}} \\ \phi_{\text{out}}: \underbrace{\mathbb{R}\times\mathbb{R}\times\mathbb{R}}_{(X\otimes Y\otimes Z)_{\text{out}}} \xrightarrow{\pi_3} \underbrace{\mathbb{R}}_{A_{\text{out}}} \end{cases}$$

Systems as algebras for the wiring diagram category

A **WD**-algebra, namely a monoidal functor

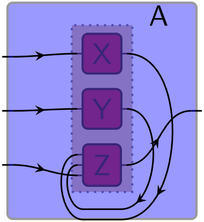
$$F\colon \mathbf{WD} \longrightarrow \mathbf{Cat}$$

$$\begin{array}{ccc} X{=}(X_{\mathrm{in}}, X_{\mathrm{out}}) & \longmapsto & FX \\ \phi \downarrow & & \downarrow F\phi \\ Y{=}(Y_{\mathrm{in}}, Y_{\mathrm{out}}) & \longmapsto & FY \end{array}$$

subsystems category

composite system functor

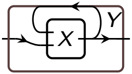
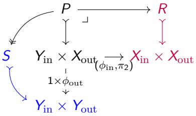
gives *semantics* to boxes, *composite operation* to wiring diagrams and *parallelizing operation* to subsystems via $FX \times FY \to F(X \otimes Y)$



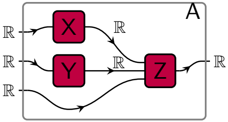
Case study: Algebra of Contracts

- To each box X_{in} -[X]- X_{out} assign category of *contracts*, i.e. relations

$$R \subseteq X_{\text{in}} \times X_{\text{out}}$$

- To each wiring diagram  assign formula 

that, given contracts on subsystems, produces contract on composite.



$R_X = [4,5] \times [4,5]$, $R_Y = [8,9] \times [8,9]$
$R_Z = [3,5] \times [7,9] \times \mathbb{R} \times [0,1]$ produce
$R_A = [4,5] \times [8,9] \times \mathbb{R} \times [0,1]$

Algebra machinery is concrete, flexible and scalable

- composite contract is completely decided by $R_{X,Y,Z}$ and wiring
- can easily replace some subcontract by any other
- computes regardless of number of boxes or ‘difficulty’ of contracts

...and the story just began

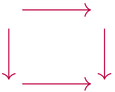
The theory has been developing, using various categorical tools

- main algebra examples: discrete & continuous dynamical systems, subalgebras e.g. linear time invariant systems, safety contracts etc.
- algebra maps are also important: translation between semantics
- *abstract systems*: a very general, all-inclusive framework
- time is incorporated: discrete, continuous and hybrid versions

Move towards interdisciplinary settings

- analysis of Cyber-Physical Systems using this formalism
- connections to other areas e.g. biological and chemical systems
- transfer of perspective and results between WD, databases (*lenses*), logic (*Dialectica*) and machine learning (*learners*)

Thank you for your attention!



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