GENERALIZATION OF ALGEBRAIC OPERATIONS VIA ENRICHMENT

Christina Vasilakopoulou

Trinity College
and
Department of Pure Mathematics and Mathematical Statistics
University of Cambridge

This dissertation is submitted for the degree of
Doctor of Philosophy

March 2014

This dissertation is the result of my own work and includes nothing that is the outcome of work done in collaboration except where specifically indicated in the text.

This dissertation is not substantially the same as any that I have submitted for a degree or diploma or any other qualification at any other university.

Christina Vasilakopoulou
27 March 2014

Generalization of Algebraic Operations via Enrichment

Christina Vasilakopoulou

SUMMARY

In this dissertation we examine enrichment relations between categories of dual structure and we sketch an abstract framework where the theory of fibrations and enriched category theory are appropriately united.

We initially work in the context of a monoidal category, where we study an enrichment of the category of monoids in the category of comonoids under certain assumptions. This is induced by the existence of the *universal measuring comonoid*, a notion originally defined by Sweedler in [**Swe69**] in vector spaces. We then consider the fibred category of modules over arbitrary monoids, and we establish its enrichment in the opfibred category of comodules over arbitrary comonoids. This is now exhibited via the existence of the *universal measuring comodule*, introduced by Batchelor in [**Bat00**].

We then generalize these results to their 'many-object' version. In the setting of the bicategory of $\mathcal{V}$-enriched matrices (see [**KL01**]), we investigate an enrichment of $\mathcal{V}$-categories in $\mathcal{V}$-*cocategories* as well as of $\mathcal{V}$-modules in $\mathcal{V}$-*comodules*. This part constitutes the core of this treatment, and the theory of fibrations and adjunctions between them plays a central role in the development. The newly constructed categories are described in detail, and they appropriately fit in a picture of duality, enrichment and fibrations as in the previous case.

Finally, we introduce the concept of an *enriched fibration*, aimed to provide a formal description for the above examples. Related work in this direction, though from a different perspective and with dissimilar outcomes, has been realized by Shulman in [**Shu13**]. We also discuss an abstraction of this picture in the environment of double categories, concerning categories of *monoids* and *modules* therein. Relevant ideas can be found in [**FGK11**].

Acknowledgements

I would like to begin by thanking my supervisor Professor Martin Hyland for his patient guidance and encouragement throughout my studies. His continuous support and inspiration have been extremery valuable to me. I would also like to thank Dr. Ignacio Lopez Franco, for all the stimulating discussions and constructive suggestions, several of which were decisive in the formation of this thesis.

I would like to offer my special thanks to Professor Panagis Karazeris for all his help and motivation during and after my undergraduate studies at the University of Patras, as well as Professor Georgios Dassios for his support and advice.

This thesis would not have been possible without the mathematical and non-mathematical assistance of the Category Theory ‘gang’: Achilleas, Guilherme, Tamara and the rest of my academic family. Also, to all my fellow PhD students and friends, Gabriele, Giulio, John, Liyan, Michał, Beth, Anastasia, Richard, Robert, Jerome, Rachel, Eleni and particularly Stefanos, special thanks. You all made Cambridge such a wonderful place to be, full of happy moments.

I am grateful to Trinity College, EPSRC and DPMMS for financially supporting my studies. I would also like to thank the Propondis Foundation and the Leventis Foundation for providing me with PhD scholarships that gave me the opportunity to realize my research project.

Special thanks must be attributed to my dear friends Elpida, Ioanna, Artemis, Irida, Aggeliki, Evi and Andreas, who have proved companions for life. I would not have made it that far if it weren’t for you.

Finally, I owe my deepest gratitude to my beloved family: my parents Giorgos and Magda and my brother Paris for their unconditional loyalty and support, as well as Dimitris for his love and dedication. Thank you for always being there for me.

Contents

CHAPTER 1

Introduction

Algebras and their modules, as well as coalgebras and their comodules, are amongst the simplest and most fundamental structures in abstract mathematics. Formally, algebras are dual to coalgebras and modules are dual to comodules, but in practice that point of view is very limited. The initial motivation for the material included in the present thesis was a more striking relation between these notions: in natural circumstances, the mere category of algebras is enriched in the category of coalgebras, and that of modules in comodules. These enrichments encapsulate some very rich algebraic structure, that of the so-called measuring coalgebras and comodules.

More specifically, the notion of the *universal measuring coalgebra* $P(A,B)$ was first introduced by Sweedler in [**Swe69**], and has been employed as a way of giving sense to an idea of generalized maps between algebras. Examples of this point of view and applications are given by Marjorie Batchelor in [**Bat91**] and [**Bat94**]. It was Gavin Wraith in the 1970's, who first suggested that this coalgebra gives an enrichment of the category of algebras in the category of coalgebras, however for a long time there was no explicit treatment of Wraith's idea in the literature. Furthermore, this idea can be appropriately extended to give an enrichment of a global category of modules in a global category of comodules, via the *universal measuring comodule* $Q(M,N)$ introduced by Batchelor in [**Bat00**]. These objects have also found applications on their own, analytically presented in the provided references.

Independently of questions of enrichment, there is a well-known fibration of the global category of modules over algebras in addition to an opfibration of the comodules over coalgebras. This extra structure seems to point torwards a picture that integrates the two classical notions, enrichment and fibration, which generally do not go well together. One of the basic objectives of this thesis is to successfully describe what could be called an *enriched fibration*.

Inspired by the above, we are led to consider the 'many-object' generalization of the previous situation. Since an algebra is evidently a (linear) category with one object, the categories of interest on this next step are naturally those of enriched categories and enriched modules, on the one hand. For the analogues of coalgebras and comodules, we proceed to the definitions of an *enriched cocategory* and *enriched comodule*. After setting up the theory of these new categories and exploring some of their more pertinent properties, we establish an enrichment of $\mathcal{V}$-categories in $\mathcal{V}$-cocategories, and of $\mathcal{V}$-modules in $\mathcal{V}$-comodules. The similarities with the base case of (co)algebras and (co)modules are expressed primarily by the methodology and the series of arguments followed. However, this generalization reveals more advanced ideas and certain patterns of expected behaviour of the categories involved. This newly acquired perspective urges us to develop a theoretic frame in which a general machinery, certain aspects of which were described in detail for the two particular cases, would always result in the speculated enriched fibration picture.

Thus, another central aim of this dissertation is to identify this abstract framework which leads to instances of the enriched fibration notion, with starting point a monoidal bicategory or even more closely related, a monoidal *pseudo double category*. In fact, the longer term goal of such a development was its possible application to different contexts, and in particular to the theory of operads. In more detail, if we replace the bicategory of $\mathcal{V}$-matrices (which is the starting point for the duality and enrichment relations for $\mathcal{V}$-categories and $\mathcal{V}$-modules) with the bicategory of $\mathcal{V}$*-symmetries* (see also [**GJ14**]), there is strong evidence that we can establish an analogous enriched fibration which merges symmetric $\mathcal{V}$-operads and operad modules and their duals. Moreover, both coloured and non-coloured versions can be included in this plan. This indicates a fruitful area for future work.

The thesis is divided in two parts: the material in Part I is mostly well-known, serving as the background for the development that follows, while the material in Part II is mostly new. We assume familiarity with the basic theory of categories, as in the standard textbook [**ML98**] by MacLane.

In Chapter 2, we review the basic definitions and features of the theory of bicategories and 2-categories, with particular emphasis on the concepts of monads/comonads and their modules/comodules in this abstract setting. Classic references on the main notions are [**Bén67, Gra74, Str80, Bor94a, KS74**]. Coherence for bicategories, very briefly mentioned here, is discussed in [**GPS95, MLP85, Pow89**], and of course MacLane's coherence theorem for monoidal categories preceded it ([**ML63, Kel64, JS93**]). Monads in a 2-category have been widely studied, with basic reference Ross Street's [**Str72**]. Categories of modules, more commonly referred to as algebras especially in the 2-category **Cat**, are formed as categories of Eilenberg-Moore algebras on the hom-categories $\mathcal{K}(A,B)$ of a bicategory $\mathcal{K}$.

Chapter 3 summarizes basic concepts related to monoidal categories, following some of the many standard references such as [**ML98, JS93, Str07**]. Categories of monoids and modules will play a very important role for the development of this dissertation, hence extra attention has been given to the presentation of their properties. In particular, questions regarding the existence of the free monoid and the cofree comonoid constructions have been of primary interest. Certain papers by Hans Porst [**Por08c, Por08b, Por08a**] have addressed this issue from a particular point of view, in the context of locally presentable categories (see [**AR94**]). Specific methods, especially the ones related to local presentability of the categories of dual objects, are carefully exhibited here and in some cases generalized a bit further.

The main definitions and elementary features of the theory of enriched categories are summarized in Chapter 4, with standard references [**Kel05, EK66**]. Since enriched modules are essential for the generalization of the monoids and modules correlation to a $\mathcal{V}$-categories and $\mathcal{V}$-modules one, we devote a section to some of their aspects needed for our purposes, see [**Bén73, Law73**]. In the last part, we recall parts of the theory of actions of monoidal categories on ordinary categories, which lead to a particular enrichment, as described also in Janelidze and Kelly's [**JK02**]. In fact, this constitutes a special case of a more general result discussed in [**GP97**], namely that there is an equivalence between the 2-category of tensored $\mathcal{W}$-categories and the 2-category of closed $\mathcal{W}$-representations, for $\mathcal{W}$ a right-closed bicategory.

In Chapter 5, the key material about fibred category theory is reviewed. Central notions and results are presented, including the correspondence between cloven fibrations and indexed categories due to Grothendieck. The notion of a fibration was first introduced in [**Gro61**], and suitable references on the subject are [**Gra66, Jac99, Joh02b**] and Hermida's work as can be found in, for example, [**Her93, Her94**]. Finally, we move to the topic of fibred adunctions and fibrewise limits, where the main constructions and ideas can be found in [**Her94**] and [**Bor94b**]. Presently, we develop the issue a bit further: we examine conditions not only for adjunctions between fibrations over the same basis, but also for general fibred adjunctions, *i.e.* between fibrations over arbitrary bases. This slightly generalizes results which exist in the literature currently. This was not done aimlessly: Theorem 5.3.7 constitutes an extremely valuable tool for the establishment of the pursued enrichments later in the thesis.

Chapter 6 describes in detail the enrichment of monoids and modules, which is the motivating case for what follows. In fact, the results of this chapter in a somewhat more restricted version previously appeared in [**Vas12**], and have already been of use to a certain extent, see for example [**AJ13**]. Explicitly, we identify the more general categorical ideas underlying the existence of Sweedler's measuring coalgebra $P(A, B)$ of [**Swe69, Bat91**] and prove its existence in a much broader context. Its defining equation is in particular also provided in [**Por08a**] and observed in [**Bar74**]. Combined with the theory of actions of monoidal categories, we show how these $P(A, B)$ for any two monoids A and B induce an enrichment of the category of monoids $\mathbf{Mon}(\mathcal{V})$ in the category of comonoids $\mathbf{Comon}(\mathcal{V})$, under specific assumptions on $\mathcal{V}$. Subsequently, the 'global' categories of modules and comodules **Mod** and **Comod** are defined, fibred and opfibred respectively over monoids and comonoids. These categories have nice properties, and in particular, as hinted by Wischnewsky at the end of [**Wis75**], **Comod** is comonadic over $\mathcal{V} \times \mathbf{Comon}(\mathcal{V})$, a fact which clarifies its structure. Via the existence of an adjoint of a functor between the global categories, the universal measuring comodule $Q(M, N)$ is constructed, as a variation of the notion in [**Bat00**] in our general setting. Again through a specific action functor, we obtain an enrichment of **Mod** in **Comod**, induced by these $Q(M, N)$ for any two modules M and N as the enriched hom-objects. Parts of this work were accomplished in collaboration with Prof. Martin Hyland and Dr. Ignacio

Lopez Franco. The diagram which roughly depicts the above is the following:

$$\begin{tikzcd}
\mathbf{Mod} \arrow[r, dashed, "\text{enriched}"] \arrow[d, "\text{fibred}"'] & \mathbf{Comod} \arrow[d, "\text{opfibred}"] \\
\mathbf{Mon}(\mathcal{V}) \arrow[r, dashed, "\text{enriched}"'] & \mathbf{Comon}(\mathcal{V}).
\end{tikzcd}$$

Chapter 7 moves up a level, aiming to estabish essentially the same results as in the previous chapter but for the 'many-object' case of (co)monoids and (co)modules as explained earlier. The bicategory of $\mathcal{V}$-matrices is the base on which the categories of enriched (co)categories and (co)modules are formed, following until a certain point the development of [**BCSW83**] and [**KL01**]. The former in fact examines categories enriched in bicategories via matrices enriched in bicategories, but for our purposes we restrict to the one-object case, that of monoidal categories. This approach of employing matrices presents certain advantages: it leads to more unified results such as existence of limits and colimits, monadicity relations, local presentability for the categories of $\mathcal{V}$-graphs, $\mathcal{V}$-categories and $\mathcal{V}$-modules, avoiding explicit formulas if they are not desired. Regarding this, Wolff's much earlier [**Wol74**] contains many important explicit constructions for $\mathcal{V}$-**Grph** and $\mathcal{V}$-**Cat**, for a symmetric monoidal closed category $\mathcal{V}$. In the same underlying framework of $\mathcal{V}$-matrices, the category $\mathcal{V}$-**Cocat** of enriched cocategories is described (Definition 7.3.8). Except from generalizing the concept of comonoids for a monoidal category, $\mathcal{V}$-cocategories appear to have important applications in their own right. In papers of Lyubashenko, Keller and others (e.g. [**Lyu03, Kel06, KM07**]) A_∞-categories, which are natural generalizations of A_∞-algebras arising in connection with Floer homology and related to mirror symmetry, are defined as a special kind of differential graded cocategories. The category of $\mathcal{V}$-comodules is also accordingly defined, and the diagram which summarizes the main results of the chapter is

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Mod} \arrow[r, dashed, "\text{enriched}"] \arrow[d, "\text{fibred}"'] & \mathcal{V}\text{-}\mathbf{Comod} \arrow[d, "\text{opfibred}"] \\
\mathcal{V}\text{-}\mathbf{Cat} \arrow[r, dashed, "\text{enriched}"'] & \mathcal{V}\text{-}\mathbf{Cocat}.
\end{tikzcd}$$

Notice that both enrichments are established via adjoint functors to actions, making use of the fibrational and opfibrational structure of the categories involved (though for the bottom one, the hom-functor can be obtained directly via an adjoint functor theorem). The same holds for the simpler case of the previous chapter, for the global category of modules and comodules. This is precisely why general fibred adjunctions in Part I prove to be essential for the study of the particular examples analyzed in this thesis.

Finally, in Chapter 8 we utilize the results and theoretical patterns of the previous two chapters in order to move 'from special to general': we formulate a definition of an enriched fibration and sketch how it is possible to obtain such a formation in the context of a bicategory or double category. The structures of importance here are the categories of *monoids* and *comonoids*, *modules* and *comodules* of a (pseudo) double category. We note that the enriched fibration concept, originally mentioned in [**GG76**], has been studied from an admittedly different point of view by Mike Shulman in [**Shu13**] and also independently in [**Bun13**]. However, the main definitions and constructions diverge from the ones presented here. Other particular references for notions employed, such as monoidal bicategories (or monoidal 2-categories) and pseudomonoids therein, are for example [**Car95, GPS95, Gur07**] and [**DS97, Mar97**]. The fundamental definition of a monoidal fibration was first introduced in [**Shu08**]. Appropriate references for the theory of pseudo double categories for our purposes are [**GP99, GP04, Shu10, FGK11**], and the original concept of a double category, *i.e.* a category (weakly) internal in **Cat**, is traced back to [**Ehr63**]. This last part of the dissertation is not as detailed as it could be, due to limitations of the current treatment. In the double categories section, most definitions and proofs are only outlined, whereas enrichment in the setting of fibrations could be the starting point of an entire enriched fibred category theory. The principal function of this final chapter is to clarify the occurrence of the main results of this work in an abstract environment, and serve as a guide for future applications.

PART I

CHAPTER 2

Bicategories

The purpose of this chapter is to provide the reader with the necessary background material regarding the theory of bicategories. In this way, the related constructions and results used later in the thesis can be readily referred to herein.

The original definition of a bicategory and a lax functor ('morphism') between bicategories can be found in Bénabou's [**Bén67**]. Other references, including the definitions of transformations and modifications are [**Str96, Bor94a**]. 2-categorical notions are here presented as 'strictified' versions of the bicategorical ones, whereas in later chapters the **Cat**-enriched view is also addressed. Due to coherence for bicategories, we are often able to use 2-categorical machinery and operations such as pasting and mates correspondence, directly in the weaker context. Categories of (co)monads and (co)modules in bicategories are carefully presented in this chapter, in order to later be employed as the appropriate formalization for specific categories of interest. Regarding 2-category theory, see the indicative [**KS74, Lac10a**], whereas [**Str72**] presents the formal theory of monads in 2-categories.

With respect to the notation followed in this chapter, note that the multiplication for monads is denoted by the letter "m" rather than the usual "μ", since the latter is employed for the monad action on their modules. Similarly, we use "Δ" for comultiplication of comonads and "δ" for the coaction on comodules. We also prefer the term '(co)module' from the more common '(co)algebra' for a (co)monad.

2.1. Basic definitions

Definition 2.1.1. A *bicategory* $\mathcal{K}$ is specified by the following data:

- A collection of objects $A, B, C, ...$, also called *0-cells*.
- For each pair of objects A, B, a category $\mathcal{K}(A,B)$ whose objects are called morphisms or *1-cells* and whose arrows are called *2-cells*. The composition is called *vertical composition* of 2-cells and is denoted by

$$\begin{tikzcd}[column sep=large] A \arrow[r, bend left=50, "f"] \arrow[r, "g" description] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\alpha'"] & B \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left=40, "f"] \arrow[r, bend right=40, "h"'] \arrow[r, phantom, "\Downarrow\alpha'\cdot\alpha"] & B. \end{tikzcd}$$

The identity 2-cell for this composition is

$$\begin{tikzcd} A \arrow[r, bend left=40, "f"] \arrow[r, bend right=40, "f"'] \arrow[r, phantom, "\Downarrow 1_f"] & B. \end{tikzcd}$$

- For each triple of objects A, B, C, a functor

$$\circ : \mathcal{K}(B,C) \times \mathcal{K}(A,B) \longrightarrow \mathcal{K}(A,C)$$

called *horizontal composition*. It maps a pair of 1-cells (g,f) to $g \circ f = gf$ and a pair of 2-cells (β,α) to $\beta * \alpha$, depicted by

$$\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "u"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\beta"] & C \arrow[r, phantom, "="] & A \arrow[r, bend left, "gf"] \arrow[r, bend right, "vu"'] \arrow[r, phantom, "\Downarrow\beta*\alpha"] & C.
\end{tikzcd}$$

- For each object $A \in \mathcal{K}$, a 1-cell $1_A : A \to A$ called the *identity 1-cell* of A.
- Associativity constraint: for each quadruple of objects A, B, C, D of $\mathcal{K}$, a natural isomorphism

$$\begin{tikzcd}
\mathcal{K}(C,D) \times \mathcal{K}(B,C) \times \mathcal{K}(A,B) \arrow[r, "1\times\circ"] \arrow[d, "\circ\times1"'] \arrow[dr, phantom, "\alpha\Uparrow"] & \mathcal{K}(C,D) \times \mathcal{K}(A,C) \arrow[d, "\circ"] \\
\mathcal{K}(B,D) \times \mathcal{K}(A,B) \arrow[r, "\circ"'] & \mathcal{K}(A,D)
\end{tikzcd}$$

called the *associator*, with components invertible 2-cells

$$\alpha_{h,g,f} : (h \circ g) \circ f \xrightarrow{\sim} h \circ (g \circ f).$$

- Identity constraints: for each pair of objects A, B in $\mathcal{K}$, natural isomorphisms

$$\begin{tikzcd}
 & \mathbf{1} \times \mathcal{K}(A,B) \cong \mathcal{K}(A,B) \times \mathbf{1} \arrow[dl, bend right, "I_A\times1"'] \arrow[d, "\cong"] \arrow[dr, bend left, "1\times I_B"] \arrow[dl, phantom, "\overset{\rho}{\Rightarrow}"] \arrow[dr, phantom, "\underset{\lambda}{\Leftarrow}"] & \\
\mathcal{K}(A,A) \times \mathcal{K}(A,B) \arrow[r, "\circ"'] & \mathcal{K}(A,B) & \mathcal{K}(A,B) \times \mathcal{K}(B,B) \arrow[l, "\circ"]
\end{tikzcd}$$

called the *unitors*, with components invertible 2-cells

$$\lambda_f : 1_B \circ f \xrightarrow{\sim} f, \quad \rho_f : f \circ 1_A \xrightarrow{\sim} f.$$

Notice that the functor $I_A : \mathbf{1} \to \mathcal{K}(A,A)$ is given by 1_A on objects and 1_{1_A} on arrows.

The above are subject to the coherence conditions expressed by the following axioms: for 1-cells $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$, the diagrams

$$\begin{tikzcd}
((k \circ h) \circ g) \circ f \arrow[rr, "\alpha_{k,h,g}*1_f"] \arrow[d, "\alpha_{kh,g,f}"'] & & (k \circ (h \circ g)) \circ f \arrow[d, "\alpha_{k,hg,f}"] \\
(k \circ h) \circ (g \circ f) \arrow[dr, "\alpha_{k,h,gf}"'] & & k \circ ((h \circ g) \circ f) \arrow[dl, "1_k*\alpha_{h,g,f}"] \\
 & k \circ (h \circ (g \circ f)) &
\end{tikzcd} \tag{2.1}$$

$$\begin{tikzcd}
(g \circ 1_B) \circ f \arrow[rr, "\alpha_{g,1_B,f}"] \arrow[dr, "\rho_g * 1_f"'] & & g \circ (1_B \circ f) \arrow[dl, "1_g * \lambda_f"] \\
& g \circ f &
\end{tikzcd} \tag{2.2}$$

commute.

It follows from the functoriality of the horizontal composition that for any composable 1-cells f and g we have the equality

$$\begin{tikzcd}
A \arrow[r, bend left, "f", ""{name=U1, below}] \arrow[r, bend right, "f"', ""{name=D1, above}] \arrow[from=U1, to=D1, Rightarrow, "1_f"] & B \arrow[r, bend left, "g", ""{name=U2, below}] \arrow[r, bend right, "g"', ""{name=D2, above}] \arrow[from=U2, to=D2, Rightarrow, "1_g"] & C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, bend left, "gf", ""{name=U, below}] \arrow[r, bend right, "vu"', ""{name=D, above}] \arrow[from=U, to=D, Rightarrow, "1_{gf}"] & C
\end{tikzcd}$$

and for any 2-cells $\alpha, \alpha', \beta, \beta'$ as below we have the equality

$$\begin{tikzcd}
{} \arrow[r, bend left=60, ""{name=U1, below}] \arrow[r, ""{name=M1a, above}, ""{name=M1b, below}] \arrow[r, bend right=60, ""{name=D1, above}] \arrow[from=U1, to=M1a, Rightarrow, "\alpha"] \arrow[from=M1b, to=D1, Rightarrow, "\alpha'"] & {} \arrow[r, bend left=60, ""{name=U2, below}] \arrow[r, ""{name=M2a, above}, ""{name=M2b, below}] \arrow[r, bend right=60, ""{name=D2, above}] \arrow[from=U2, to=M2a, Rightarrow, "\beta"] \arrow[from=M2b, to=D2, Rightarrow, "\beta'"] & {}
\end{tikzcd}
=
\begin{tikzcd}
{} \arrow[rr, bend left=40, ""{name=U, below}] \arrow[rr, ""{name=Ma, above}, ""{name=Mb, below}] \arrow[rr, bend right=40, ""{name=D, above}] \arrow[from=U, to=Ma, Rightarrow, "\beta * \alpha"] \arrow[from=Mb, to=D, Rightarrow, "\beta' * \alpha'"] & & {}
\end{tikzcd}$$

also known as the *interchange law*. The above equalities can also be written

$$1_g \circ 1_f = 1_{g \circ f},$$
$$(\beta' \cdot \beta) * (\alpha' \cdot \alpha) = (\beta' * \alpha') \cdot (\beta * \alpha).$$

Given a bicategory $\mathcal{K}$, we may reverse the 1-cells but not the 2-cells and form the bicategory $\mathcal{K}^{\mathrm{op}}$, with $\mathcal{K}^{\mathrm{op}}(A, B) = \mathcal{K}(B, A)$. We may also reverse only the 2-cells and form the bicategory $\mathcal{K}^{\mathrm{co}}$ with $\mathcal{K}^{\mathrm{co}}(A, B) = \mathcal{K}(A, B)^{\mathrm{op}}$. Reversing both 1-cells and 2-cells yields a bicategory $(\mathcal{K}^{\mathrm{co}})^{\mathrm{op}} = (\mathcal{K}^{\mathrm{op}})^{\mathrm{co}}$.

EXAMPLES 2.1.2.

(1) For any category $\mathcal{C}$ with chosen pullbacks, there is the bicategory of spans $\mathbf{Span}(\mathcal{C})$. This has the same objects as $\mathcal{C}$ and hom-categories $\mathbf{Span}(X, Y)$ with objects spans $X \leftarrow A \rightarrow Y$ and arrows $\alpha : A \Rightarrow B$ commutative diagrams

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] \arrow[dd, "\alpha"'] & \\
X & & Y \\
& B \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

with obvious (vertical) composition. The horizontal composition is given by pullbacks, and their universal property defines the constraints α, ρ, λ.

(2) Suppose $\mathcal{C}$ is a regular category, *i.e.* any morphism factorizes as a strong epimorphism followed by a monomorphism, and strong epimorphisms are closed under pullbacks. The bicategory of relations $\mathbf{Rel}(\mathcal{C})$ is defined as $\mathbf{Span}(\mathcal{C})$, but its 1-cells are spans $X \leftarrow R \rightarrow Y$ with jointly monic legs, or equivalently relations $R \rightarrowtail X \times Y$. The factorization system is required in order to define composition $X \rightarrow Y \rightarrow Z$, since the resulting map from the pullback to $X \times Z$ is not necessarily monic.

(3) In the bicategory of bimodules **BMod** objects are rings, 1-cells from R to S are (R,S)-bimodules (*i.e.* abelian groups which have a left R-action and a right S-action that commute with each other), and 2-cells are bimodule maps. The horizontal composition $R \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} S \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} T$ is given by tensoring over S, constructed as in Section 3.4. This generalizes to the bicategory $\mathcal{V}$-**BMod** of $\mathcal{V}$-categories and $\mathcal{V}$-bimodules, described in Section 4.2.

(4) The bicategory of matrices **Mat** has sets as objects, $X \times Y$-indexed families of sets as 1-cells from X to Y and families of functions as 2-cells. Composition is given by 'matrix multiplication': if $A = (A_{xy}) : X \to Y$ and $B = (B_{yz}) : Y \to Z$, their composite is given by the family of sets $(AB)_{xy} = \sum_y \big(A_{xy} \times B_{yz}\big)$. The enriched version of this bicategory, $\mathcal{V}$-**Mat**, is going to be extensively employed for the needs of this thesis.

Definition 2.1.3. Given bicategories $\mathcal{K}$ and $\mathcal{L}$, a *lax functor* $\mathscr{F} : \mathcal{K} \to \mathcal{L}$ consists of the following data:

- For any object $A \in \mathcal{K}$, an object $\mathscr{F}A \in \mathcal{L}$.
- For every pair of objects $A, B \in \mathcal{K}$, a functor $\mathscr{F}_{A,B} : \mathcal{K}(A,B) \to \mathcal{L}(\mathscr{F}A, \mathscr{F}B)$.
- For every triple of objects $A, B, C \in \mathcal{K}$, a natural transformation

$$\begin{tikzcd}
\mathcal{K}(B,C) \times \mathcal{K}(A,B) \arrow[rr, "\circ"] \arrow[d, "\mathscr{F}_{B,C} \times \mathscr{F}_{A,B}"'] & \arrow[d, phantom, "\delta \Uparrow"] & \mathcal{K}(A,C) \arrow[d, "\mathscr{F}_{A,C}"] \\
\mathcal{L}(\mathscr{F}B, \mathscr{F}C) \times \mathcal{L}(\mathscr{F}A, \mathscr{F}B) \arrow[rr, "\circ"'] & & \mathcal{L}(\mathscr{F}A, \mathscr{F}C)
\end{tikzcd} \tag{2.3}$$

with components $\delta_{g,f} : (\mathscr{F}g) \circ (\mathscr{F}f) \to \mathscr{F}(g \circ f)$, for 1-cells $g : B \to C$ and $f : A \to B$.

- For every object $A \in \mathcal{K}$, a natural transformation

$$\begin{tikzcd}
1 \arrow[r, "I_A"] \arrow[dr, "I_{\mathscr{F}A}"'] \arrow[dr, phantom, bend left=20, "\gamma \Uparrow"] & \mathcal{K}(A,A) \arrow[d, "\mathscr{F}_{A,A}"] \\
& \mathcal{L}(\mathscr{F}A, \mathscr{F}A)
\end{tikzcd} \tag{2.4}$$

with components $\gamma_A : 1_{\mathscr{F}A} \to \mathscr{F}(1_A)$.

The natural transformations γ and δ have to satisfy the following coherence axioms: for 1-cells $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, the diagrams

$$\begin{tikzcd}
(\mathscr{F}h \circ \mathscr{F}g) \circ \mathscr{F}f \arrow[r, "\delta_{h,g} * 1"] \arrow[d, "\alpha"'] & \mathscr{F}(h \circ g) \circ \mathscr{F}f \arrow[d, "\delta_{hg,f}"] \\
\mathscr{F}h \circ (\mathscr{F}g \circ \mathscr{F}f) \arrow[d, "1 * \delta_{g,f}"'] & \mathscr{F}((h \circ g) \circ f) \arrow[d, "\mathscr{F}\alpha"] \\
\mathscr{F}h \circ \mathscr{F}(g \circ f) \arrow[r, "\delta_{h,gf}"'] & \mathscr{F}(h \circ (g \circ f)),
\end{tikzcd} \tag{2.5}$$

$$\begin{tikzcd}
1_{\mathscr{F}B}\circ\mathscr{F}f \arrow[r, "\gamma_B*1"] \arrow[d, "\lambda"'] & \mathscr{F}(1_B)\circ\mathscr{F}f \arrow[d, "\delta_{1_B,f}"] & , & \mathscr{F}f\circ 1_{\mathscr{F}A} \arrow[r, "1*\gamma_A"] \arrow[d, "\rho"'] & \mathscr{F}f\circ\mathscr{F}(1_A) \arrow[d, "\delta_{f,1_A}"] \\
\mathscr{F}f & \mathscr{F}(1_B\circ f) \arrow[l, "\mathscr{F}\lambda"] & & \mathscr{F}f & \mathscr{F}(f\circ 1_A) \arrow[l, "\mathscr{F}\rho"]
\end{tikzcd} \tag{2.6}$$

commute.

If γ and δ are natural isomorphisms (respectively identities), then $\mathscr{F}$ is called a *pseudofunctor* or *homomorphism* (respectively *strict functor*) of bicategories. Similarly, we can define a *colax functor* of bicategories by reversing the direction of γ and δ, sometimes also called *oplax*. All these kinds of functors between bicategories can be composed, and this composition obeys strict associativity and identity laws. Thus we obtain categories $\mathbf{Bicat}_l$, $\mathbf{Bicat}_c$, $\mathbf{Bicat}_{ps}$, $\mathbf{Bicat}_s$ with the same objects and arrows lax, colax, pseudo and strict functors respectively.

DEFINITION 2.1.4. Consider two lax functors $\mathscr{F},\mathscr{G} : \mathcal{K} \to \mathcal{L}$ between bicategories. A *lax natural transformation* $\tau : \mathscr{F} \Rightarrow \mathscr{G}$ consists of the following data:

- For each object $A \in \mathcal{K}$, a morphism $\tau_A : \mathscr{F}A \to \mathscr{G}A$ in $\mathcal{L}$.
- For any pair of objects $A, B \in \mathcal{K}$, a natural transformation

$$\begin{tikzcd}
\mathcal{K}(A,B) \arrow[r, "\mathscr{F}_{A,B}"] \arrow[d, "\mathscr{G}_{A,B}"'] \arrow[dr, phantom, "\tau\Nearrow"] & \mathcal{L}(\mathscr{F}A,\mathscr{F}B) \arrow[d, "\mathcal{L}(1{,}\tau_B)"] \\
\mathcal{L}(\mathscr{G}A,\mathscr{G}B) \arrow[r, "\mathcal{L}(\tau_A{,}1)"'] & \mathcal{L}(\mathscr{F}A,\mathscr{G}B)
\end{tikzcd} \tag{2.7}$$

with components, for any $f : A \to B$, 2-cells

$$\begin{tikzcd}
\mathscr{F}A \arrow[r, "\mathscr{F}f"] \arrow[d, "\tau_A"'] \arrow[dr, phantom, "\tau_f\Nearrow"] & \mathscr{F}B \arrow[d, "\tau_B"] \\
\mathscr{G}A \arrow[r, "\mathscr{G}f"'] & \mathscr{G}B.
\end{tikzcd} \tag{2.8}$$

These data are subject to following axioms: given any pair of arrows $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{K}$, the component $\tau_{g\circ f}$ relates to the 2-cells τ_f, τ_g by the equality

$$\begin{tikzcd}
& & & & \mathscr{F}A \arrow[rr, "\mathscr{F}(g\circ f)"] \arrow[d, "\tau_A"'] \arrow[drr, phantom, "\tau_{g\circ f}\Nearrow"] & & \mathscr{F}C \arrow[d, "\tau_C"] \\
\mathscr{F}A \arrow[r, "\mathscr{F}f"] \arrow[d, "\tau_A"'] \arrow[rr, bend left=60, "\mathscr{F}(g\circ f)"] \arrow[rr, phantom, bend left=30, "\delta_{g,f}\Uparrow"] \arrow[dr, phantom, "\tau_f\Nearrow"] & \mathscr{F}B \arrow[r, "\mathscr{F}g"] \arrow[d, "\tau_B"] \arrow[dr, phantom, "\tau_g\Nearrow"] & \mathscr{F}C \arrow[d, "\tau_C"] & = & \mathscr{G}A \arrow[rr, "\mathscr{G}(g\circ f)"'] \arrow[dr, "\mathscr{G}f"'] \arrow[rr, phantom, bend right=30, "\delta'_{g,f}\Uparrow"] & & \mathscr{G}C \\
\mathscr{G}A \arrow[r, "\mathscr{G}f"'] & \mathscr{G}B \arrow[r, "\mathscr{G}g"'] & \mathscr{G}C & & & \mathscr{G}B \arrow[ur, "\mathscr{G}g"'] &
\end{tikzcd} \tag{2.9}$$

expressing the compatibility of τ with composition. Also, for any object $A \in \mathcal{K}$ we have the equality

$$\begin{tikzcd}
\mathscr{F}A \arrow[r, "\mathscr{F}1_A"] \arrow[d, "\tau_A"'] \arrow[dr, phantom, "\tau_{1_A}\Nearrow"] & \mathscr{F}A \arrow[d, "\tau_A"] & & \mathscr{F}A \arrow[r, bend left=50, "\mathscr{F}1_A"] \arrow[r, phantom, bend left=25, "\gamma_A\Uparrow"] \arrow[r, "1_{\mathscr{F}A}"'] \arrow[d, "\tau_A"'] \arrow[dr, "\tau_A"] \arrow[dr, phantom, "\cong", near start, shift right=3ex] & \mathscr{F}A \arrow[d, "\tau_A"] \arrow[dl, phantom, "\cong", near start, shift left=1ex] \\
\mathscr{G}A \arrow[r, "\mathscr{G}1_A"] \arrow[r, bend right=50, "1_{\mathscr{G}A}"'] \arrow[r, phantom, bend right=25, "\gamma'_A\Uparrow"] & \mathscr{G}A & = & \mathscr{G}A \arrow[r, "1_{\mathscr{G}A}"'] & \mathscr{G}A
\end{tikzcd} \tag{2.10}$$

expressing the compatibility of τ with units.

Remark.

(1) The naturality for the transformation (2.7) can be expressed by the equality

$$\begin{tikzcd}
\mathscr{F}A \arrow[r, bend left=20, "\mathscr{F}g"] \arrow[r, bend right=20, "\mathscr{F}f"'] \arrow[r, phantom, "\Uparrow\mathscr{F}\alpha"] \arrow[d, "\tau_A"'] \arrow[dr, phantom, "\tau_f\Nearrow"] & \mathscr{F}B \arrow[d, "\tau_B"] & & \mathscr{F}A \arrow[r, "\mathscr{F}g"] \arrow[d, "\tau_A"'] \arrow[dr, phantom, "\tau_g\Nearrow"] & \mathscr{F}B \arrow[d, "\tau_B"] \\
\mathscr{G}A \arrow[r, "\mathscr{G}f"'] & \mathscr{G}B & = & \mathscr{G}A \arrow[r, bend left=20, "\mathscr{G}g"] \arrow[r, bend right=20, "\mathscr{G}f"'] \arrow[r, phantom, "\Uparrow\mathscr{G}\alpha"] & \mathscr{G}B
\end{tikzcd}$$

for any 2-cell $\alpha : f \Rightarrow f$.

(2) Using pasting operations properties (see Section 2.3), the equality (2.9) can be expressed by the commutativity of

$$\begin{tikzcd}
\mathscr{G}g \circ (\mathscr{G}f \circ \tau_A) \arrow[r, "\mathscr{G}g * \tau_f"] \arrow[d, "\alpha^{-1}"'] & \mathscr{G}g \circ (\tau_B \circ \mathscr{F}f) \arrow[r, "\alpha^{-1}"] & (\mathscr{G}g \circ \tau_B) \circ \mathscr{F}f \arrow[d, "\tau_g * \mathscr{F}f"] \\
(\mathscr{G}g \circ \mathscr{G}f) \circ \tau_A \arrow[d, "\delta'_{g,f} * \tau_A"'] & & (\tau_C \circ \mathscr{F}g) \circ \mathscr{F}f \arrow[d, "\alpha"] \\
\mathscr{G}(g \circ f) \circ \tau_A \arrow[dr, "\tau_{g\circ f}"'] & & \tau_C \circ (\mathscr{F}g \circ \mathscr{F}f) \arrow[dl, "\tau_C * \delta_{g,f}"] \\
& \tau_C \circ \mathscr{F}(g \circ f) &
\end{tikzcd}$$

inside the hom-category $\mathcal{L}(\mathscr{F}A, \mathscr{G}C)$.

(3) Similarly, the equality (2.10) can be expressed by the commutativity of

$$\begin{tikzcd}
1_{\mathscr{G}A} \circ \tau_A \arrow[r, "\gamma'_A * \tau_A"] \arrow[d, "\lambda"'] & \mathscr{G}(1_A) \circ \tau_A \arrow[dd, "\tau_{1_A}"] \\
\tau_A \arrow[d, "\rho^{-1}"'] & \\
\tau_A \circ 1_{\mathscr{F}A} \arrow[r, "\tau_A * \gamma_A"'] & \tau_A \circ \mathscr{F}(1_A).
\end{tikzcd}$$

A lax natural transformation τ is a *pseudonatural* transformation (respectively *strict*) when all the 2-cells τ_f as in (2.8) are isomorphisms (respectively identities).

Also, a *colax* (or *oplax*) natural transformation is equipped with a natural transformation in the opposite direction of (2.7). Note that between either lax or colax functors $\mathscr{F},\mathscr{G} : \mathcal{K} \to \mathcal{L}$ of bicategories, we can consider both lax and colax natural transformations.

Definition 2.1.5. Consider lax functors $\mathscr{F},\mathscr{G} : \mathcal{K} \to \mathcal{L}$ between bicategories, and $\tau, \sigma : \mathscr{F} \Rightarrow \mathscr{G}$ two lax natural transformations. A *modification* $m : \tau \Rrightarrow \sigma$ is a family of 2-cells

$$\begin{tikzcd}
\mathscr{F}A \arrow[r, bend left, "\tau_A", ""'{name=U}] \arrow[r, bend right, "\sigma_A"', ""{name=D}] & \mathscr{G}A
\arrow[from=U, to=D, Rightarrow, "m_A"]
\end{tikzcd}$$

for every object A of $\mathcal{K}$, such that

$$\begin{tikzcd}
\mathscr{F}A \arrow[r, "\mathscr{F}f"] \arrow[d, bend right=50, "\tau_A"', ""{name=L1}] \arrow[d, bend left=50, "\sigma_A", ""'{name=R1}] & \mathscr{F}B \arrow[d, "\sigma_B"] \arrow[dl, phantom, "\sigma_f \Nearrow"] \\
\mathscr{G}A \arrow[r, "\mathscr{G}f"'] & \mathscr{G}B
\arrow[from=L1, to=R1, Rrightarrow, "m_A"]
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{F}A \arrow[r, "\mathscr{F}f"] \arrow[d, "\tau_A"'] & \mathscr{F}B \arrow[d, bend right=50, "\tau_B"', ""{name=L2}] \arrow[d, bend left=50, "\sigma_B", ""'{name=R2}] \arrow[dl, phantom, "\tau_f \Nearrow"] \\
\mathscr{G}A \arrow[r, "\mathscr{G}f"'] & \mathscr{G}B.
\arrow[from=L2, to=R2, Rrightarrow, "m_B"]
\end{tikzcd}
\tag{2.11}$$

It is not hard to define composition of natural transformations and modifications, and respective identities. Therefore, for any two bicategories $\mathcal{K}, \mathcal{L}$ there is a functor bicategory $\mathbf{Lax}(\mathcal{K}, \mathcal{L}) = \mathbf{Bicat}_l(\mathcal{K}, \mathcal{L})$ of lax functors, lax natural transformations and modifications, and it has a sub-bicategory $\mathbf{Hom}(\mathcal{K}, \mathcal{L}) = \mathbf{Bicat}_{ps}(\mathcal{K}, \mathcal{L})$ of pseudofunctors, pseudonatural transformations and modifications. In fact, the *tricategory* **Hom** is a very important 3-dimensional category of bicategories (see [**GPS95, Gur13**]). Notice that $\mathbf{Hom}(\mathcal{K}, \mathcal{L})$ is a strict bicategory, *i.e.* 2-category when $\mathcal{L}$ is a 2-category.

2.2. Monads and modules in bicategories

Definition 2.2.1. A *monad* in a bicategory $\mathcal{K}$ consists of an object B together with an endomorphism $t : B \to B$ and 2-cells $\eta : 1_B \Rightarrow t$, $m : t \circ t \Rightarrow t$ called the *unit* and *multiplication* respectively, such that the diagrams

$$\begin{tikzcd}
(t \circ t) \circ t \arrow[rr, "\alpha_{t,t,t}"] \arrow[d, "m \circ 1"'] & & t \circ (t \circ t) \arrow[d, "1 \circ m"] \\
t \circ t \arrow[dr, "m"'] & & t \circ t \arrow[dl, "m"] \\
& t &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
1_B \circ t \arrow[r, "\eta \circ 1"] \arrow[dr, "\lambda_t"'] & t \circ t \arrow[d, "m"] & t \circ 1_B \arrow[l, "1 \circ \eta"'] \arrow[dl, "\rho_t"] \\
& t &
\end{tikzcd}$$

commute.

Equivalently, a monad in a bicategory $\mathcal{K}$ is a lax functor $\mathscr{F} : \mathbf{1} \to \mathcal{K}$, where $\mathbf{1}$ is the terminal bicategory with a unique 0-cell $\star$ (one 1-cell and one 2-cell). This amounts to an object $\mathscr{F}(\star) = B \in \mathcal{K}$ and a functor

$$\mathscr{F}_{\star,\star} : \mathbf{1}(\star, \star) \to \mathcal{K}(B, B)$$

which picks up an endoarrow $t : B \to B$ in $\mathcal{K}$. The natural transformations δ and γ of the lax functor give the multiplication and the unit of t

$$m \equiv \delta_{1_\star,1_\star} : t \circ t \to t \quad \text{and} \quad \eta \equiv \gamma_\star : 1_B \to t$$

and the axioms for $\mathscr{F}$ give the monad axioms for (t, m, η).

Remark 2.2.2. As mentioned earlier, lax functors between bicategories compose. Therefore if $\mathscr{G} : \mathcal{K} \to \mathcal{L}$ is a lax functor between bicategories, the composite

$$\mathbf{1} \xrightarrow{\mathscr{F}} \mathcal{K} \xrightarrow{\mathscr{G}} \mathcal{L}$$

is itself a lax functor from $\mathbf{1}$ to $\mathcal{L}$, hence defines a monad. In other words, if $t : B \to B$ is a monad in the bicategory $\mathcal{K}$, then $\mathscr{G}t : \mathscr{G}B \to \mathscr{G}B$ is a monad in the bicategory $\mathcal{L}$, *i.e.* lax functors preserve monads.

For an object B in the bicategory $\mathcal{K}$ and a monad $t : B \to B$, there is an induced ordinary monad (*i.e.* in **Cat**) on the hom-categories, namely 'post-composition with t'. Explicitly, for any 0-cell A we have an endofunctor

$$\mathcal{K}(A, t) : \mathcal{K}(A, B) \longrightarrow \mathcal{K}(A, B)$$

which is the mapping

$$\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\alpha"] & B \arrow[r, mapsto] & A \arrow[r, bend left, "f"{name=U2}] \arrow[r, bend right, "g"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\alpha"] & B \arrow[r, "t"] & B \end{tikzcd}$$

for objects and morphisms in $\mathcal{K}(A, B)$. The multiplication and unit of the monad $\bar{m}$ and $\bar{\eta}$, are natural transformations with components, for each $f : A \to B$ in $\mathcal{K}(A, B)$,

$$\bar{m}_f \;=\; \begin{tikzcd} & & B \arrow[dr, "t"] \arrow[d, phantom, "\Downarrow m"] & \\ A \arrow[r, "f"] & B \arrow[ur, "t"] \arrow[rr, bend right, "t"'] & {} & B, \end{tikzcd} \qquad \bar{\eta}_f \;=\; \begin{tikzcd} A \arrow[rr, bend left, "f"] \arrow[dr, "f"'] & \Downarrow \rho_f^{-1} & B. \\ & B \arrow[ur, bend left, "1_B"] \arrow[ur, bend right, "t"'] \arrow[ur, phantom, "\Downarrow \eta"] & \end{tikzcd}$$

Now, consider the Eilenberg-Moore category $\mathcal{K}(A, B)^{\mathcal{K}(A,t)}$ of $\mathcal{K}(A, t)$-algebras. It has as objects 1-cells $f : A \to B$ equipped with an *action* $\mu : \mathcal{K}(A, t)(f) \Rightarrow f$, *i.e.* a 2-cell

$$\begin{tikzcd} & B \arrow[dr, "t"] \arrow[d, phantom, "\Downarrow \mu"] & \\ A \arrow[ur, "f"] \arrow[rr, bend right, "f"'] & {} & B \end{tikzcd} \tag{2.12}$$

compatible with the multiplication and unit of the monad $\mathcal{K}(A, t)$:

$$\begin{tikzcd} & B \arrow[r, bend left, "t"] \arrow[ddrr, bend right, "t"'] & B \arrow[dr, "t"] \arrow[d, phantom, "\Downarrow m"] & \\ & & {} & B \\ A \arrow[uur, "f"] \arrow[urrr, bend right, "t"'] & \Downarrow \mu & & \end{tikzcd} \;=\; \begin{tikzcd} & B \arrow[r, bend left, "t"] \arrow[d, phantom, "\Downarrow \mu"] & B \arrow[dr, "t"] & \\ A \arrow[ur, "f"] \arrow[urr, bend right, "f"'] \arrow[rrr, bend right, "t"'] & {} & \Downarrow \mu & B, \end{tikzcd} \tag{2.13}$$

$$\begin{tikzcd}
& B \arrow[dr, bend left, "1_B"] \arrow[dr, "t"'] \arrow[dr, phantom, bend left=15, "\Downarrow\eta"] & \\
A \arrow[ur, bend left, "f"] \arrow[rr, bend right, "f"'] \arrow[rr, phantom, "\Downarrow\mu"] & & B
\end{tikzcd}
=
\begin{tikzcd}
& B \arrow[dr, bend left, "1_B"] & \\
A \arrow[ur, bend left, "f"] \arrow[rr, bend right, "f"'] \arrow[rr, phantom, "\Downarrow\rho_f"] & & B.
\end{tikzcd}$$

Such an 1-cell f together with an action μ is called a *t-module* or *t-algebra.* An arrow $(f, \mu) \xrightarrow{\tau} (g, \mu')$ is a 2-cell $\tau : f \Rightarrow g$ in $\mathcal{K}$ compatible with the actions, *i.e.* such that

$$\begin{tikzcd}
& B \arrow[dr, bend left, "t"] & \\
A \arrow[ur, bend left, "f"] \arrow[ur, bend right, "g"'] \arrow[ur, phantom, "\Downarrow\tau"] \arrow[rr, bend right=60, "g"'] \arrow[rr, phantom, "\Downarrow\mu'"] & & B
\end{tikzcd}
=
\begin{tikzcd}
& B \arrow[dr, bend left, "t"] & \\
A \arrow[ur, bend left, "f"] \arrow[rr, "f"'] \arrow[rr, phantom, bend left=30, "\Downarrow\mu"] \arrow[rr, bend right=60, "g"'] \arrow[rr, phantom, bend right=30, "\Downarrow\tau"] & & B,
\end{tikzcd} \tag{2.14}$$

called a *morphism of t-modules.*

Definition 2.2.3. The category of Eilenberg-Moore algebras $\mathcal{K}(A, B)^{\mathcal{K}(A,t)}$ for $t : B \to B$ a monad in the bicategory $\mathcal{K}$ is the *category of left t-modules* with domain A, denoted by ${}^{A}_{t}\mathbf{Mod}$.

We may similarly define the category $\mathbf{Mod}^B_s$ of *right s-modules* with codomain B. It is the category of Eilenberg-Moore algebras $\mathcal{K}(A, B)^{\mathcal{K}(s,B)}$ for $s : A \to A$ a monad in the bicategory $\mathcal{K}$, where $\mathcal{K}(s, B)$ is the monad 'pre-composition with s'. Moreover, the above endofunctors combined define the monad

$$\begin{array}{rl}
\mathcal{K}(s,t): & \mathcal{K}(A,B) \longrightarrow \mathcal{K}(A,B) \\
& (A \xrightarrow{f} B) \longmapsto (A \xrightarrow{s} A \xrightarrow{f} B \xrightarrow{t} B)
\end{array}$$

on $\mathcal{K}(A, B)$, and the category of algebras $\mathcal{K}(A, B)^{\mathcal{K}(s,t)}$ is now called the category of *right s/left t-bimodules*, ${}_t\mathbf{Mod}_s$.

Remark 2.2.4. In the classical case where $\mathcal{K}$=**Cat**, the term left (respectively right) 't-algebra' is more commonly restricted to those with domain (respectively codomain) the unit category $\mathbf{1}$. A left t-module with domain $\mathbf{1}$, *i.e.* a functor $f : \mathbf{1} \to B$, is then identified with the corresponding object X in the category B, and the actions $\mu : t(X) \to X$ and maps $\tau : X \to Y$ are morphisms in B. The category $\mathcal{K}(\mathbf{1}, B)^{\mathcal{K}(\mathbf{1},t)}$ is then denoted by B^t.

Notice that in the above presentation, there is a certain circularity in the definition of modules for a monad in an arbitrary bicategory $\mathcal{K}$. More precisely, the Eilenberg-Moore category of algebras which is used in the very definition of the category of modules in this abstract setting (Definition 2.2.3), is in reality a particular example of a category of modules for a monad in $\mathcal{K}$ = **Cat**. However, this could be easily avoided: in Kelly-Street's [**KS74**], an action of a monad t in a 2-category is defined to be a 2-cell as in (2.12) satisfying the specified axioms, and maps are defined accordingly. Hence, the fact that we now identify from the beginning this structure with the Eilenberg-Moore category for an ordinary monad does not affect the level of generality.

Dually to the above, we have the following definitions.

Definition 2.2.5. A *comonad* in a bicategory $\mathcal{K}$ consists of an object A together with an endoarrow $u : A \to A$ and 2-cells $\Delta : u \Rightarrow u \circ u$, $\varepsilon : u \Rightarrow 1_A$ called the *comultiplication* and *counit* respectively, such that the diagrams

$$\begin{tikzcd}
& u \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
u \circ u \arrow[d, "\Delta\circ 1"'] & & u \circ u \arrow[d, "1\circ\Delta"] \\
(u \circ u) \circ u \arrow[rr, "\alpha_{u,u,u}"'] & & u \circ (u \circ u),
\end{tikzcd}
\qquad
\begin{tikzcd}
1_A \circ u \arrow[dr, "\lambda_u"'] & u \circ u \arrow[l, "\varepsilon\circ 1"'] \arrow[r, "1\circ\varepsilon"] & u \circ 1_A \arrow[dl, "\rho_u"] \\
& u \arrow[u, "\Delta"'] &
\end{tikzcd}$$

commute.

Notice that a comonad in the bicategory $\mathcal{K}$ is precisely a monad in the bicategory $\mathcal{K}^{\mathrm{co}}$. Similarly to before, for an object A and a comonad $u : A \to A$ in a bicategory $\mathcal{K}$, there is an induced comonad in **Cat** between hom-categories

$$\mathcal{K}(u, B) : \mathcal{K}(A, B) \longrightarrow \mathcal{K}(A, B)$$

which precomposes objects and arrows in $\mathcal{K}(A, B)$ with the 1-cell $u : A \to A$. The axioms for a comonad follow again from those of u, hence we can form the category of coalgebras $\mathcal{K}(A, B)^{\mathcal{K}(u,B)}$. Its objects are 1-cells $h : A \to B$ equipped with a *coaction* $\delta : h \Rightarrow \mathcal{K}(u, B)(h)$, *i.e.* a 2-cell

$$\begin{tikzcd}
A \arrow[rr, bend left, "h"] \arrow[dr, "u"'] & \Downarrow\delta & B \\
& A \arrow[ur, "h"'] &
\end{tikzcd}$$

compatible with the comultiplication and counit of $\mathcal{K}(u, B)$, and arrows $\sigma : (h, \delta) \to (k, \delta')$ are 2-cells $\sigma : h \Rightarrow k$ compatible with the coactions δ and δ'.

Definition 2.2.6. The category of Eilenberg-Moore coalgebras $\mathcal{K}(A, B)^{\mathcal{K}(u,B)}$ for a comonad $u : A \to A$ in the bicategory $\mathcal{K}$ is the *category of right u-comodules* or *coalgebras* with codomain B, denoted by $\mathbf{Comod}_u^B$.

Similarly, for a comonad $v : B \to B$ we can define the category ${}_v^A\mathbf{Comod}$ of *left v-comodules* with domain A as the category $\mathcal{K}(A, B)^{\mathcal{K}(A,v)}$ as well as the category of *right u/left v-bicomodules* ${}_v\mathbf{Comod}_u$ as the category of coalgebras of the comonad 'pre-composition with u and post-composition with v', $\mathcal{K}(u, v)$, on $\mathcal{K}(A, B)$.

Remark. As mentioned in Remark 2.2.4, for the classical case $\mathcal{K} = \mathbf{Cat}$ the term 'v-coalgebra' is more commonly restricted to the case that the domain of a left v-comodule (or respectively the codomain of a right u-comodule) is the unit category $\mathbf{1}$. The coalgebra $h : \mathbf{1} \to B$ is then identified with the object Z of the category which is picked out by the functor h, and we denote $\mathcal{K}(\mathbf{1}, B)^{\mathcal{K}(\mathbf{1},v)}$ for a comonad v as B^v.

Definition 2.2.7. A *(lax) monad functor* between two monads $t : B \to B$ and $s : C \to C$ in a bicategory consists of an 1-cell $f : B \to C$ between the 0-cells of the monads together with a 2-cell

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[d, "t"'] & C \arrow[d, "s"] \arrow[dl, phantom, "\Swarrow\psi"] \\
B \arrow[r, "f"'] & C
\end{tikzcd}$$

satisfying compatibility conditions with multiplications and units:

$$\begin{tikzcd}
 & C \arrow[dr, "s"] & & & & & & C \arrow[dr, "s"] & \\
B \arrow[ur, bend left, "f"] \arrow[dr, "t"] \arrow[drr, bend right=40, "t"'] & \Downarrow\psi & C \arrow[r, "s"] & C & = & B \arrow[r, "f"] \arrow[drr, phantom, ""] \arrow[dr, "t"'] & C \arrow[ur, "s"] \arrow[rr, bend right, "s"'] & \Downarrow m' & C \\
 & B \arrow[ur, "f"] \arrow[dr, "t"] & \Downarrow\psi & & & & \Downarrow\psi & & \\
 & \Swarrow m & B \arrow[uur, "f"'] & & & & B \arrow[uurr, "f"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & C \arrow[dr, bend left, "1_C"] \arrow[dr, bend right, "s"'] \arrow[dr, phantom, "\Swarrow\eta'"] & & & & & \\
B \arrow[ur, "f"] \arrow[dr, "t"'] & \Downarrow\psi & C & = & B \arrow[r, bend left, "1_B"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\eta"] & B \arrow[r, "f"] & C \\
 & B \arrow[ur, "f"'] & & & & &
\end{tikzcd}$$

If the 2-cell ψ is in the opposite direction, and the diagrams are accordingly modified, we have a *colax* monad functor (or monad *opfunctor*) between two monads. There are also appropriate notions of monad natural transformations for monads in bicategories, not essential for the purposes of this thesis, which can be found in detail in [**Str72**]. Because of the correspondence between monads and lax functors from the terminal bicategory, we obtain a bicategory $\mathbf{Mnd}(\mathcal{K}) \equiv [\mathbf{1}, \mathcal{K}]_l$.

In search of an induced functor between categories of modules, we will need some well-known results related to maps of monads on ordinary categories. The following definition is just a special case of the above definition for $\mathcal{K} = \mathbf{Cat}$.

Definition 2.2.8. Let $T = (T, m, \eta)$ be a monad on a category $\mathcal{C}$ and $T' = (T', m', \eta')$ a monad on a category $\mathcal{C}'$. A *lax map of monads* $(\mathcal{C}, T) \to (\mathcal{C}', T')$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ together with a natural transformation

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "T"] \arrow[d, "F"'] & \mathcal{C} \arrow[d, "F"] \\
\mathcal{C}' \arrow[r, "T'"'] \arrow[ur, phantom, "\psi\Nearrow"] & \mathcal{C}'
\end{tikzcd}$$

making the diagrams

$$\begin{tikzcd}
T'T'F \arrow[r, "T'\psi"] \arrow[d, "m'F"'] & T'FT \arrow[r, "\psi T"] & FTT \arrow[d, "Fm"] \\
T'F \arrow[rr, "\psi"'] & & FT,
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[rr, "F\eta"] \arrow[dr, "\eta' F"'] & & FT \\
 & T'F \arrow[ur, "\psi"'] &
\end{tikzcd}$$

commute. A *strong* or *pseudo* (respectively *strict*) map of monads is a lax map (F, ψ) in which ψ is an isomorphism (respectively the identity).

A very important property of lax maps of monads is that they give rise to maps between categories of algebras: a lax map $(F,\psi):(\mathcal{C},T)\to(\mathcal{C}',T')$ induces a functor

$$\begin{tikzcd} F_*: & \mathcal{C}^T \arrow[r] & \mathcal{C}'^{T'} \\ & (X,a) \arrow[r, mapsto] & (FX, Fa\circ\psi_X) \end{tikzcd}$$

which means that if $TX \xrightarrow{a} X$ is the action of the T-algebra X, then $T'FX \xrightarrow{\psi_X} FTX \xrightarrow{Fa} FX$ is the action which makes FX into a T'-algebra. In fact, there is a bijection between the two structures.

Lemma 2.2.9. *Let T and T' be monads on categories $\mathcal{C}$ and $\mathcal{C}'$. There is a one-to-one correspondence betweeen lax maps of monads $(\mathcal{C},T)\to(\mathcal{C}',T')$ and pairs of functors (K,F) such that the square*

$$\begin{tikzcd} \mathcal{C}^T \arrow[r, "K"] \arrow[d, "U"'] & \mathcal{C}'^{T'} \arrow[d, "U'"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{C}' \end{tikzcd}$$

commutes, where U,U' are the forgetful functors. Explicitly, a lax map (F,ψ) corresponds bijectively to the pair (F_*,F).

We can apply this lemma to obtain functors between the categories of modules for a monad in a bicategory as described above. More specifically, by Remark 2.2.2 lax functors between bicategories preserve monads, and this in a sense carries over to the categories of their modules.

Proposition 2.2.10. *If $\mathscr{F}:\mathcal{K}\to\mathcal{L}$ is a lax functor between two bicategories and $t:B\to B$ a monad in $\mathcal{K}$, there is an induced functor*

$$\mathbf{Mod}(\mathscr{F}_{A,B}):\mathcal{K}(A,B)^{\mathcal{K}(A,t)}\longrightarrow\mathcal{L}(\mathscr{F}A,\mathscr{F}B)^{\mathcal{L}(\mathscr{F}A,\mathscr{F}t)}$$

between the category of left t-modules in $\mathcal{K}$ and the category of left $\mathscr{F}t$-modules in $\mathcal{L}$, which maps a t-module $f:A\to B$ to the $\mathscr{F}t$-module $\mathscr{F}f:\mathscr{F}A\to\mathscr{F}B$. Moreover, the following diagram commutes:

$$\begin{tikzcd} \mathcal{K}(A,B)^{\mathcal{K}(A,t)} \arrow[r, "\mathbf{Mod}(\mathscr{F}_{A,B})"] \arrow[d, "U"'] & \mathcal{L}(\mathscr{F}A,\mathscr{F}B)^{\mathcal{L}(\mathscr{F}A,\mathscr{F}t)} \arrow[d, "U"] \\ \mathcal{K}(A,B) \arrow[r, "\mathscr{F}_{A,B}"'] & \mathcal{L}(\mathscr{F}A,\mathscr{F}B) \end{tikzcd} \tag{2.15}$$

Proof. The endofunctor $\mathcal{K}(A,t)$ is an ordinary monad on the hom-category $\mathcal{K}(A,B)$, and since $\mathscr{F}t$ is a monad in $\mathcal{L}$, the endofunctor $\mathcal{L}(\mathscr{F}A,\mathscr{F}t)$ is also a monad on the hom-category $\mathcal{L}(\mathscr{F}A,\mathscr{F}B)$.

In order to apply Lemma 2.2.9, we need to exhibit a map of monads as in Definition 2.2.8. In fact, we have a functor

$$\mathscr{F}_{A,B}:\mathcal{K}(A,B)\to\mathcal{L}(\mathscr{F}A,\mathscr{F}B)$$

and also a natural transformation

$$\begin{tikzcd}
\mathcal{K}(A,B) \arrow[r, "\mathcal{K}(A,t)"] \arrow[d, "\mathscr{F}_{A,B}"'] \arrow[dr, phantom, "\psi\Nearrow"] & \mathcal{K}(A,B) \arrow[d, "\mathscr{F}_{A,B}"] \\
\mathcal{L}(\mathscr{F}A,\mathscr{F}B) \arrow[r, "\mathcal{L}(\mathscr{F}A,\mathscr{F}t)"'] & \mathcal{L}(\mathscr{F}A,\mathscr{F}B)
\end{tikzcd}$$

with components, for any 1-cell $f : A \to B$,

$$\begin{tikzcd}
 & \mathscr{F}B \arrow[dr, "\mathscr{F}t"] & \\
\mathscr{F}A \arrow[ur, "\mathscr{F}f"] \arrow[rr, bend right, "\mathscr{F}(t\circ f)"'] & \Downarrow\delta_{t,f} & FB
\end{tikzcd}$$

where $\mathscr{F}_{A,B}$ and δ come from the definition of a lax functor. Hence, we do have a map of monads

$$(\mathscr{F}_{A,B}, \psi) : (\mathcal{K}(A,B), \mathcal{K}(A,t)) \longrightarrow (\mathcal{L}(\mathscr{F}A,\mathscr{F}B), \mathcal{L}(\mathscr{F}A,\mathscr{F}t))$$

which induces a functor between the categories of algebras

$$(\mathscr{F}_{A,B})_* \equiv \mathbf{Mod}(\mathscr{F}_{A,B})$$

such that the diagram (2.15) commutes. ☐

In a completely dual way, we can verify that colax functors between bicategories preserve comonads, and that they also induce functors between the corrresponding categories of comodules.

2.3. 2-categories

A (strict) *2-category* is a bicategory in which all constraints are identities, *i.e.* $\alpha, \rho, \lambda = 1$. In this case, the horizontal composition is strictly associative and unitary and the axioms (2.1), (2.2) hold automatically. Consequently, the collection of 0-cells and 1-cells form a category on its own.

EXAMPLES.

1. The collection of all (small) categories, functors and natural transformations forms the 2-category **Cat**, which is a leading example in category theory.
2. Monoidal categories, (strong) monoidal functors and monoidal natural transformations form the 2-category **MonCat** (see Chapter 3).
3. If $\mathcal{V}$ is a monoidal category, $\mathcal{V}$-enriched categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations form the 2-category $\mathcal{V}$-**Cat** (see Chapter 4).
4. Fibrations and opfibrations over $\mathbb{X}$, (op)fibred functors and (op)fibred natural transformations form the 2-categories **Fib**($\mathbb{X}$) and **OpFib**($\mathbb{X}$) (see Chapter 5).

(5) Suppose $\mathbb{E}$ is a category with finite limits. There is a 2-category $\mathbf{Cat}(\mathbb{E})$ with objects categories internal to $\mathbb{E}$, which have an $\mathbb{E}$-object of objects and an $\mathbb{E}$-object of morphisms. Instances of this are ordinary categories ($\mathbb{E} = \mathbf{Set}$), double categories ($\mathbb{E} = \mathbf{Cat}$) and crossed modules ($\mathbb{E} = \mathbf{Grp}$).

A (strict) *2-functor* is a strict functor between 2-categories, whereas a (strict) *2-natural transformation* is a strict natural transformation between 2-functors.

Since a 2-category is a special case of a bicategory, all kinds of functors (and natural transformations) described in Section 2.1 can be defined in this context. They now give rise to categories $\mathbf{2\text{-}Cat}, \mathbf{2\text{-}Cat}_{ps}, \mathbf{2\text{-}Cat}_{l}, \mathbf{2\text{-}Cat}_{c}$. Moreover, for $\mathcal{K}, \mathcal{L}$ 2-categories, there are various kinds of functor 2-categories: $[\mathcal{K}, \mathcal{L}]$ with 2-functors, 2-natural transformations and modifications, $\mathbf{Lax}(\mathcal{K}, \mathcal{L})_s$ with lax functors, strict 2-natural transformations and modifications, $[\mathcal{K}, \mathcal{L}]_{\mathrm{ps}}$ with pseudofunctors, pseudonatural transformations and modifications etc. Evidently, this implies that all flavours of categories with objects 2-categories are in reality 2-categories themselves, and moreover $\mathbf{2\text{-}Cat}$ is a paradigmatic example of a *3-category.*

Remark 2.3.1. We saw earlier how bicategories and lax/colax/pseudo functors form ordinary categories, and also how structures like $\mathbf{Lax}(\mathcal{K}, \mathcal{L})$ or $\mathbf{Hom}(\mathcal{K}, \mathcal{L})$ of appropriate functors, natural transformations and modifications are in fact bicategories themselves (or functor 2-categories in the strict case like above). However bicategories, lax functors and (co)lax natural transformations fail to form a 2-category. Even restricting from bicategories to 2-categories and from lax functors to 2-functors does not suffice in order to form a 2-dimensional structure with a weaker notion of natural transformation. This is due to problems arising regarding the vertical and horizontal composition of 2-cells.

The above is thoroughly discussed in Lack's [**Lac10b**], where *icons* are employed so that bicategories and lax functors can be the objects and 1-cells of a 2-category $\mathbf{Bicat}_2$. More precisely, the 2-cells $\tau : \mathscr{F} \Rightarrow \mathscr{G}$ are colax natural transformations (see Definition 2.1.4) whose components $\tau_A : \mathscr{F}A \to \mathscr{G}A$ are identities, hence the name *I*denitity *C*omponent *O*plax *N*atural transformation. That reduces the natural transformation in the opposite direction of (2.7) to the simpler

$$\begin{tikzcd}
\mathcal{K}(A,B) \arrow[r, bend left, "\mathscr{F}_{A,B}"] \arrow[r, bend right, "\mathscr{G}_{A,B}"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathcal{L}(\mathscr{F}A, \mathscr{F}B)
\end{tikzcd}$$

which satisfies accordingly simplified axioms. Icons were firstly introduced in [**LP08**] and they allow the study of bicategories in a plain 2-dimensional setting, with applications in various contexts.

In many cases, various concepts used in ordinary category theory are special instances of abstract notions defined in an arbitrary 2-category or bicategory. For example, the usual notion of equivalence of categories is just a special case of the following notion of (internal) equivalence in any bicategory, applied to $\mathbf{Cat}$.

Definition. A 1-cell $f : A \to B$ in a bicategory $\mathcal{K}$ is an *equivalence* when there exist another 1-cell $g : B \to A$ and invertible 2-cells

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, bend right, "1_A"'] & \Downarrow\cong & A, \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dr, "f"] & \\ B \arrow[ur, "g"] \arrow[rr, bend right, "1_B"'] & \Downarrow\cong & B \end{tikzcd}$$

i.e. isomorphisms $gf \cong 1_A$ and $fg \cong 1_B$ in $\mathcal{K}$. We write $f \simeq g$.

Just as the notion of equivalence of categories can be internalized in any 2-category, the notion of equivalence for 2-categories can be internalized in any 3-category in an appropriate way, hence we obtain the following definition for **2-Cat**.

Definition. A 2-functor $T : \mathcal{K} \to \mathcal{L}$ between two 2-categories $\mathcal{K}$ and $\mathcal{L}$ is a *(strict) 2-equivalence* if there is some 2-functor $S : \mathcal{L} \to \mathcal{K}$ and isomorphisms $\mathbf{1} \cong TS$, $ST \cong \mathbf{1}$. We write $\mathcal{K} \backsimeq \mathcal{L}$.

There is a well-known proposition which gives conditions for a 2-functor to be a 2-equivalence.

Proposition 2.3.2. *The 2-functor $T : \mathcal{K} \to \mathcal{L}$ is an equivalence if and only if T is fully faithful,* i.e. $T_{A,A'} : \mathcal{A}(A, A') \to \mathcal{B}(TA, TA')$ *is an isomorphism of categories for every $A, A' \in \mathcal{A}$, and essentially surjective on objects,* i.e. *every object $B \in \mathcal{L}$ is isomorphic to TA for some $A \in \mathcal{A}$.*

The appropriate weaker version for the notion of equivalence in the context of bicategories is the following.

Definition. A *biequivalence* between bicategories $\mathcal{K}$ and $\mathcal{L}$ consists of two pseudofunctors $\mathscr{F} : \mathcal{K} \to \mathcal{L}$ and $\mathscr{G} : \mathcal{L} \to \mathcal{K}$ and pseudonatural transformations $\mathscr{G}\mathscr{F} \to 1_{\mathcal{K}}$, $1_{\mathcal{L}} \to \mathscr{F}\mathscr{G}$ which are invertible up to isomorphism. Equivalently, $\mathscr{F} : \mathcal{K} \to \mathcal{L}$ is a biequivalence if and only if it is locally an equivalence, *i.e.* each $\mathscr{F}_{A,B} : \mathcal{K}(A, B) \to \mathcal{L}(\mathscr{F}A, \mathscr{F}B)$ is an equivalence of categories, and every $B \in \mathcal{L}$ is equivalent to $\mathscr{F}A$ for some A.

Notice that the second statement in fact is equivalent to the first, only if the axiom of choice is assumed. This has to do with the fact that in general, there exist notions of *strong* and *weak* equivalence between categories, and every weak equivalence being a strong one is equivalent to the axiom of choice.

The coherence theorems for bicategories and their homomorphisms are of great importance, and have been fundamental for the development of higher category theory. In particular, it is asserted that certain diagrams involving the constraint isomorphisms of bicategories will always commute. Coherence allows us to replace any bicategory with an appropriate strict 2-category, so that various situations are greatly simplified. This ensures for example that the pasting diagrams, commonly used when working with 2-categories, can also be used for bicategories.

Theorem 2.3.3. *Every bicategory is biequivalent to a 2-category.*

The proof is based on a bicategorical generalization of the Yoneda Lemma (see Street's [**Str80**]), which states that the embedding

$$\begin{tikzcd}
\mathcal{K} \arrow[r] & \mathbf{Hom}(\mathcal{K}^{\mathrm{op}}, \mathbf{Cat}) \\
A \arrow[r, mapsto] & \mathcal{K}^{\mathrm{op}}(A, -)
\end{tikzcd}$$

is locally an equivalence, hence any bicategory $\mathcal{K}$ is biequivalent to a full sub-2-category of $\mathbf{Hom}(\mathcal{K}^{\mathrm{op}}, \mathbf{Cat})$.

Using the notion of *category enriched graph*, which is a particular case of a $\mathcal{V}$-graph studied in detail in Section 7.2 and originates from Wolff's [**Wol74**], we can actually construct a strict functor of bicategories between $\mathcal{K}$ and a 2-category, which is a biequivalence. Hence the coherence theorem can be stated in the following more conventional way.

Theorem 2.3.4 (Coherence for Bicategories). *In a bicategory, every 2-cell diagram made up of expanded instances of α, λ, ρ and their inverses must commute.*

A more detailed description of coherence for bicategories and homomorphisms and further references can be found in [**MLP85, GPS95, Gur13**]. Also, the approach of Joyal-Street in [**JS93**] for monoidal categories can be modified to show the above result.

We now turn to composition of 2-cells in a general 2-category. Additionally to the usual vertical and horizontal composition, we consider a special case of horizontal composition which acts on a 1-cell and a 2-cell and produces a 2-cell. Explicitly, if we identify any morphism $f : A \to B$ with its identity 2-cell 1_f, we can form the composite 2-cell

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, bend left, "g", ""{name=U, below}] \arrow[r, bend right, "h"', ""{name=L, above}] \arrow[Rightarrow, from=U, to=L, "\alpha"] & C \arrow[r, "k"] & D \arrow[r, phantom, "\equiv"] & A \arrow[r, bend left, "f", ""{name=U1, below}] \arrow[r, bend right, "f"', ""{name=L1, above}] \arrow[Rightarrow, from=U1, to=L1, "1_f"] & B \arrow[r, bend left, "g", ""{name=U2, below}] \arrow[r, bend right, "h"', ""{name=L2, above}] \arrow[Rightarrow, from=U2, to=L2, "\alpha"] & C \arrow[r, bend left, "k", ""{name=U3, below}] \arrow[r, bend right, "k"', ""{name=L3, above}] \arrow[Rightarrow, from=U3, to=L3, "1_k"] & D
\end{tikzcd}$$

called *whiskering* α by f and k. It is denoted by $k\alpha f : kgf \Rightarrow khf$ and really is the horizontal composite $1_k * \alpha * 1_f$.

The various kinds of composition can be combined to give a more general operation of *pasting* (see [**Bén67, KS74, Str07**]). The two basic situations are

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "h"'] & \Downarrow\alpha & {} \arrow[dr, "g"] & \\
& {} \arrow[ur, "l"] \arrow[rr, "k"'] & \Downarrow\beta & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& {} \arrow[rr, "r"] \arrow[dr, "u"] & \Downarrow\delta & D \\
C \arrow[ur, "p"] \arrow[rr, "s"'] & \Downarrow\gamma & {} \arrow[ur, "t"'] &
\end{tikzcd}$$

For the first, we can first whisker α by g and also β by h,

$$\begin{tikzcd}
A \arrow[r, bend left, "f", ""{name=U, below}] \arrow[r, bend right, "lh"', ""{name=L, above}] \arrow[Rightarrow, from=U, to=L, "\alpha"] & {} \arrow[r, "g"] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "h"] & {} \arrow[r, bend left, "gl", ""{name=U, below}] \arrow[r, bend right, "k"', ""{name=L, above}] \arrow[Rightarrow, from=U, to=L, "\beta"] & B
\end{tikzcd}$$

in order to obtain two vertically composed 2-cells

$$\begin{tikzcd}
A \arrow[r, bend left=60, "gf"] \arrow[r, "glh" description] \arrow[r, bend right=60, "kh"'] \arrow[r, phantom, bend left=30, "\Downarrow g\alpha"] \arrow[r, phantom, bend right=30, "\Downarrow \beta h"] & B & = & A \arrow[r, bend left=40, "gf"] \arrow[r, bend right=40, "kh"'] \arrow[r, phantom, "\Downarrow \beta h\cdot g\alpha"] & B
\end{tikzcd}$$

which is called the *pasted composite* of the original diagram. Following a similar procedure, we can deduce that the pasted composite of the second diagram is the 2-cell

$$\begin{tikzcd}
C \arrow[r, bend left=40, "rp"] \arrow[r, bend right=40, "ts"'] \arrow[r, phantom, "\Downarrow t\gamma\cdot\delta p"] & D.
\end{tikzcd}$$

One can generalize the pasting operation further, in order to compute multiple composites like

$$\begin{tikzcd}
 & & & {} \arrow[r] \arrow[d, phantom, "\Downarrow"] & {} \arrow[dd] \arrow[dr] & \\
A \arrow[urrr, bend left=15] \arrow[rr] \arrow[dr] & & {} \arrow[ur] \arrow[urr, bend right=10] \arrow[u, phantom, "\Downarrow"] & & & {} \arrow[l, phantom, "\Downarrow"] \\
 & {} \arrow[ur] \arrow[rrr] \arrow[u, phantom, "\Downarrow"] & & & {} \arrow[ur] \arrow[ull, phantom, "\Downarrow"] &
\end{tikzcd}$$

It is a general fact that the result of pasting is independent of the choice of the order in which the composites are taken, *i.e.* of the way it is broken down into basic pasting operations. This is clear in simple cases, and can be proved inductively in the general case, after an appropriate formalization in terms of polygonal decompositions of the disk. A formal 2-categorical pasting theorem, showing that the operation is well-defined using Graph Theory, can be found in Power's [**Pow90**].

We finish this section with some classical notions in 2-categories and their properties, which are going to be of use later in the thesis.

DEFINITION 2.3.5. An *adjunction* in a 2-category $\mathcal{K}$ consists of 0-cells A and B, 1-cells $f : A \to B$ and $g : B \to A$ and 2-cells $\eta : 1_A \Rightarrow g \circ f$ and $\varepsilon : f \circ g \Rightarrow 1_B$ subject to the usual triangle equations:

$$\begin{tikzcd}
 & A \arrow[dr, "1_A"] \arrow[dd, bend left=20, "f"'] & & & & \\
B \arrow[ur, "g"] \arrow[dr, "1_B"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & & A \arrow[l, phantom, "\Downarrow\eta"] & = & B \arrow[r, bend left=30, "g"] \arrow[r, bend right=30, "g"'] \arrow[r, phantom, "\Downarrow 1_g"] & A, \\
 & B \arrow[ur, "g"'] & & & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & B \arrow[dr, "f"] & & & & \\
A \arrow[ur, "1_A"] \arrow[dr, "f"'] \arrow[r, phantom, "\Downarrow\eta"] & & B \arrow[l, phantom, "\Downarrow\varepsilon"] & = & A \arrow[r, bend left=30, "f"] \arrow[r, bend right=30, "f"'] \arrow[r, phantom, "\Downarrow 1_f"] & B \\
 & B \arrow[ur, "1_B"'] \arrow[uu, bend right=20, "g"] & & & &
\end{tikzcd}$$

which can be written as $(g\varepsilon)\cdot(\eta g) = 1_g$ and $(\varepsilon f)\cdot(f\eta) = 1_f$.

The standard notation for an adjunction is $f \dashv g : B \to A$. The same definition applies in case $\mathcal{K}$ is a bicategory, with the associativity and identity constraints suppressed because of coherence.

Remark. Suppose that $f \dashv g$ is an adjunction in a 2-category (or bicategory) $\mathcal{K}$ and $\mathscr{F} : \mathcal{K} \to \mathcal{L}$ is a pseudofunctor. Then $\mathscr{F}f \dashv \mathscr{F}g$ in $\mathcal{L}$, with unit

$$1_{\mathscr{F}A} \cong \mathscr{F}(1_A) \xrightarrow{\mathscr{F}\eta} \mathscr{F}(gf) \cong \mathscr{F}g \circ \mathscr{F}f$$

and counit

$$\mathscr{F}f \circ \mathscr{F}g \cong \mathscr{F}(fg) \xrightarrow{\mathscr{F}\varepsilon} \mathscr{F}(1_B) \cong 1_{\mathscr{F}B}$$

where the isomorphisms are components of the constraints γ and δ of the pseudofunctor $\mathscr{F}$. In other words, pseudofunctors preserve adjunctions.

In particular, we can apply the representable 2-functor $\mathcal{K}(X, -) : \mathcal{K} \to \mathbf{Cat}$ for any 0-cell X and obtain an adjunction in $\mathbf{Cat}$

$$\begin{tikzcd}
\mathcal{K}(X, A) \arrow[r, shift left, "f\circ\text{-}"] \arrow[r, phantom, "\perp"] & \mathcal{K}(X, B) \arrow[l, shift left, "g\circ\text{-}"]
\end{tikzcd}$$

with bijections $\phi_{h,k} : \mathcal{K}(X, B)(f \circ h, k) \cong \mathcal{K}(X, A)(h, g \circ k)$ natural in h and k. We can also apply the contravariant representable 2-functor $\mathcal{K}(-, X) : \mathcal{K}^{\mathrm{op}} \to \mathbf{Cat}$ which produces an (ordinary) adjunction $(\text{-} \circ g) \dashv (\text{-} \circ f)$. This is sometimes called the *local approach* to adjunctions, and of course by usual Yoneda lemma arguments we can reobtain the *global approach* of Definition 2.3.5.

Definition 2.3.6. Suppose that $f \dashv g : B \to A$ and $f' \dashv g' : B' \to A'$ are two adjunctions in a 2-category $\mathcal{K}$. A *map of adjunctions* from $(f \dashv g)$ to $(f' \dashv g')$ consists of a pair of 1-cells $(h : A \to A', k : B \to B')$ such that both squares

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[r, "g"] \arrow[d, "k"] & A \arrow[d, "h"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & A'
\end{tikzcd}$$

commute, and $h\eta = \eta' h$ or equivalently $k\varepsilon = \varepsilon' k$ for the units and counits of the adjunctions.

The equivalence of the two conditions becomes evident as a particular case of the mate correspondence described below.

Proposition 2.3.7. *Let $f \dashv g : A \to B$ and $f' \dashv g' : B' \to A'$ be two adjunctions in a 2-category (or bicategory) $\mathcal{K}$, and $h : A \to A'$, $k : B \to B'$ 1-cells. There is a natural bijection between 2-cells*

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow m"] & A' \arrow[d, "f'"] & \text{and} & B \arrow[r, "g"] \arrow[d, "k"'] \arrow[dr, phantom, "\swarrow \nu"] & A \arrow[d, "h"] \\
B \arrow[r, "k"'] & B' & & B' \arrow[r, "g'"'] & A'
\end{tikzcd}$$

where ν is given by the composite

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[dr, bend right, "1_B"'] \arrow[dr, phantom, "\Swarrow\varepsilon"] & A \arrow[r, "h"] \arrow[d, "f"] \arrow[dr, phantom, "\Swarrow m"] & A' \arrow[d, "f'"] \arrow[dr, bend left, "1_{A'}"] \arrow[dr, phantom, "\Swarrow\eta'"] & \\
& B \arrow[r, "k"'] & B' \arrow[r, "g'"'] & A'
\end{tikzcd}$$

and m is given by the composite

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[dr, bend right, "f"'] \arrow[dr, phantom, "\Searrow\eta"] & A \arrow[r, "h"] \arrow[dr, phantom, "\Searrow\nu"] & A' \arrow[dr, bend left, "f'"] \arrow[dr, phantom, "\Searrow\varepsilon'"] & \\
& B \arrow[u, "g"'] \arrow[r, "k"'] & B' \arrow[u, "g'"'] \arrow[r, "1_{B'}"'] & B'.
\end{tikzcd}$$

We call the 2-cells *mates* under the adjunctions $f \dashv g$ and $f' \dashv g'$. In particular, for $h = k = 1$, there is a bijection between 2-cells $\mu : f \Rightarrow f'$ and $\nu : g \Rightarrow g'$.

Using pasting operation, we can deduce that the 2-cells above are explicitly given by the composites

$$\nu : hg \xRightarrow{\eta' hg} g'f'hg \xRightarrow{g'\mu g} g'kfg \xRightarrow{g'k\varepsilon} g'k, \tag{2.16}$$

$$\mu : f'h \xRightarrow{f'h\eta} f'hgf \xRightarrow{f'\nu f} f'g'kf \xRightarrow{\varepsilon' kf} kf. \tag{2.17}$$

In Section 2.2 we studied monads and modules in bicategories. In the special case when $\mathcal{K}$ is the 2-category **2-Cat**, the monad t is usually called a *doctrine* (or 2-*monad*) and consists of a 2-functor $D : \mathcal{B} \to \mathcal{B}$ with 2-natural transformations $\eta : 1_{\mathcal{B}} \to D$, $m : D^2 \to D$ satisfying the usual axioms. A D-algebra is considered in the strict sense, although most often the 2-functor has domain **1** so it is identified with an object A in $\mathcal{B}$, as explained in Remark 2.2.4. For morphisms of D-algebras, however, the lax ones are the more usual to appear in nature.

Explicitly, for D-algebras (A, μ) and (A', μ'), a *lax morphism* (or *lax D-functor*) is a pair $(f, \bar{f})$ where $f : A \to A'$ is a morphism in $\mathcal{B}$ and $\bar{f}$ is a 2-cell

$$\begin{tikzcd}
DA \arrow[r, "\mu"] \arrow[d, "Df"'] \arrow[dr, phantom, "\bar{f}\Nearrow"] & A \arrow[d, "f"] \\
DA' \arrow[r, "\mu'"'] & A'
\end{tikzcd}$$

satisfying compatibility axioms with the multiplication and unit of D. If $\bar{f}$ is an isomorphism, then this is a *strong* morphism of D-algebras, whereas if $\bar{f}$ is the identity then we have *strict* morphism which coincides with the 'D-modules morphism' as defined in the previous section. If we reverse the direction of $\bar{f}$ and accordingly in the axioms, we have a *colax* morphism. Clearly a strong morphism of D-algebras is both lax and colax.

With appropriate notions of D-natural transformations, we can form 2-categories D-$\mathbf{Alg}_l$ with lax, D-$\mathbf{Alg}_c$ with colax, D-$\mathbf{Alg}_s$ with strong and D-$\mathbf{Alg} \equiv \mathcal{B}^D$ with strict morphisms. All the above can be found in detail in [**KS74, BKP89**], and the main results come from the so-called *doctrinal adjunction*.

THEOREM 2.3.8. *Let $f \dashv g$ be an adjunction in a 2-category $\mathcal{C}$ and let D be a 2-monad on $\mathcal{C}$. There is a bijective correspondance between 2-cells $\bar{g}$ which make $(g, \bar{g})$ into a lax D-morphism and 2-cells $\bar{f}$ which make $(f, \bar{f})$ into a colax D-morphism.*

PROPOSITION 2.3.9. *There is an adjunction $(f, \bar{f}) \dashv (g, \bar{g})$ in the 2-category D-$\mathbf{Alg}_l$ if and only if $f \dashv g$ in the 2-category $\mathcal{C}$ and $\bar{f}$ is invertible.*

The inverse of $\bar{f}$ is in fact the mate of $\bar{g}$, and both proofs rely solely on the properties of the mates correspondence. More precisely, 2-cells of the form

$$\begin{tikzcd}
DB \arrow[r, "\beta"] \arrow[d, "Dg"'] \arrow[dr, phantom, "\bar{g}\Nearrow"] & B \arrow[d, "g"] \\
DA \arrow[r, "\alpha"'] & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
DA \arrow[r, "\alpha"] \arrow[d, "Df"'] \arrow[dr, phantom, "\Swarrow\bar{f}"] & A \arrow[d, "f"] \\
DB \arrow[r, "\beta"'] & B
\end{tikzcd}$$

which are mates under the adjunctions $Df \dashv Dg$ and $f \dashv g$ are considered, and all details can be found in [**Kel74a**].

An application of these facts is going to be exhibited in the next chapter, for the 2-monad D on $\mathbf{Cat}$ which gives rise to monoidal categories.

CHAPTER 3

Monoidal Categories

This chapter presents the basic theory of monoidal categories, with particular emphasis on the categories of monoids/comonoids and modules/comodules. These structures are of central importance for our purposes, since ultimately they form a first example of the enriched fibration notion (see Chapter 6). Key references are [**JS93, Str07, Por08c**], and the monoidal category $\mathcal{V} = \mathbf{Mod}_R$ of R-modules and R-linear maps for a commutative ring R serves as a motivating illustration of our results.

A recurrent process in this treatment is the establishment of the existence of certain adjoints for various purposes, such as monoidal closed structures, free monoid and cofree comonoid constructions, enriched hom-functors etc. This also justifies the significance of locally presentable categories (see [**AR94**]) in our context, since their properties allow the application of adjoint functor theorems in a straightforward way. Below we quote some relevant, well-known results which will be employed throughout the thesis, so that we do not interrupt the main progress.

The following simple adjoint functor theorem which can be found in Max Kelly's [**Kel05**, 5.33] ensures that any cocontinuous functor with domain a locally presentable category has a right adjoint.

Theorem 3.0.1. *If the cocomplete $\mathcal{C}$ has a small dense subcategory, every cocontinuous $S : \mathcal{C} \to \mathcal{B}$ has a right adjoint.*

The standard way of determining adjunctions via representing objects is connected with the following 'Adjunctions with a parameter' theorem (see [**ML98**, Theorem IV.7.3]), which defines the important notion of a *parametrized adjunction.*

Theorem 3.0.2. *Suppose that, for a functor of two variables $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$, there exists an adjunction*

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "F(-,B)"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift left, "G(B,-)"] \end{tikzcd} \tag{3.1}$$

for each object $B \in \mathcal{B}$, with an isomorphism $\mathcal{C}(F(A,B),C) \cong \mathcal{A}(A,G(B,C))$, natural in A and C. Then, there is a unique way to assign an arrow

$$G(h,1) : G(B',C) \longrightarrow G(B,C)$$

for each $h : B \to B'$ in $\mathcal{B}$ and $C \in \mathcal{C}$, so that G becomes a functor of two variables $\mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{A}$ for which the above bijection is natural in all three variables A, B, C.

The unique choice of $G(h,-)$ to realize the above, coming from the fact that it is a *conjugate* natural transformation to $F(-,h) : F(-,B) \Rightarrow F(-,B')$, is given for example by the commutative

$$\begin{tikzcd}
G(B',-) \arrow[rr, dashed, "G(h,-)"] \arrow[d, "\eta"'] & & G(B,-) \\
G(B,F(G(B',-),B)) \arrow[rr, "G(1{,}F(G(1{,}-){,}h))"] & & G(B,F(G(B',-),B')) \arrow[u, "G(1{,}\varepsilon')"']
\end{tikzcd} \tag{3.2}$$

where η is the unit of $F(-,B) \dashv G(B,-)$ and ε' the counit of $F(-,B') \dashv G(B',-)$.

The first instance of a parametrized adjoint in this chapter is the internal hom in a monoidal category, which will play a decisive role. In [**CGR12**], more advanced ideas on multivariable adjunctions are presented.

3.1. Basic definitions

Definition. A *monoidal category* $(\mathcal{V}, \otimes, I, a, l, r)$ is a category $\mathcal{V}$ equipped with a functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ called the *tensor product*, an object I of $\mathcal{V}$ called the *unit object*, and natural isomorphisms with components

$$a_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C),$$

$$r_A : A \otimes I \xrightarrow{\sim} A, \quad l_A : I \otimes A \xrightarrow{\sim} A$$

called the *associativity* constraint, the *right unit* constraint and the *left unit* constraint respectively, subject to two coherence axioms: the following diagrams

$$\begin{tikzcd}
 & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "a_{A,B,C\otimes D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "a_{A\otimes B,C,D}"] \arrow[d, "a_{A,B,C}\otimes 1"'] & & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "a_{A,B\otimes C,D}"] & & A \otimes ((B \otimes C) \otimes D), \arrow[u, "1\otimes a_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "a_{A,I,B}"] \arrow[dr, "r_A\otimes 1"'] & & A \otimes (I \otimes B) \arrow[dl, "1\otimes l_B"] \\
 & A \otimes B &
\end{tikzcd}$$

commute.

Given a monoidal category $\mathcal{V}$, we can define a bicategory $\mathcal{K}$ with one object $\star$ by setting $\mathcal{K}(\star,\star) = \mathcal{V}$, $\circ_{\star,\star,\star} = \otimes$ and α, λ, ρ given by the constraints of the monoidal category. Conversely, any such one-object bicategory yields a monoidal category. In fact, for any object A in a bicategory $\mathcal{K}$, the hom-category $\mathcal{K}(A,A)$ is equipped with a monoidal structure induced by the horizontal composition of the bicategory:

$$\otimes : \quad \mathcal{K}(A,A) \times \mathcal{K}(A,A) \longrightarrow \mathcal{K}(A,A) \tag{3.3}$$

$$(A \xrightarrow{g} A,\ A \xrightarrow{f} A) \longmapsto A \xrightarrow{g\otimes f := g\circ f} A$$

The unit object is the identity 1-cell $I = 1_A$ and the associativity and left/right unit constraints come from the associator and the left/right unitors of the bicategory $\mathcal{K}$. The coherence axioms follow in a straightforward way from those of a bicategory.

Due to this correspondence, various results of the previous chapter are of relevance to the theory of monoidal categories. In particular, coherence for bicategories (Theorems 2.3.3 and 2.3.4) ensures that monoidal categories are also 'coherent'. The coherence theorem for monoidal categories first appeared in Mac Lane's [**ML63**]. A formulation of it states that every diagram which consists of arrows obtained by repeated applications of the functor $\otimes$ to instances of a, r, l and their inverses (the so-called 'expanded instances') and 1 commutes. This essentially allows one to work as if a, r, l are all identities. This is derived from the fact that any monoidal category is monoidally equivalent (via a strict monoidal functor) to a *strict* monoidal category, where a, r, l are identities.

Notice that if $\mathcal{V}$ is a monoidal category, then its opposite category $\mathcal{V}^{\mathrm{op}}$ is also monoidal with the same tensor product $\otimes^{\mathrm{op}}$. Some authors call 'opposite monoidal category' the *reverse* category $\mathcal{V}^{\mathrm{rev}}$, which is $\mathcal{V}$ with $A \otimes^{\mathrm{rev}} B = B \otimes A$, $a^{\mathrm{rev}} = a^{-1}$, $l^{\mathrm{rev}} = l$ and $r^{\mathrm{rev}} = r$.

A *braiding* c for a monoidal category $\mathcal{V}$ is a natural isomorphism

$$\begin{tikzcd}
\mathcal{V} \times \mathcal{V} \arrow[r, "\otimes"] \arrow[d, "\mathrm{sw}"'] & \mathcal{V} \\
\mathcal{V} \times \mathcal{V} \arrow[ur, bend right, "\otimes"'] \arrow[ur, phantom, "\Swarrow c"] &
\end{tikzcd}$$

with components invertible arrows $c_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$ for all $A, B \in \mathcal{V}$, where *sw* switches the entries of the pair. These isomorphisms satisfy the coherence axioms expressed by the commutativity of

$$\begin{tikzcd}
& A \otimes (B \otimes C) \arrow[r, "c_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[dr, "a_{B,C,A}"] & \\
(A \otimes B) \otimes C \arrow[ur, "a_{A,B,C}"] \arrow[dr, "c_{A,B}\otimes 1"'] & & & B \otimes (C \otimes A) \\
& (B \otimes A) \otimes C \arrow[r, "a_{B,A,C}"'] & B \otimes (A \otimes C), \arrow[ur, "1\otimes c_{A,C}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (A \otimes B) \otimes C \arrow[r, "c_{A\otimes B,C}"] & C \otimes (A \otimes B) \arrow[dr, "a^{-1}_{C,A,B}"] & \\
A \otimes (B \otimes C) \arrow[ur, "a^{-1}_{A,B,C}"] \arrow[dr, "1\otimes c_{B,C}"'] & & & (C \otimes A) \otimes B \\
& A \otimes (C \otimes B) \arrow[r, "a^{-1}_{A,C,B}"'] & (A \otimes C) \otimes B. \arrow[ur, "c_{A,C}\otimes 1"'] &
\end{tikzcd}$$

A *braided monoidal category* is a monoidal category with a chosen braiding. A *symmetry* s for a monoidal category $\mathcal{V}$ is a braiding s with components

$$s_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$$

which also satisfies the commutativity of

$$\begin{tikzcd}
A\otimes B \arrow[rr, "="] \arrow[dr, "s_{A.B}"'] & & A\otimes B \\
& B\otimes A, \arrow[ur, "s_{B,A}"'] &
\end{tikzcd}$$

which expresses that $s^{-1}_{A,B} = s_{B,A}$. Because of this, only the one hexagon from the definition of the braiding is needed to define a symmetry.

A monoidal category with a chosen symmetry is called *symmetric*. Coherence theorems for braided and symmetric monoidal categories again state that any (braided) symmetric monoidal category is (braided) symmetric monoidally equivalent to a *strict* (braided) symmetric monoidal category, see [**JS93**].

Examples. (1) A special collection of examples called *cartesian monoidal categories* is given by considering any category with finite products, taking $\otimes = \times$ and $I = 1$ the terminal object. The constraints a, l, r are the canonical isomorphisms induced by the universal property of products. Important particular cases of this are the categories **Set** of (small) sets, **Cat** of categories, **Gpd** of groupoids, **Top** of topological spaces etc. All these examples are in fact symmetric monoidal categories.

(2) The category **Ab** of abelian groups and group homomorphisms is a symmetric monoidal category with the usual tensor product $\otimes$ of abelian groups and the additive group of integers $\mathbb{Z}$ as unit object. The associativity and unit constraints come from the respective canonical isomorphisms for the tensor of abelian groups. Notice that there is also a different symmetric monoidal structure on the cocomplete **Ab**, namely $(\mathbf{Ab}, \oplus, 0)$ where $\oplus$ is the direct product.

(3) The category $\mathbf{Mod}_R$ of modules over a commutative ring R and R-module homomorphisms is a symmetric monoidal category with tensor the usual tensor product $\otimes_R$ of R-modules. The unit object is the ring R and the associativity and unit constraints are the canonical ones. The symmetry s has components the canonical isomorphisms $A \otimes_R B \cong B \otimes_R A$. Clearly the category of k-vector spaces and k-linear maps $\mathbf{Vect}_k$ for a field k is again a symmetric monoidal category.

(4) For any bicategory $\mathcal{K}$, the hom-categories $(\mathcal{K}(A,A), \circ, 1_A)$ for any 0-cell A are monoidal categories as explained earlier, but not necessarily symmetric. As a special case for $\mathcal{K} = \mathbf{Cat}$, the category $\mathrm{End}(\mathcal{C})$ of endofunctors on a category $\mathcal{C}$ is a monoidal category with composition as the tensor product and $1_{\mathcal{C}}$ as the unit.

Definition. If $\mathcal{V}$ and $\mathcal{W}$ are monoidal categories, a *lax monoidal functor* between them consists of a functor $F : \mathcal{V} \to \mathcal{W}$ together with natural transformations

$$\begin{tikzcd}
\mathcal{V}\times\mathcal{V} \arrow[r, "F\times F"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\Swarrow\phi"] & \mathcal{W}\times\mathcal{W} \arrow[d, "\otimes"] \\
\mathcal{V} \arrow[r, "F"'] & \mathcal{W}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbf{1} \arrow[r, "I_{\mathcal{V}}"] \arrow[dr, bend right, "I_{\mathcal{W}}"'] \arrow[dr, phantom, "\Swarrow\phi_0"] & \mathcal{V} \arrow[d, "F"] \\
& \mathcal{W}
\end{tikzcd}
\qquad (3.4)$$

with components

$$\phi_{A,B} : FA \otimes FB \to F(A \otimes B)$$
$$\phi_0 : I \to FI$$

satisfying the associativity and unitality axioms: the diagrams

$$\begin{tikzcd}
FA \otimes FB \otimes FC \arrow[r, "\phi_{A,B}\otimes 1"] \arrow[d, "1\otimes\phi_{B,C}"'] & F(A \otimes B) \otimes FC \arrow[d, "\phi_{A\otimes B,C}"] \\
FA \otimes F(B \otimes C) \arrow[r, "\phi_{A,B\otimes C}"'] & F(A \otimes B \otimes C),
\end{tikzcd} \tag{3.5}$$

$$\begin{tikzcd}
FA \arrow[r, "1\otimes\phi_0"] \arrow[d, "\phi_0\otimes 1"'] \arrow[dr, dashed, "1"] & FA \otimes FI \arrow[d, "\phi_{A,I}"] \\
FI \otimes FA \arrow[r, "\phi_{I,A}"'] & FA
\end{tikzcd}$$

commute, where the constraints α, l, r have been suppressed.

In the case where $\phi_{A,B}, \phi_0$ are isomorphisms, the functor F is called *(strong) monoidal*, whereas if they are identities F is called *strict monoidal.* Dually, F is a *colax monoidal functor* when it is equipped with with natural families in the opposite direction, $\psi_{A,B} : F(A \otimes B) \to FA \otimes FB$ and $\psi_0 : FI \to I$. Notice how these definitions follow from Definition 2.1.3 for the one-object bicategory case.

A functor $F : \mathcal{V} \to \mathcal{W}$ between braided monoidal categories $\mathcal{V}$ and $\mathcal{W}$ is *braided monoidal* if it is monoidal and also makes the diagram

$$\begin{tikzcd}
FA \otimes FB \arrow[r, "c_{FA,FB}"] \arrow[d, "\phi_{A,B}"'] & FB \otimes FA \arrow[d, "\phi_{B,A}"] \\
F(A \otimes B) \arrow[r, "F(c_{A,B})"'] & F(B \otimes A)
\end{tikzcd}$$

commute, for all $A, B \in \mathcal{V}$. If $\mathcal{V}$ and $\mathcal{W}$ are symmetric, then F is a *symmetric monoidal functor* with no extra conditions.

DEFINITION. If $F, G : \mathcal{V} \to \mathcal{W}$ are lax monoidal functors, a *monoidal natural transformation* $\tau : F \Rightarrow G$ is an (ordinary) natural transformation such that the following two diagrams commute:

$$\begin{tikzcd}
FA \otimes FB \arrow[r, "\phi_{A,B}"] \arrow[d, "\tau_A\otimes\tau_B"'] & F(A \otimes B) \arrow[d, "\tau_{A\otimes B}"] \\
GA \otimes GB \arrow[r, "\phi'_{A,B}"'] & G(A \otimes B),
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\phi_0"] \arrow[dr, "\phi'_0"'] & FI \arrow[d, "\sigma_I"] \\
& GI.
\end{tikzcd} \tag{3.6}$$

A *braided* or *symmetric* monoidal natural transformation is just a monoidal natural transformation between braided or symmetric monoidal functors.

It is not hard to verify that the different kinds of monoidal functors compose. Depending on the monoidal structure that the functors are equipped with, we have the 2-categories $\mathbf{MonCat}_s$, $\mathbf{MonCat}$, $\mathbf{MonCat}_l$ and $\mathbf{MonCat}_c$ of monoidal categories, strict/strong/lax/colax monoidal functors and monoidal natural transformations. If the functors are moreover braided or symmetric, we have different versions of 2-categories **BrMonCat** and **SymmMonCat**.

Remark 3.1.1. The category **MonCat** is itself a cartesian monoidal category. For $\mathcal{V}$, $\mathcal{W}$ two monoidal categories, their product $\mathcal{V}\times\mathcal{W}$ has the structure of a monoidal category with tensor product the composite

$$\begin{tikzcd}
\mathcal{V}\times\mathcal{W}\times\mathcal{V}\times\mathcal{W} \arrow[rr, dashed, "\otimes_{(\mathcal{V}\times\mathcal{W})}"] \arrow[d, "1\times\mathrm{sw}\times 1"'] & & \mathcal{V}\times\mathcal{W} \\
\mathcal{V}\times\mathcal{V}\times\mathcal{W}\times\mathcal{W} \arrow[urr, bend right, "\otimes_{\mathcal{V}}\times\otimes_{\mathcal{W}}"'] & &
\end{tikzcd}$$

and unit the pair $(I_{\mathcal{V}}, I_{\mathcal{W}})$. On objects, the above operation explicitly gives

$$((A,B),(A',B'))\mapsto(A\otimes A', B\otimes B').$$

Similarly $F\times G$ is a monoidal functor when F and G are. The terminal category $\mathbf{1}$ is the unit monoidal category, hence $(\mathbf{MonCat},\times,\mathbf{1})$ is in fact a monoidal category.

Definition. The monoidal category $\mathcal{V}$ is said to be *(left) closed* when, for each $A\in\mathcal{V}$, the functor $-\otimes A:\mathcal{V}\to\mathcal{V}$ has a right adjoint $[A,-]:\mathcal{V}\to\mathcal{V}$ with a bijection

$$\mathcal{V}(C\otimes A,B)\cong\mathcal{V}(C,[A,B]). \tag{3.7}$$

natural in C and B. We call $[A,B]$ the *(left) internal hom* of A and B.

If also every $A\otimes -$ has a right adjoint $[A,-]'$, we say that the monoidal category $\mathcal{V}$ is *right closed.* When $\mathcal{V}$ is a braided monoidal category, each left internal hom gives a right internal hom $[A,B]=[A,B]'$. A monoidal category is called *closed* (or biclosed) when it is left and right closed.

For example, the symmetric monoidal category $\mathbf{Mod}_R$ is a monoidal closed category, by the well-known adjunction

$$\begin{tikzcd}
\mathbf{Mod}_R \arrow[r, shift left, "-\otimes_R M"] \arrow[r, phantom, "\perp"] & \mathbf{Mod}_R \arrow[l, shift left, "\mathrm{Hom}_R(M,-)"]
\end{tikzcd}$$

where Hom_R is the linear hom functor.

By ‘adjunctions with a parameter’ theorem 3.0.2, the definition of the internal hom for a monoidal closed category $\mathcal{V}$ implies that there is a unique way of making it into a functor of two variables

$$[-,-]:\mathcal{V}^{\mathrm{op}}\times\mathcal{V}\longrightarrow\mathcal{V}$$

such that the bijection (3.7) is natural in all three variables. Explicitly, if $f:C\to A$ and $g:B\to D$ are arrows of $\mathcal{V}$, there is a unique arrow $[f,g]:[A,B]\to[C,D]$ such that the diagram

$$\begin{tikzcd}
{[A,B]\otimes C} \arrow[rr, "{[f,g]\otimes 1}"] \arrow[d, "1\otimes f"'] & & {[C,D]\otimes C} \arrow[d, "\mathrm{ev}^C_D"] \\
{[A,B]\otimes A} \arrow[r, "\mathrm{ev}^A_B"'] & B \arrow[r, "g"'] & D
\end{tikzcd}$$

commutes, where ev^A is the counit of the adjunction $-\otimes A \dashv [A,-]$ usually called the *evaluation*. In other words, the internal hom bifunctor $[-,-]$ is the *parametrized adjoint* of the tensor bifunctor $(-\otimes -)$.

Notice that in any parametrized adjunction as in (3.1) with natural isomorphisms $\mathcal{C}((F(A,B),C)\cong\mathcal{A}(A,G(B,C))$, the counit is a collection of components

$$\varepsilon^B_A : F(G(B,A),B)\longrightarrow A$$

which is natural in A and also *dinatural* or *extranatural* in B. This is expressed by the commutativity of

$$\begin{tikzcd}
F(G(B',A),B) \arrow[r, "{F(1,f)}"] \arrow[d, "{F(G(f,1),1)}"'] & F(G(B',A),B') \arrow[d, "\varepsilon^{B'}_A"] \\
F(G(B,A),B) \arrow[r, "\varepsilon^B_A"'] & A
\end{tikzcd} \tag{3.8}$$

for any arrow $f : B\to B'$. Dinaturality is discussed in detail in [**ML98**, IX.4].

Finally, in any symmetric monoidal closed category $\mathcal{V}$ we also have an adjunction

$$\begin{tikzcd}
\mathcal{V} \arrow[r, shift left, "{[-,A]^{\mathrm{op}}}"] \arrow[r, phantom, "\perp"] & \mathcal{V}^{\mathrm{op}} \arrow[l, shift left, "{[-,A]}"]
\end{tikzcd} \tag{3.9}$$

with a natural isomorphism $\mathcal{V}^{\mathrm{op}}([V,A],W)\cong\mathcal{V}(V,[W,A])$, explicitly given by the following bijective correspondences:

$$\begin{array}{cl}
W \longrightarrow [V,A] & \text{in } \mathcal{V} \\
\hline
\begin{tikzcd}[row sep=small] W\otimes V \arrow[r] \arrow[d, phantom, "\wr\!\shortparallel"] & A \\ V\otimes W \arrow[ur] & \end{tikzcd} & \text{in } \mathcal{V} \\
\hline
V \longrightarrow [W,A] & \text{in } \mathcal{V}.
\end{array}$$

3.2. Doctrinal adjunction for monoidal categories

As mentioned briefly at the end of Section 2.3, monoidal categories are (strict) algebras for a specific 2-monad D on **Cat**, which arise from *clubs.* Details of these facts and structures can be found in [**Kel72, Kel74a, Kel74b, Web04**]. In this context, lax morphisms of D-algebras turn out to be lax monoidal functors and D-natural transformations are monoidal natural transformations. Therefore, by doctrinal adjunction we can see how lax and colax monoidal structures on adjoint functors between monoidal categories relate to each other.

Depending on which 2-category of monoidal categories we are working in, Definition 2.3.5 gives us different notions of *monoidal adjunctions*. For example, an adjunction in the 2-category $\mathbf{MonCat}_l$ is an adjunction between monoidal categories

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left=1.5, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left=1.5, "G"] \end{tikzcd}$$

where F and G are lax monoidal functors and the unit and the counit are monoidal natural transformations.

Now, suppose that $F \dashv G$ is an ordinary adjunction between two monoidal categories $\mathcal{C}$ and $\mathcal{D}$, where the left adjoint F has the structure of a colax monoidal functor, *i.e.* it is equipped with 2-cells ψ, ψ_0 in the opposite direction of (3.4). Consider the diagram

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r, shift left=1.5, "F\times F"] \arrow[r, phantom, "\perp"] \arrow[d, "\otimes"'] & \mathcal{D} \times \mathcal{D} \arrow[l, shift left=1.5, "G\times G"] \arrow[d, "\otimes"] \\ \mathcal{C} \arrow[r, shift left=1.5, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left=1.5, "G"] \end{tikzcd}$$

which illustrates two adjunctions and two functors between the categories involved. Then, by Proposition 2.3.7 which gives the mate correspondance, the 2-cell ψ corresponds uniquely to a 2-cell ϕ via

$$\begin{tikzcd} & \mathcal{D} \times \mathcal{D} \arrow[dl, bend right, "G\times G"'] \arrow[d, "1"] \arrow[dl, phantom, shift left=3, "\overset{\varepsilon\times\varepsilon}{\Rightarrow}"] \\ \mathcal{C} \times \mathcal{C} \arrow[r, "F\times F"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\overset{\psi}{\Rightarrow}"] & \mathcal{D} \times \mathcal{D} \arrow[d, "\otimes"] \\ \mathcal{C} \arrow[r, "F"] \arrow[d, "1"'] & \mathcal{D} \arrow[dl, bend left, "G"] \arrow[dl, phantom, shift right=3, "\overset{\eta}{\Rightarrow}"] \\ \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{D} \times \mathcal{D} \arrow[r, "\otimes"] \arrow[d, "G\times G"'] \arrow[dr, phantom, "\phi \Nearrow"] & \mathcal{D} \arrow[d, "G"] \\ \mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"] & \mathcal{C}. \end{tikzcd} \tag{3.10}$$

In terms of components via pasting, $\phi_{A,B}$ is expressed as the composite

$$GA \otimes GB \xrightarrow{\eta_{GA\otimes GB}} GF(GA \otimes GB) \xrightarrow{G\psi_{GA,GB}} G(FGA \otimes FGB) \xrightarrow{G(\varepsilon_A \otimes \varepsilon_B)} G(A \otimes B).$$

Similarly, the 2-cell ψ_0 corresponds uniquely to a 2-cell ϕ_0 via

$$\begin{tikzcd} \mathbf{1} \arrow[r, "1"] \arrow[d, "I_{\mathcal{C}}"'] \arrow[dr, phantom, "\overset{\psi_0}{\Rightarrow}"] & \mathbf{1} \arrow[d, "I_{\mathcal{D}}"] \\ \mathcal{C} \arrow[r, "F"] \arrow[d, "1"'] & \mathcal{D} \arrow[dl, bend left, "G"] \arrow[dl, phantom, shift right=3, "\overset{\eta}{\Rightarrow}"] \\ \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathbf{1} \arrow[r, "I_{\mathcal{D}}"] \arrow[dd, "I_{\mathcal{C}}"'] & \mathcal{D} \arrow[ddl, bend left, "G"] \\ \arrow[r, phantom, "\overset{\phi_0}{\Rightarrow}"] & \\ \mathcal{C} \end{tikzcd} \tag{3.11}$$

and in terms of components, ϕ_0 is the composite

$$I \xrightarrow{\eta_I} GFI \xrightarrow{G\psi_0} GI.$$

Moreover, the arrows $\phi_{A,B}$ and ϕ_0 turn out to satisfy the axioms (3.5) thus they constitute a *lax monoidal structure* for the right adjoint G.

On the other hand, if we start with a lax monoidal structure (ϕ, ϕ_0) on G, again due to the bijective correspondance of mates we end up with a colax structure (ψ, ψ_0) on the left adjoint F, given by the composites

$$\begin{tikzcd}
F(A\otimes B) \arrow[r, "F(\eta\otimes\eta)"] \arrow[rrd, dashed, "\psi_{A,B}"'] & F(GFA\otimes GFB) \arrow[r, "F\phi"] & FG(FA\otimes FB) \arrow[d, "\varepsilon"] \\
 & & FA\otimes FB,
\end{tikzcd}
\qquad
\begin{tikzcd}
FI \arrow[r, "F\phi_0"] \arrow[rd, dashed, "\psi_0"'] & FGI \arrow[d, "\varepsilon"] \\
 & I.
\end{tikzcd}
\tag{3.12}$$

The above establish the following result.

Proposition 3.2.1. *Suppose we have two (ordinary) adjoint functors $F \dashv G$ between monoidal categories. Then, colax monoidal structures on the left adjoint F correspond bijectively, via mates, to lax monoidal structures on the right adjoint G.*

Of course this is a special case of Theorem 2.3.8 for $\mathcal{K} = \mathbf{Cat}$ and D the 2-monad whose algebras are monoidal categories. Proposition 2.3.9 also applies.

Proposition 3.2.2. *A functor F equipped with a lax monoidal structure has a right adjoint in $\mathbf{MonCat}_l$ if and only if F has a right adjoint in $\mathbf{Cat}$ and its lax monoidal structure is a strong monoidal structure.*

Proof. '⇒' Suppose $F \dashv G$ is an adjunction in $\mathbf{MonCat}_l$ and (ϕ, ϕ_0), (ϕ', ϕ'_0) are the lax structure maps of F and G. By the above corollary, the lax monoidal structure of the right adjoint G it induces a colax structure (ψ, ψ_0) on the left adjoint F, given by the composites (3.12).

In order for F to be a strong monoidal functor, it is enough to show that this colax structure induced from G is the two-sided inverse to the lax structure of F.

- $\psi_{A,B} \circ \phi_{A,B} = 1_{FA\otimes FB}$:

$$\begin{tikzcd}
FA\otimes FB \arrow[r, "\phi_{A,B}"] \arrow[rd, dotted, "F\eta_A\otimes F\eta_B"'] \arrow[rrrdd, bend right, "1_{FA\otimes FB}"'] & F(A\otimes B) \arrow[rr, "F(\eta_A\otimes\eta_B)"] \arrow[d, phantom, "(i)"] & & F(GFA\otimes GFB) \arrow[d, "F\phi'_{FA,FB}"] \\
 & FGFA\otimes FGFB \arrow[rru, dotted, "\phi_{GFA,GFB}"'] \arrow[rrd, dotted, "\varepsilon_{FA}\otimes\varepsilon_{FB}"'] \arrow[d, phantom, "(iii)"] \arrow[r, phantom, "(ii)"] & & FG(FA\otimes FB) \arrow[d, "\varepsilon_{FA\otimes FB}"] \\
 & {} & & FA\otimes FB
\end{tikzcd}$$

where (i) commutes by naturality of ϕ, (ii) by the fact that $\varepsilon : FG \Rightarrow 1_{\mathcal{D}}$ is a monoidal natural transformation between lax monoidal functors, and (iii) by one of the triangular identities.

- $\psi_0 \circ \phi_0 = 1_I$:

$$\begin{tikzcd}
I \arrow[r, "\phi_0"] \arrow[rrd, "1_I"'] & FI \arrow[r, "F\phi'_0"] & FGI \arrow[d, "\varepsilon_I"] \\
 & & I
\end{tikzcd}$$

which commutes by the axioms (3.6) for the monoidal counit ε of the adjunction.

By forming similar diagrams we can see how $\phi_{A,B} \circ \psi_{A,B} = 1_{F(A \otimes B)}$ and $\psi_0 \circ \phi_0 = id_I$, hence F is equipped with a strong monoidal structure.

'$\Leftarrow$' Suppose that F has the structure of a strong monoidal functor (ϕ, ϕ_0) and it has an ordinary right adjoint G. Clearly F has a lax monoidal structure and a colax monoidal structure (ϕ^{-1}, ϕ_0^{-1}). Therefore it induces a lax monoidal structure on the right adjoint G given by the composites (3.10), (3.11).

What is left to show is that the unit η and the counit ε of the adjunction are monoidal natural transformations, *i.e.* they satisfy the commutativity of the diagrams (3.6). For example, the first diagram for $\eta : 1_{\mathcal{C}} \Rightarrow GF$ becomes

$$\begin{tikzcd}
A \otimes B \arrow[rr, "\eta_{A \otimes B}"] \arrow[d, "\eta_A \otimes \eta_B"'] & {} \arrow[d, phantom, "(i)"] & GF(A \otimes B) \arrow[ddll, dotted, "GF(\eta_A \otimes \eta_B)"] & \\
GFA \otimes GFB \arrow[d, "\eta_{GFA \otimes GFB}"'] & {} & (ii) & G(FA \otimes FB) \arrow[ul, "G\phi_{A,B}"'] \arrow[dll, dotted, "G(F\eta_A \otimes F\eta_B)"'] \\
GF(GFA \otimes GFB) \arrow[r, shift right, "G\phi^{-1}_{GFA,GFB}"'] & G(FGFA \otimes FGFB) \arrow[l, dotted, shift right, "G\phi_{GFA,GFB}"'] \arrow[urr, "G(\varepsilon_{FA} \otimes \varepsilon_{FB})"'] & &
\end{tikzcd}$$

where (i) commutes by naturality of η, and (ii) by naturality of ϕ and one of the triangular identities. Notice that the lower composite from $GFA \otimes GFB$ to $GF(A \otimes B)$ is the lax structure map $\phi''_{A,B}$ of the composite lax monoidal functor GF.

The second diagram commutes trivially, and in a very similar way we can show that ε is also a monoidal natural transformation. Hence, the adjunction can be lifted in $\mathbf{MonCat}_l$. $\square$

The above propositions generalize to the case of parametrized adjoints. For example, if the functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ between monoidal categories has a colax structure

$$\psi_{(A,B),(A',B')} : F(A \otimes A', B \otimes B') \to F(A,B) \otimes F(A',B')$$
$$\psi_0 : F(I_{\mathcal{A}}, I_{\mathcal{B}}) \to I_{\mathcal{C}},$$

then its parametrized adjoint $G : \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{A}$ obtains a lax structure via the composites

$$\begin{tikzcd}
G(B,C) \otimes G(B',C') \arrow[r, "\eta^{B \otimes B'}_{G(B,C) \otimes G(B',C')}"] \arrow[ddr, dashed, bend right, "\phi_{(B,C),(B',C')}"'] & G(B \otimes B', F(G(B,C) \otimes G(B',C'), B \otimes B')) \arrow[d, "{G(1,\psi_{(G(B,C),B),(G(B',C'),B')})}"] \\
& G(B \otimes B', F(G(B,C),B) \otimes F(G(B',C'),B')) \arrow[d, "{G(1,\varepsilon^B_C \otimes \varepsilon^{B'}_{C'})}"] \\
& G(B \otimes B', C \otimes C'),
\end{tikzcd}$$

$$\begin{tikzcd}
I_{\mathcal{A}} \arrow[r, "\eta_{I_{\mathcal{A}}}^{I_{\mathcal{B}}}"] \arrow[dr, dashed, "\phi_0"'] & G(I_{\mathcal{B}}, F(I_{\mathcal{A}}, I_{\mathcal{B}})) \arrow[d, "G(1,\psi_0)"] \\
& G(I_{\mathcal{B}}, I_{\mathcal{C}}).
\end{tikzcd}$$

The respective axioms are satisfied by naturality and dinaturality of the unit and counit η, ε of the parametrized adjunction and the axioms for (ψ, ψ_0) of F.

Proposition 3.2.3. *Suppose $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{A}$ are parametrized adjoints between monoidal categories,* i.e. *$F(-, B) \dashv G(B, -)$ for all $B \in \mathcal{B}$. Then, colax monoidal structures on F correspond bijectively to lax monoidal structures on G.*

As an application, consider the case of a symmetric monoidal closed category $\mathcal{V}$, with symmetry s. The tensor product functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ from the monoidal $\mathcal{V} \times \mathcal{V}$ (see Remark 3.1.1) is equipped with a strong monoidal structure, namely

$$\phi_{(A,B),(A',B')} : A \otimes B \otimes A' \otimes B' \xrightarrow{1 \otimes s_{B,A'} \otimes 1} A \otimes A' \otimes B \otimes B',$$

$$\phi_0 : I \xrightarrow{r_I^{-1}} I \otimes I.$$

Therefore Proposition 3.2.3 applies and its parametrized adjoint obtains the structure of a lax monoidal functor.

Proposition 3.2.4. *In a symmetric monoidal closed category $\mathcal{V}$, the internal hom functor $[-,-] : \mathcal{V}^{\mathrm{op}} \otimes \mathcal{V} \to \mathcal{V}$ has the structure of a lax monoidal functor, with structure maps*

$$\chi_{(A,B),(A',B')} : [A, B] \otimes [A', B'] \to [A \otimes A', B \otimes B'],$$

$$\chi_0 : I \to [I, I]$$

which correspond, under the adjunction $- \otimes A \dashv [A, -]$, to the morphisms

$$[A, B] \otimes [A', B'] \otimes A \otimes A' \xrightarrow{1 \otimes s \otimes 1} [A, B] \otimes A \otimes [A', B'] \otimes A' \xrightarrow{\mathrm{ev} \otimes \mathrm{ev}} B \otimes B',$$

$$I \otimes I \xrightarrow{l_I = r_I} I.$$

3.3. Categories of monoids and comonoids

A *monoid* in a monoidal category $\mathcal{V}$ is an object A equipped with arrows

$$m : A \otimes A \to A \qquad \text{and} \qquad \eta : I \to A$$

called the *multiplication* and the *unit*, satisfying the associativity and identity conditions: the diagrams

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "1 \otimes m"] \arrow[d, "m \otimes 1"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"'] & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
I \otimes A \arrow[r, "\eta \otimes 1"] \arrow[dr, "l_A"'] & A \otimes A \arrow[d, "m"] & A \otimes I \arrow[l, "1 \otimes \eta"'] \arrow[dl, "r_A"] \\
& A &
\end{tikzcd} \tag{3.13}$$

commute, where the associativity constraint is suppressed from the first diagram. A *monoid morphism* between two monoids (A, m, η) and (A', m', η') is an arrow $f : A \to A'$ in $\mathcal{V}$ such that the diagrams

$$\begin{tikzcd}
A \otimes A \arrow[r, "m"] \arrow[d, "f\otimes f"'] & A \arrow[d, "f"] \\
A' \otimes A' \arrow[r, "m'"'] & A'
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
I \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & A \arrow[d, "f"] \\
& A'
\end{tikzcd} \tag{3.14}$$

commute. We obtain a category $\mathbf{Mon}(\mathcal{V})$ of monoids and monoid morphisms. Furthermore, a 2-cell $\alpha : f \Rightarrow g$ is defined to be an arrow $\alpha : I \to B$ such that

$$\begin{tikzcd}
A \arrow[r, "\alpha\otimes f"] \arrow[d, "g\otimes\alpha"'] & B \otimes B \arrow[d, "m"] \\
B \otimes B \arrow[r, "m"'] & B
\end{tikzcd} \tag{3.15}$$

commutes, thus $\mathbf{Mon}(\mathcal{V})$ is a 2-category.

Dually, there is a 2-category of *comonoids* $\mathbf{Comon}(\mathcal{V})$ with objects triples (C, Δ, ϵ) where C is an object in $\mathcal{V}$, $\Delta : C \to C \otimes C$ is the *comultiplication* and $\epsilon : C \to I$ is the *counit*, such that dual diagrams to (3.13) commute. *Comonoid morphisms* $(C, \Delta, \epsilon) \to (C', \Delta', \epsilon')$ are arrows $g : C \to C'$ in $\mathcal{V}$ such that the dual of (3.14) commutes, and 2-cells $\beta : f \Rightarrow g$ are arrows $\beta : C \to I$ satisfying dual diagrams to (3.15).

For the purposes of this dissertation, the 2-dimensional structure of the categories of monoids and comonoids (and modules and comodules later) will not be employed. Notice that as categories, $\mathbf{Comon}(\mathcal{V}) = \mathbf{Mon}(\mathcal{V}^{\mathrm{op}})^{\mathrm{op}}$.

Remark 3.3.1. We saw in Section 3.1 how, for any object B in a bicategory $\mathcal{K}$, the hom-category $\mathcal{K}(B, B)$ obtains the structure of a monoidal category, with tensor product the horizontal composition and unit the identity 1-cell. From this viewpoint, the data that define the notion of a monad $t : B \to B$ in a bicategory (Definition 2.2.1) equivalently define a monoid in the monoidal category $(\mathcal{K}(B, B), \circ, 1_B)$. Dually, a comonad $u : A \to A$ in a bicategory $\mathcal{K}$ as in Definition 2.2.5 is precisely a comonoid in the monoidal $\mathcal{K}(A, A)$.

If the monoidal category $\mathcal{V}$ is braided, we can define a monoid structure on the tensor product $A \otimes B$ of two monoids A, B via

$$A \otimes B \otimes A \otimes B \xrightarrow{1\otimes c\otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{m\otimes m} A \otimes B$$

$$I \xrightarrow{r_I^{-1}} I \otimes I \xrightarrow{\eta\otimes\eta} A \otimes B$$

where the constraints are again suppressed. This induces a monoidal structure on the category $\mathbf{Mon}(\mathcal{V})$, such that the forgetful functor to $\mathcal{V}$ is a strict monoidal functor. The braiding/symmetry of $\mathcal{V}$ lifts to its category of monoids, so $\mathbf{Mon}(\mathcal{V})$ is a braided/symmetric monoidal category when $\mathcal{V}$ is. This happens because $\mathbf{Mon}(\mathcal{V}) \to$

$\mathcal{V}$ always reflects isomorphisms. Dually, $\mathbf{Comon}(\mathcal{V})$ also inherits the monoidal structure from $\mathcal{V}$, via

$$C \otimes D \xrightarrow{\delta\otimes\delta} C \otimes C \otimes D \otimes D \cong C \otimes D \otimes C \otimes D, \quad C \otimes D \xrightarrow{\epsilon\otimes\epsilon} I \otimes I \cong I.$$

The monoidal unit in both cases is I, with trivial monoid and comonoid structure via r_I.

For example, the category of monoids in the symmetric monoidal category $(\mathbf{Ab}, \otimes, \mathbb{Z})$ is the category of rings $\mathbf{Rng}$, and in the symmetric cartesian monoidal category $(\mathbf{Cat}, \times, \mathbf{1})$ it is the category of strict monoidal categories $\mathbf{MonCat}_{st}$. Also, the category of monoids in the symmetric monoidal category $\mathbf{Mod}_R$ for a commutative ring R is the category of R-algebras $\mathbf{Alg}_R$ and the category of comonoids is the category of R-coalgebras $\mathbf{Coalg}_R$.

An important property of lax monoidal functors is that they map monoids to monoids. More precisely, if $F : \mathcal{V} \to \mathcal{W}$ is a lax monoidal functor between monoidal categories $\mathcal{V}$ and $\mathcal{W}$, there is an induced functor

$$\begin{aligned} \mathbf{Mon}(F):\ \mathbf{Mon}(\mathcal{V}) &\longrightarrow \mathbf{Mon}(\mathcal{W}) \\ (A, m, \eta) &\longmapsto (FA, m', \eta') \end{aligned} \tag{3.16}$$

which gives FA the structure of a monoid in $\mathcal{W}$, with multiplication and unit

$$m' : FA \otimes FA \xrightarrow{\phi_{A,A}} F(A \otimes A) \xrightarrow{Fm} FA$$
$$\eta' : I \xrightarrow{\phi_0} FI \xrightarrow{F\eta} FA$$

where $\phi_{A,A}$ and ϕ_0 are the structure maps of F. The associativity and identity conditions are satisfied because of naturality of ϕ, ϕ_0 and the fact that A is a monoid. Dually, if $G : \mathcal{V} \to \mathcal{W}$ is colax monoidal functor, it maps comonoids to comonoids via an induced functor

$$\begin{aligned} \mathbf{Comon}(F):\ \mathbf{Comon}(\mathcal{V}) &\longrightarrow \mathbf{Comon}(\mathcal{W}) \\ (C, \delta, \epsilon) &\longmapsto (GC, \psi \circ G\delta, \psi \circ G\epsilon). \end{aligned}$$

For example, in a symmetric monoidal closed category $\mathcal{V}$, the internal hom functor $[-,-] : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$ is lax monoidal by Proposition 3.2.4. The category of monoids of the monoidal category $\mathcal{V}^{\mathrm{op}} \times \mathcal{V}$ is

$$\mathbf{Mon}(\mathcal{V}^{\mathrm{op}} \times \mathcal{V}) \cong \mathbf{Mon}(\mathcal{V}^{\mathrm{op}}) \times \mathbf{Mon}(\mathcal{V}) \cong \mathbf{Comon}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Mon}(\mathcal{V}),$$

so there is an induced functor betweem the categories of monoids

$$\begin{aligned} \mathbf{Mon}[-,-]:\ \mathbf{Comon}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Mon}(\mathcal{V}) &\longrightarrow \mathbf{Mon}(\mathcal{V}) \\ (\ C\ ,\ A\ ) &\longmapsto [C, A]. \end{aligned} \tag{3.17}$$

The concrete content of this observation is that whenever C is a comonoid and A a monoid, the object $[C, A]$ obtains the structure of a monoid, with unit $I \to [C, A]$

which is the transpose under the adjunction $-\otimes C \dashv [C,-]$ of

$$C \xrightarrow{\epsilon} I \xrightarrow{\eta} A$$

and with multiplication $[C,A]\otimes[C,A]\to[C,A]$ the transpose of the composite

$$\begin{tikzcd}
{[C,A]\otimes[C,A]\otimes C} \arrow[r, "1\otimes\Delta"] \arrow[rrdd, dashed] & {[C,A]\otimes[C,A]\otimes C\otimes C} \arrow[r, "1\otimes s\otimes 1"] & {[C,A]\otimes C\otimes[C,A]\otimes C} \arrow[d, "\mathrm{ev}\otimes\mathrm{ev}"] \\
& & A\otimes A \arrow[d, "m"] \\
& & A.
\end{tikzcd}$$

Remark 3.3.2. For the symmetric monoidal closed category $\mathbf{Mod}_R$, the internal hom

$$[-,-]=\mathrm{Hom}_R(-,-):\mathbf{Mod}_R^{\mathrm{op}}\times\mathbf{Mod}_R\longrightarrow\mathbf{Mod}_R$$

has the structure of a lax monoidal functor by Proposition 3.2.4. Therefore it induces a functor

$$\begin{tikzcd}
\mathbf{Mon}(\mathrm{Hom}_R):\ \mathbf{Coalg}_R^{\mathrm{op}}\times\mathbf{Alg}_R \arrow[r] & \mathbf{Alg}_R \\
(\ C\ ,\ A\ ) \arrow[r, mapsto] & \mathrm{Hom}_R(C,A)
\end{tikzcd}$$

between the categories of coalgebras and algebras. This implies the well-known fact that for C an R-coalgebra and A an R-algebra, the set $\mathrm{Hom}_R(C,A)$ of the linear maps between them obtains the structure of an R-algebra under the *convolution* structure

$$(f*g)(c)=\sum_{(c)}f(c_1)g(c_2)\quad\text{and}\quad 1=\eta\circ\epsilon$$

where $*$ is expressed using the 'sigma notation' for the coalgebra comultiplication $\Delta(c)=\sum_i c_{1i}\otimes c_{i2}:=\sum_{(c)}c_{(1)}\otimes c_{(2)}$ introduced in [**Swe69**].

Another example of a functor induced between categories of monoids is the following.

Lemma 3.3.3. *If $\mathscr{F}:\mathcal{K}\to\mathcal{L}$ is a lax functor between two bicategories, there is an induced functor*

$$\mathbf{Mon}\mathscr{F}_{A,A}:\mathbf{Mon}\mathcal{K}(A,A)\longrightarrow\mathbf{Mon}\mathcal{L}(\mathscr{F}A,\mathscr{F}A) \tag{3.18}$$

for each object A in $\mathcal{K}$, which is the functor $\mathscr{F}_{A,A}$ restricted to the category of monoids of the monoidal category $(\mathcal{K}(A,A),\circ,1_A)$.

Proof. Since $\mathscr{F}$ is a lax functor between bicategories, we have a functor $\mathscr{F}_{A,B}:\mathcal{K}(A,B)\to\mathcal{L}(\mathscr{F}A,\mathscr{F}B)$ between the hom-categories for all $A,B\in\mathcal{K}$. In particular, there is a functor

$$\mathscr{F}_{A,A}:\mathcal{K}(A,A)\to\mathcal{L}(\mathscr{F}A,\mathscr{F}A)$$

which maps the 1-cell $f:A\to A$ to $\mathscr{F}f:\mathscr{F}A\to\mathscr{F}A$ and a 2-cell $\alpha:f\Rightarrow g$ to $\mathscr{F}\alpha:\mathscr{F}f\Rightarrow\mathscr{F}g$. If we regard $\mathcal{K}(A,A)$ and $\mathcal{L}(\mathscr{F}A,\mathscr{F}A)$ as monoidal categories with respect to the horizontal composition as in (3.3), $\mathscr{F}_{A,A}$ has the structure of a lax monoidal functor. Indeed, it is equipped with natural transformations with components, for each $f, g \in \mathcal{K}(A, A)$,

$$\phi_{f,g} : \mathscr{F} f \otimes \mathscr{F} g \to \mathscr{F}(f \otimes g) \quad \text{and} \quad \phi_0 : I_{\mathcal{L}(FA,FA)} \to \mathscr{F} I_{\mathcal{K}(A,A)}$$

which are precisely the components $\delta_{f,g}$ and γ_A of the natural transformations (2.3, 2.4) that the lax functor $\mathscr{F}$ is equipped with, since $\otimes \equiv \circ$ and $I_{\mathcal{K}(A,A)} \equiv 1_A$. The axioms follow from those of δ and γ. Hence a functor (3.18) between the categories of monoids is induced. □

In Remark 3.3.1 we saw how a monad $t : A \to A$ in a bicategory $\mathcal{K}$ is actually a monoid in $\mathcal{K}(A, A)$. The above lemma states that if $\mathscr{F}$ is a lax functor, then $\mathscr{F} t : \mathscr{F} A \to \mathscr{F} A$ is a monoid in $\mathcal{L}(\mathscr{F} A, \mathscr{F} A)$, *i.e.* $\mathscr{F} t$ is a monad in the bicategory $\mathcal{L}$. Therefore we re-discover the fact that lax functors between bicategories preserve monads, from a different point of view than Remark 2.2.2, where a monad was identified with a lax functor from the terminal bicategory to $\mathcal{K}$.

For any monoidal category $\mathcal{V}$, there are forgetful functors

$$S : \mathbf{Mon}(\mathcal{V}) \longrightarrow \mathcal{V} \quad \text{and} \quad U : \mathbf{Comon}(\mathcal{V}) \longrightarrow \mathcal{V}$$

which just discard the (co)multiplication and the (co)unit. A crucial issue for our needs is the assumptions under which these functors have a left or right adjoint accordingly. In other words, we are interested in the conditions on $\mathcal{V}$ that allow the *free monoid* and the *cofree comonoid* construction.

The existence of a free monoid functor is quite frequent, since the monoidal structures that arise in practice may well be closed, so that the tensor product preserves colimits in both arguments. In particular, the following is true.

Proposition 3.3.4. *Suppose that $\mathcal{V}$ is a monoidal category with countable coproducts which are preserved by $\otimes$ on either side. The forgetful* $\mathbf{Mon}(\mathcal{V}) \to \mathcal{V}$ *has a left adjoint L, and the free monoid on an object X is given by the 'geometric series'*

$$LX = \coprod_{n \in \mathbb{N}} X^{\otimes n}.$$

There are various sets of conditions, stronger or weaker, that guarantee the existence of free monoids and are connected with the different kinds of settings where they apply, such as free monads, free algebras, free operads etc. There are many classical references on these constructions, for example by Kelly, Dubuc, Barr and others, and most are outlined in Lack's [**Lac10c**].

On the other hand, the existence of a cofree comonoid functor is more problematic. In Sweedler's [**Swe69**], the cofree coalgebra on a vector space V is constructed as a certain subcoalgebra of $T(V^*)^o$, where T gives the tensor algebra of the linear dual of V, and $(-)^o$ is the dual algebra functor as described later in Remark 6.1.2. In [**BL85**], a new description of the cofree coalgebra is given, still in $\mathbf{Vect}_k$ for a field k. In Barr's [**Bar74**], it is shown that the forgetful $\mathbf{Coalg}_R \to \mathbf{Mod}_R$ for a commutative ring R has a right adjoint, and in Fox's [**Fox93**] two constructions on the cofree coalgebra on an R-module are presented. Finally in [**Haz03**], connections of cofree coalgebras in $\mathbf{Mod}_R$ with the notion of multivariable recursiveness are examined.

We are here interested in the generalization from $\mathbf{Vect}_k$ and $\mathbf{Mod}_R$ to the existence of such cofree objects (comonoids) in an arbitrary monoidal category $\mathcal{V}$. Hans Porst in a series of papers [**Por06, Por08a, Por08b, Por08c**] studied the categories of monoids and comonoids (also the categories of modules and comodules for them) and their various categorical properties, with emphasis on the local presentability structure inherited from the initial monoidal category. We are going to employ many of those strategies for our purposes, so at this point we briefly describe the most basic parts of this theory. A standard reference for locally presentable categories is Adamek-Rosicky's [**AR94**].

Recall that a small full subcategory $\mathcal{A}$ of a category $\mathcal{C}$ is called *dense* provided that every object of $\mathcal{C}$ is a canonical colimit of objects of $\mathcal{A}$, *i.e.* the colimit of the forgetful $(\mathcal{A} \downarrow C) \to \mathcal{C}$. Also, an object in a category $\mathcal{C}$ is called λ*-presentable* for λ a regular cardinal, provided that its hom-functor $\mathcal{C}(C, -)$ preserves λ-filtered limits. For $\lambda = \aleph_0$, we have the notion of a *finitely presentable object.*

Definition. (1) A *locally* λ*-presentable* category $\mathcal{C}$ is a cocomplete category which has a set $\mathcal{A}$ of λ-presentable objects, such that every object is a λ-filtered colimit of objects from $\mathcal{A}$. A category is called *locally presentable* when it is locally λ-presentable for some regular cardinal λ, and *locally finitely presentable* for $\lambda = \aleph_0$.

(2) A λ*-accessible* category is a category with λ-filtered colimits and a set of λ-presentable objects, such that every object is a λ-filtered colimit of those. A category is called *accessible* if it is λ-accessible for some regular cardinal λ.

Notice that in a locally λ-presentable category $\mathcal{C}$, all λ-presentable objects have a set of representatives (with respect to isomorphism). Any such set is denoted by $\mathbf{Pres}_\lambda\mathcal{C}$ and is a small dense full subcategory of $\mathcal{K}$, hence also a strong generator. Recall that a *generator* is a family of objects $\mathcal{G}$ such that for pairs $A \overset{f}{\underset{g}{\rightrightarrows}} B$ with $f \neq g$, there exists $G \in \mathcal{G}$ and $h : G \to A$ with $fh \neq gh$. It is *strong* if for any A and a proper subobject, there exists $G \in \mathcal{G}$ and $G \to A$ which doesn't factorize through the subobject.

Other useful properties of locally presentable categories are completeness, wellpoweredness and co-wellpoweredness. Obviously, an accessible category with all colimits is locally presentable, but so is an accessible category with all limits (see [**AR94**, 2.47]). A functor F between λ-accessible categories is *accessible* if it preserves λ-filtered colimits, whereas a *finitary* functor in general preserves all filtered colimits.

In [**Por08c**] the class of *admissible monoidal categories* is introduced. These are locally presentable symmetric monoidal categories $\mathcal{V}$, such that for each object A the functor $A \otimes -$ preserves filtered colimits. Examples are the category $\mathbf{Mod}_R$ for a commutative ring R, every locally presentable category with respect to binary products, and every monoidal closed category which is locally presentable. However, the results exhibited below also hold for small variations from the above conditions. For example, the symmetry can be replaced with $\otimes$ preserving filtered colimits on both entries.

The notion of *functor algebras* and *functor coalgebras* for an endofunctor are of importance in the proofs below. Given an endofunctor on any category $F : \mathcal{C} \to \mathcal{C}$, the category $\mathbf{Alg}F$ of F-algebras has objects pairs $(A, \alpha : FA \to A)$ and morphisms $(A, \alpha) \to (A', \alpha')$ are arrows $f : A \to A'$ making the diagram

$$\begin{tikzcd}
FA \arrow[r, "\alpha"] \arrow[d, "Ff"'] & A \arrow[d, "f"] \\
FA' \arrow[r, "\alpha'"] & A'
\end{tikzcd}$$

commute. The category $\mathbf{Coalg}F = (\mathbf{Alg}F^{\mathrm{op}})^{\mathrm{op}}$ is defined dually, with objects pairs $(C, \beta : C \to FC)$ and arrows $g : C \to C'$ making the diagram

$$\begin{tikzcd}
C \arrow[r, "\beta"] \arrow[d, "g"'] & FC \arrow[d, "Fg"] \\
C' \arrow[r, "\beta'"] & FC'
\end{tikzcd}$$

commute. More about these categories and their properties can be found in [**AR94, AP03**]. The most useful facts are the following:

(i) The forgetful functor $\mathbf{Alg}F \to \mathcal{C}$ creates all limits and and those colimits which are preserved by F.
(ii) The forgetful functor $\mathbf{Coalg}F \to \mathcal{C}$ creates all colimits and those limits which are preserved by F.
(iii) If $\mathcal{C}$ is locally presentable and F preserves filtered colimits, the categories $\mathbf{Alg}F$ and $\mathbf{Coalg}F$ are locally presentable.

Notably, these categories can be expressed as specific *inserters* $\mathbf{Alg}F = \mathbf{Ins}(F, \mathrm{id}_{\mathcal{C}})$ and $\mathbf{Coalg}F = \mathbf{Ins}(\mathrm{id}_{\mathcal{C}}, F)$. Fact (*iii*) thus follows from the more general 'Weighted Limit Theorem' by Makkai and Paré [**MP89**, 5.1.6], which in particular asserts that the above inserters are accessible categories when $\mathcal{C}$ and F are accessible. For details about these constructions, see [**AR94**, Theorem 2.72].

In the applications where $\mathbf{Alg}F$ and $\mathbf{Coalg}F$ for specific endofunctors are studied, they usually turn out to be monadic and comonadic respectively over $\mathcal{C}$. Since coequalizers of split pairs are absolute colimits, *i.e.* preserved by any functor, monadicity and comonadicity are established as soon as the forgetful functor has a left or right adjoint respectively.

PROPOSITION 3.3.5. [**Por08c**, 2.6-2.7] *Suppose $\mathcal{V}$ is an admissible category.*

(1) $\mathbf{Mon}(\mathcal{V})$ *is finitary monadic over $\mathcal{V}$ and locally presentable.*
(2) $\mathbf{Comon}(\mathcal{V})$ *is a locally presentable category and comonadic over $\mathcal{V}$.*

PROOF. (Sketch) The idea is to view both categories of monoids and comonoids as subcategories of the functor algebras and functor coalgebras categories, for specific endofunctors on $\mathcal{V}$.

Consider the functors T_+ and $T_\times$ on our admissible category $\mathcal{V}$ given by

$$T_+(C) = (C \otimes C) + I, \qquad T_\times(C) = (C \otimes C) \times I.$$

These are finitary functors, because the 'n-th tensor power' functor $T_n = (-)^{\otimes n}$ preserves filtered colimits, and $(- \times I)$ preserves filtered colimits for any locally presentable category (where finite limits commute with filtered colimits).

We deduce that $\mathbf{Alg}T_+$ is finitary monadic over $\mathcal{V}$, locally presentable and contains $\mathbf{Mon}(\mathcal{V})$ as a full subcategory, and also $\mathbf{Coalg}T_\times$ is comonadic over $\mathcal{V}$, locally presentable and contains $\mathbf{Comon}(\mathcal{V})$ as a full subcategory. Moreover, the categories of monoids and comonoids are closed under limits and colimits respectively.

The first part of the proposition regarding $\mathbf{Mon}(\mathcal{V})$ follows from general arguments for monadicity and local presentability of categories of algebras for a finitary monad (see [**GU71**, Satz 10.3]). On the other hand, these arguments cannot be dualized for $\mathbf{Comon}(\mathcal{V})$. For example, the dual of a locally presentable category is not locally presentable (unless it is a small complete lattice).

Therefore a different approach is followed, using the notion of an equifier of a family of natural transformations. The decisive fact then is that if all functors involved are accessible, then the equifier is an accessible category (see [**AR94**, 2.76]).

Definition. Let $F_1^i, F_2^i : \mathcal{A} \to \mathcal{B}_i$ be a family of functors, and for each $i \in I$, $(\phi^i, \psi^i) : F_1^i \to F_2^i$ be a pair of natural transformations. Then, the full subcategory of $\mathcal{A}$ spanned by those object A which satisfy $\phi^i_A = \psi^i_A$ for all i is called the *equifier* of the above family of natural transformations, denoted by

$$\mathbf{Eq}(\phi^i, \psi^i)_{\{i \in I\}}.$$

More explicitly, three pairs (ϕ^i, ψ^i) of natural transformations between composites of the forgetful $\mathbf{Coalg}T_\times \to \mathcal{V}$ and the 'tensor power functor' $\otimes^n$ are defined, the equality of which give precisely the coassociativity and coidentity conditions of the definition of a comonoid. Hence $\mathbf{Comon}(\mathcal{V}) = \mathbf{Eq}((\phi^i, \psi^i)_{i=1,2,3})$, and for $\mathcal{V}$ admissible this implies that $\mathbf{Comon}(\mathcal{V})$ is locally presentable.

Now comonadicity of $\mathbf{Comon}(\mathcal{V})$ over $\mathcal{V}$ follows: in the commutative triangle

$$\begin{tikzcd}
\mathbf{Comon}(\mathcal{V}) \arrow[r, hook] \arrow[dr, dashed, "U"'] & \mathbf{Coalg}F \arrow[d] \\
& \mathcal{V}
\end{tikzcd}$$

where all categories are locally presentable, both forgetful functors to $\mathcal{V}$ have a right adjoint by Theorem 3.0.1, since they are cocontinuous. Moreover, the right leg is comonadic by basic facts for functor coalgebras, and the inclusion preserves and reflects all limits from the complete full subcategory $\mathbf{Comon}(\mathcal{V})$ to the complete $\mathbf{Coalg}F$. Therefore it creates equalizers of split pairs and so does U, which then satisfies the conditions of Precise Monadicity Theorem. In particular, the existence of the *cofree comonoid functor* $R : \mathcal{V} \to \mathbf{Comon}(\mathcal{V})$ is established. □

Another property which $\mathbf{Comon}(\mathcal{V})$ inherits from the monoidal category $\mathcal{V}$ is monoidal closedness.

Proposition 3.3.6. [**Por08c**, 3.2] *If $\mathcal{V}$ is a symmetric monoidal closed category which is locally presentable, then the category of comonoids* $\mathbf{Comon}(\mathcal{V})$ *is a locally presentable symmetric monoidal closed category as well.*

Proof. The symmetric monoidal structure of $\mathbf{Comon}(\mathcal{V})$ was described earlier. In order to prove the existence of a right adjoint to

$$-\otimes C : \mathbf{Comon}(\mathcal{V}) \to \mathbf{Comon}(\mathcal{V}) \tag{3.19}$$

for any comonoid C in $\mathcal{V}$, we can use the adjoint functor theorem 3.0.1. The category $\mathbf{Comon}(\mathcal{V})$ is cocomplete and has a small dense subcategory, since it is locally presentable by Proposition 3.3.5. Moreover, the functor (3.19) preserves all colimits by the commutativity of

$$\begin{tikzcd}
\mathbf{Comon}(\mathcal{V}) \arrow[r, "-\otimes C"] \arrow[d, "U"'] & \mathbf{Comon}(\mathcal{V}) \arrow[d, "U"] \\
\mathcal{V} \arrow[r, "-\otimes UC"'] & \mathcal{V}
\end{tikzcd}$$

where the comonadic forgetful U creates all colimits and $-\otimes UC$ preserves them since $\mathcal{V}$ is monoidal closed. Hence we have an adjunction

$$\begin{tikzcd}
\mathbf{Comon}(\mathcal{V}) \arrow[r, shift left, "(-\otimes C)"] \arrow[r, phantom, "\perp"] & \mathbf{Comon}(\mathcal{V}) \arrow[l, shift left, "\textsc{Hom}(C,-)"]
\end{tikzcd}$$

where Hom denotes the internal hom of $\mathbf{Comon}(\mathcal{V})$. ☐

Corollary. For a commutative ring R, the category of R-algebras $\mathbf{Alg}_R$ is monadic over $\mathbf{Mod}_R$ and locally presentable, and the category of R-coalgebras $\mathbf{Coalg}_R$ is comonadic over $\mathbf{Mod}_R$, locally presentable and monoidal closed.

The fact that $\mathbf{Coalg}_R$ is locally presentable in fact generalizes the *Fundamental Theorem of Coalgebras*, which states that every k-coalgebra for a field k is a filtered colimit of finite dimensional coalgebras, *i.e.* whose underlying vector space is finite dimensional (see [**Swe69, DNR01**]). These are precisely the finitely presentable objects in $\mathbf{Coalg}_k$, hence we obtain an analogous statement for $\mathbf{Coalg}_R$ for a commutative ring R.

3.4. Categories of modules and comodules

If (A, m, η) is a monoid in a monoidal category $\mathcal{V}$, a *(left) A-module* is an object M of $\mathcal{V}$ equipped with an arrow $\mu : A\otimes M \to M$ called *action*, such that the diagrams

$$\begin{tikzcd}
A\otimes A\otimes M \arrow[r, "m\otimes 1"] \arrow[d, "1\otimes\mu"'] & A\otimes M \arrow[d, "\mu"] \\
A\otimes M \arrow[r, "\mu"'] & M
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
 & A\otimes M \arrow[dr, "\mu"] & \\
I\otimes M \arrow[ur, "\eta\otimes 1"] \arrow[rr, "l_M"'] & & M
\end{tikzcd} \tag{3.20}$$

commute, where a is suppressed. An *A-module morphism* $(M,\mu) \to (M',\mu')$ is an arrow $f : M \to M'$ in $\mathcal{V}$ such that the diagram

$$\begin{tikzcd}
A \otimes M \arrow[r, "1\otimes f"] \arrow[d, "\mu"'] & A \otimes M' \arrow[d, "\mu'"] \\
M \arrow[r, "f"'] & M'
\end{tikzcd} \tag{3.21}$$

commutes. Thus for any monoid A in $\mathcal{V}$, there is a category $\mathbf{Mod}_{\mathcal{V}}(A)$ of left A-modules and A-module morphisms.

Dually, a *(right) C-comodule* for (C,Δ,ϵ) a comonoid in $\mathcal{V}$ is an object X in $\mathcal{V}$ together with the *coaction* $\delta : X \to X \otimes C$, satisfying compatibility conditions with the comultiplication and counit. A *C-comodule morphism* $(X,\delta) \to (X',\delta')$ is a arrow $g : X \to X'$ in $\mathcal{V}$ which respects the coactions. There is a category of right C-comodules $\mathbf{Comod}_{\mathcal{V}}(C)$ for every comonoid C in a monoidal category $\mathcal{V}$.

In a very similar way, we can define categories of *right A-modules* and *left C-comodules.* If $\mathcal{V}$ is a symmetric monoidal category, there is an obvious isomorphism between categories of left and right A-modules and left and right C-comodules, so usually there is no distinction in the notation between left and right modules and comodules.

For example, in the monoidal category of abelian groups $\mathbf{Ab}$, the category of modules for a ring $R \in \mathbf{Mon}(\mathbf{Ab})$ is precisely the category of R-modules $\mathbf{Mod}_R$. Moreover, for $\mathcal{V} = \mathbf{Mod}_R$ itself, we denote by $\mathbf{Mod}_A$ the category of those R-modules which are equipped with the structure of an A-module for an R-algebra $A \in \mathbf{Mon}(\mathbf{Mod}_R)$. Similarly, $\mathbf{Comod}_C$ is the category of C-comodules for an R-coalgebra $C \in \mathbf{Comon}(\mathbf{Mod}_R)$.

Recall how, when a monoidal category $\mathcal{V}$ is viewed as the hom-category $\mathcal{K}(\star,\star)$ of a bicategory $\mathcal{K}$ with one object $\star$, a monoid A in $\mathcal{V}$ is precisely a monad in $\mathcal{K}$ (Remark 3.3.1). This analogy carries over to modules for a monoid in $\mathcal{V}$. In Definition 2.2.3, the category of left t-modules for a monad t in the bicategory $\mathcal{K}$ was defined to be the category of Eilenberg-Moore algebras for the monad 'post-composition with t'. For the one-object case, since the tensor product of $\mathcal{K}(\star,\star)$ is just horizontal composition, the following well-known fact is immediately implied.

Proposition 3.4.1. *For any monoid A and any comonoid C in a monoidal category $\mathcal{V}$, the categories of A-modules $\mathbf{Mod}_{\mathcal{V}}(A)$ and C-comodules $\mathbf{Comod}_{\mathcal{V}}(C)$ are respectively monadic and comonadic over $\mathcal{V}$.*

Explicitly, the category of (left) modules for a monoid (A,m,η) is the category of algebras for the monad $(A\otimes -, \eta\otimes -, m\otimes -)$ on $\mathcal{V}$, and the category of (right) comodules for a comonoid (C,Δ,ϵ) is the category of coalgebras for the comonad $(-\otimes C, -\otimes\epsilon, -\otimes\Delta)$ on $\mathcal{V}$.

In the previous section, it was demonstrated how a lax monoidal functor between monoidal categories $F : \mathcal{V} \to \mathcal{W}$ induces a functor $\mathbf{Mon}F$ between their categories of monoids, as in (3.16). Furthermore, for any monoid A in $\mathcal{V}$, there is an induced functor between the categories of modules

$$\begin{aligned} \mathbf{Mod}F:\ \mathbf{Mod}_{\mathcal{V}}(A) &\longrightarrow \mathbf{Mod}_{\mathcal{W}}(FA) \\ (M,\mu) &\longmapsto (FM,\mu') \end{aligned} \tag{3.22}$$

where the object FM in $\mathcal{W}$ obtains the structure of a FA-module via the action

$$\mu' : FA \otimes FM \xrightarrow{\phi_{A,M}} F(A \otimes M) \xrightarrow{F\mu} FM$$

with $\phi_{A.M}$ the lax structure map of F.

As an application, consider the internal hom functor $[-,-] : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$ in a symmetric monoidal closed category $\mathcal{V}$. By Proposition 3.2.4 it is lax monoidal, as the parametrized adjoint of the strong monoidal $(- \otimes -)$, and it induces the functor $\mathbf{Mon}[-,-]$ as in (3.17). Now, a monoid in $\mathcal{V}^{\mathrm{op}} \times \mathcal{V}$ is a pair (C, A) where C is a comonoid and A a monoid, and also

$$\mathbf{Mod}_{\mathcal{V}^{\mathrm{op}}\times\mathcal{V}}((C,A)) \cong \mathbf{Mod}_{\mathcal{V}^{\mathrm{op}}}(C) \times \mathbf{Mod}_{\mathcal{V}}(A) \cong \mathbf{Comod}_{\mathcal{V}}(C)^{\mathrm{op}} \times \mathbf{Mod}_{\mathcal{V}}(A).$$

Hence the induced functor (3.22) in this case is

$$\begin{aligned} \mathbf{Mod}[-,-]:\ \mathbf{Comod}_{\mathcal{V}}(C)^{\mathrm{op}} \times \mathbf{Mod}_{\mathcal{V}}(A) &\longrightarrow \mathbf{Mod}_{\mathcal{V}}([C,A]) \\ (\ (X,\delta)\ ,\ (M,\mu)\ ) &\longmapsto ([X,M],\mu'). \end{aligned} \tag{3.23}$$

This concretely means that whenever X is a C-comodule and M is an A-module, the object $[X, M]$ obtains the structure of a $[C, A]$-module, with action

$$\mu' : [C,A] \otimes [X,M] \to [X,M]$$

which is the transpose under $- \otimes X \dashv [X,-]$ of the composite

$$\begin{tikzcd}
{[C,A] \otimes [X,M] \otimes X} \arrow[r, "1\otimes\delta"] \arrow[ddrr, dashed] & {[C,A] \otimes [X,M] \otimes X \otimes C} \arrow[r, "1\otimes s"] & {[C,A] \otimes C \otimes [X,M] \otimes X} \arrow[d, "\mathrm{ev}\otimes\mathrm{ev}"] \\
 & & A \otimes M \arrow[d, "\mu"] \\
 & & M.
\end{tikzcd} \tag{3.24}$$

Corollary. For A an R-algebra and C an R-coalgebra for a commutative ring R, there is an induced map

$$\begin{aligned} \mathbf{Mod}(\mathrm{Hom}_R):\ \mathbf{Comod}_C^{\mathrm{op}} \times \mathbf{Mod}_A &\longrightarrow \mathbf{Mod}_{\mathrm{Hom}_R(C,A)} \\ (\ X\ ,\ M\ ) &\longmapsto \mathrm{Hom}_R(X,M) \end{aligned}$$

which endows the R-module of linear maps between X and M with the structure of a $\mathrm{Hom}_R(C, A)$-module.

In the previous section, it turned out that for the class of admissible monoidal categories, the categories of monoids and comonoids had very useful properties (see Proposition 3.3.5). As far as the categories of modules and comodules are concerned, $\mathbf{Comod}_{\mathcal{V}}(C)$ is again more particular than $\mathbf{Mod}_{\mathcal{V}}(A)$ and similar techniques as for

$\mathbf{Comon}(\mathcal{V})$ can be used. The following generalizes the results for comodules over a coalgebra in $\mathcal{V} = \mathbf{Mod}_R$ of [**Por06**].

Proposition 3.4.2. *Suppose $\mathcal{V}$ is a locally presentable monoidal category, such that $\otimes$ preserves filtered colimits in both variables. Then*

(1) $\mathbf{Mod}_{\mathcal{V}}(A)$ *for a monoid A is finitary monadic over $\mathcal{V}$ and so locally presentable.*
(2) $\mathbf{Comod}_{\mathcal{V}}(C)$ *for a comonoid C is a locally presentable category.*

Proof. By Proposition 3.4.1, the endofunctor on $\mathcal{V}$ which induces the monad for which the algebras are (left) A-modules is $(A\otimes -)$, which is finitary by assumptions.

Similarly, the endofunctor which gives rise to the comonadic $\mathbf{Comod}_{\mathcal{V}}(C)$ over $\mathcal{V}$ is $F_C = - \otimes C$, which is also finitary. Imitating the proof of Proposition 3.3.5, consider the category of functor F_C-coalgebras which contains $\mathbf{Comod}_{\mathcal{V}}(C)$ as its full subcategory, closed under formation of colimits. Then $\mathbf{Coalg}F_C$ is comonadic over $\mathcal{V}$ and locally presentable itself. Now define pairs of natural transformations

$$\begin{tikzcd}
\phi^1,\psi^1 :\ \mathbf{Coalg}F_C \arrow[r, bend left, "U"{name=A}] \arrow[r, bend right, "F_CF_CU"'{name=B}] \arrow[from=A, to=B, Rightarrow, phantom, "\Downarrow"] & \mathcal{V}\ ,\quad \phi^2,\psi^2 :\ \mathbf{Coalg}F_C \arrow[r, bend left, "U"{name=C}] \arrow[r, bend right, "(-\otimes I)U"'{name=D}] \arrow[from=C, to=D, Rightarrow, phantom, "\Downarrow"] & \mathcal{V}
\end{tikzcd}$$

with components

$$\begin{array}{lcl}
\phi^1_X : X \xrightarrow{\beta} X\otimes C \xrightarrow{\beta\otimes 1} X\otimes C\otimes C & \text{and} & \phi^2_X : X \xrightarrow{\beta} X\otimes C \xrightarrow{1\otimes\epsilon} X\otimes I \\
\psi^1_X : X \xrightarrow{\beta} X\otimes C \xrightarrow{1\otimes\Delta} X\otimes C\otimes C & & \psi^2_X : X \xrightarrow{r^{-1}} X\otimes I
\end{array}$$

where $\beta : X \to X\otimes C$ is the structure map of the functor F_C-coalgebra X, and Δ, ϵ are the comultiplication and counit of the comonoid C. Since all categories and functors involved are accessible, the equifier of this family of natural transformations is accessible as well. It is not hard to see that

$$\mathbf{Eq}((\phi^i,\psi^i)_{i=1,2}) = \mathbf{Comod}_{\mathcal{V}}(C)$$

so the category of comodules is accessible and moreover cocomplete, thus locally presentable. ☐

The above proposition indicates the structure that finitary monadic and finitary comonadic categories over locally presentable categories inherit. We note that Gabriel and Ulmer's result in [**GU71**] for algebras of finitary monads does not seem to dualize, but by following a similar approach to Adámek and Rosický's 'Locally presentable and accessible categories', we obtain the following result.

Theorem 3.4.3. *Suppose that $\mathcal{C}$ is a locally presentable category.*

- *If (T,m,η) is a finitary monad on $\mathcal{C}$, the category of algebras $\mathcal{C}^T$ is locally presentable.*
- *If (S,Δ,ϵ) is a finitary comonad on $\mathcal{C}$, the category of coalgebras $\mathcal{C}^S$ is locally presentable.*

Proof. The category of Eilenberg-Moore algebras $\mathcal{C}^T$ is always a full subcategory of the locally presentable category of endofunctor algebras $\mathbf{Alg}T$ (see previous section). More precisely, it is expressed as an equifier of natural transformations between accessible functors $\mathbf{Alg}T \to \mathcal{C}$ hence is accessible as in [**AR94**, 2.78], and by default is also complete.

On the other hand, the category of coalgebras $\mathcal{C}^S$ is a full subcategory of the locally presentable category of endofunctor coalgebras $\mathbf{Coalg}T$, expressed as the equifier $\mathcal{C}^S = \mathbf{Eq}\big((\phi^t, \psi^t)_{t=1,2}\big)$ for

$$\begin{tikzcd} \mathbf{Coalg}S \arrow[r, bend left, "U"] \arrow[r, bend right, "SSU"'] \arrow[r, phantom, "\Downarrow_{\phi^1,\psi^1}"] & \mathcal{C} \end{tikzcd}$$

with

$$\begin{aligned} \phi^1_{(C,\beta)} &: C \xrightarrow{\beta} SC \xrightarrow{S\beta} SSC \\ \psi^1_{(C,\beta)} &: C \xrightarrow{\beta} SC \xrightarrow{\Delta_C} SSC \end{aligned}$$

$$\begin{tikzcd} \mathbf{Coalg}S \arrow[r, bend left, "U"] \arrow[r, bend right, "U"'] \arrow[r, phantom, "\Downarrow_{\phi^2,\psi^2}"] & \mathcal{C} \end{tikzcd}$$

with

$$\begin{aligned} \phi^2_{(C,\beta)} &: C \xrightarrow{\beta} SC \xrightarrow{\epsilon_C} C \\ \psi^2_{(C,\beta)} &: C \xrightarrow{\quad 1_C \quad} C. \end{aligned}$$

All categories and functors involved are accessible, hence $\mathcal{C}^S$ is an accessible category, with all colimits created from $\mathcal{C}$. □

Proposition 3.4.2 could directly be established from the above. Notice that the assumptions on $\mathcal{V}$ could of course be changed to 'locally presentable, symmetric monoidal category, such that $B \otimes -$ preserves filtered colimits', *i.e.* admissible monoidal category. As mentioned earlier, symmetry allows us to identify in a sense the categories of left and right modules and comodules, without distinguishing cases in the respective proofs. Even in the non-symmetric case though, the results hold for all four cases separately.

Corollary. If A is an R-algebra and C an R-coalgebra for a commutative ring R, the categories $\mathbf{Mod}_A$ and $\mathbf{Comod}_C$ are locally presentable.

Notably, many useful properties and constructions for $\mathbf{Comod}_C$ in the category $\mathbf{Mod}_R$ are included in Wischnewsky's [**Wis75**].

So far we have studied categories of modules and comodules for fixed monoids and comonoids in a monoidal category $\mathcal{V}$. Since a (co)module is just an object in $\mathcal{V}$ with extra structure, relative to some (co)monoid, it could be expected that the same object is possible to be endowed with (co)module structures relating it with different (co)monoids.

Suppose that A, B are two monoids in the monoidal category $\mathcal{V}$. Each monoid morphism $f : A \to B$ between them determines a functor

$$f^* : \mathbf{Mod}_{\mathcal{V}}(B) \longrightarrow \mathbf{Mod}_{\mathcal{V}}(A) \tag{3.25}$$

which makes every B-module (N, μ) into an A-module f^*N via the action

$$A \otimes N \xrightarrow{f\otimes 1} B \otimes N \xrightarrow{\mu} N.$$

This functor is sometimes called *restriction of scalars* along f. Also, each B-module arrow becomes an A-module arrow (*i.e.* commutes with the A-actions), and so we have a commutative triangle of categories and functors

$$\begin{tikzcd}
\mathbf{Mod}_{\mathcal{V}}(B) \arrow[rr, "f^*"] \arrow[dr] & & \mathbf{Mod}_{\mathcal{V}}(A) \arrow[dl] \\
& \mathcal{V}. &
\end{tikzcd} \tag{3.26}$$

On the other hand, if C and D are two comonoids in $\mathcal{V}$, each comonoid arrow $g : C \to D$ induces a functor

$$g_! : \mathbf{Comod}_{\mathcal{V}}(C) \longrightarrow \mathbf{Comod}_{\mathcal{V}}(D) \tag{3.27}$$

which makes every C-comodule (X, δ) into a D-comodule $g_!X$ via the coaction

$$X \xrightarrow{\delta} X \otimes C \xrightarrow{1\otimes g} X \otimes D,$$

called *corestriction of scalars* along g. The respective commutative triangle is

$$\begin{tikzcd}
\mathbf{Comod}_{\mathcal{V}}(C) \arrow[rr, "g_!"] \arrow[dr] & & \mathbf{Comod}_{\mathcal{V}}(D) \arrow[dl] \\
& \mathcal{V}. &
\end{tikzcd} \tag{3.28}$$

Notice that by the above triangles, where the legs are monadic and comonadic respectively, f^* is a continuous functor and $g_!$ is a cocontinuous functor when $\mathcal{V}$ is (co)complete.

It is often of interest to deduce the existence of adjoints of the functors f^* and $g_!$. This is why the last part of this section is a digression, devoted to the identification of certain assumptions on the monoidal category $\mathcal{V}$ which permit the explicit construction of such adjoints. Most of the constructions are well-known in particular categories, like $\mathcal{V}$=**Ab** for the categories of modules for rings, which is also our motivating example.

If A, B are two monoids in $\mathcal{V}$, define a *left A/right B-bimodule M* to be an object in $\mathcal{V}$ with a left A-action $A \otimes M \xrightarrow{\lambda} M$ and a right B-action $M \otimes B \xrightarrow{\rho} B$ such that the actions commute, and denote it by ${}_AM_B$. In a dual way, we can define a *left C/right D-bicomodule* ${}_CX_D$.

i) In an arbitrary monoidal category $\mathcal{V}$, the *tensor product* of the bimodules ${}_AM_B$, ${}_BN_{A'}$ over B is the coequalizer

$$\begin{tikzcd}
M \otimes B \otimes N \arrow[r, shift left, "1\otimes\lambda_N"] \arrow[r, shift right, "\rho_M\otimes 1"'] & M \otimes N \arrow[r, two heads] & M \otimes_B N
\end{tikzcd} \tag{3.29}$$

where ρ_M is the right B-action on M and λ_N is the left B-action on N. Dually, the *cotensor product* for bicomodules ${}_CX_D$, ${}_DY_{C'}$ over D is the equalizer

$$\begin{tikzcd}
X\square_D Y \arrow[r, tail] & X \otimes Y \arrow[r, shift left, "r_X\otimes 1"] \arrow[r, shift right, "1\otimes l_Y"'] & X \otimes D \otimes Y
\end{tikzcd}$$

where r_X is the right D-coaction on X and l_Y is the left D-coaction on Y.

ii) In a symmetric monoidal closed category $\mathcal{V}$, we can form $\mathrm{Hom}_A(M,N)$ for two A-modules M,N as the equalizer

$$\begin{tikzcd}\mathrm{Hom}_A(M,N) \arrow[r, tail] & {[M,N]} \arrow[r, shift left, "t"] \arrow[r, shift right, "k"'] & {[A,[M,N]]}\end{tikzcd}$$

where t corresponds under $-\otimes X \dashv [X,-]$ to

$$[M,N]\otimes A\otimes M \xrightarrow{1\otimes s} [M,N]\otimes M\otimes A \xrightarrow{\mathrm{ev}\otimes 1} N\otimes A \xrightarrow{\rho_N} N$$

and k corresponds to

$$[M,N]\otimes A\otimes M \xrightarrow{1\otimes\lambda_M} [M,N]\otimes M \xrightarrow{\mathrm{ev}} N.$$

Proposition 3.4.4. *Suppose that the monoidal category $\mathcal{V}$ has coequalizers and the functor $B\otimes -$ preserves them for any monoid B. Then the functor f^* has a left adjoint, for any monoid morphism f. Dually, if $\mathcal{V}$ has equalizers and the functor $-\otimes C$ preserves them for any comonoid C, then $g_!$ has a right adjoint for any comonoid morphism g.*

Proof. Firstly notice that any monoid A can be considered as a left and right A-module via multiplication, and any comonoid C is a left and right C-comodule via comultiplication.

When B is viewed as a left B/right A-bimodule via restriction of scalars along $f: A\to B$, there exists a natural bijection

$$\mathbf{Mod}_{\mathcal{V}}(B)(B\otimes_A M, N)\cong \mathbf{Mod}_{\mathcal{V}}(A)(M, f^*N)$$

for any left A-module M and left B-module N, which establishes an adjunction

$$\begin{tikzcd}\mathbf{Mod}_{\mathcal{V}}(A) \arrow[r, shift left=1.5, "B\otimes_A -"] \arrow[r, phantom, "\perp"] & \mathbf{Mod}_{\mathcal{V}}(B). \arrow[l, shift left=1.5, "f^*"]\end{tikzcd}$$

Notice that the left B-action on $B\otimes_A M$ is induced by universality of the top coequalizer, since $B\otimes -$ preserves them:

$$\begin{tikzcd}
B\otimes B\otimes A\otimes M \arrow[rr, "1\otimes 1\otimes\lambda_M"] \arrow[dr, "1\otimes 1\otimes f\otimes 1"] \arrow[dd, "m\otimes 1\otimes 1"'] & & B\otimes B\otimes M \arrow[r, two heads] \arrow[dd, "m\otimes 1"] & B\otimes B\otimes_A M \arrow[dd, dashed, "\exists!\lambda_{B\otimes_A M}"] \\
& B\otimes B\otimes B\otimes M \arrow[ur, "1\otimes m\otimes 1"'] & & \\
B\otimes A\otimes M \arrow[rr, "1\otimes\mu"] \arrow[dr, "1\otimes f\otimes 1"'] & & B\otimes M \arrow[r, two heads] & B\otimes_A M. \\
& B\otimes B\otimes M \arrow[ur, "m\otimes 1"'] & &
\end{tikzcd}$$

Dually, for a comonoid arrow $g: C\to D$ we have the adjunction

$$\begin{tikzcd}\mathbf{Comod}_{\mathcal{V}}(C) \arrow[r, shift left=1.5, "g_!"] \arrow[r, phantom, "\perp"] & \mathbf{Mod}_{\mathcal{V}}(D) \arrow[l, shift left=1.5, "-\square_D C"]\end{tikzcd}$$

when C is viewed as a left D-comodule via corestriction along g. □

Remark. By the adjoint lifting theorem (see for example [**Joh02a**, 1.1.3]), we can deduce the sheer existence of a left adjoint for f^* and a right adjoint for $g_!$ if

$\mathbf{Mod}_{\mathcal{V}}(B)$ and $\mathbf{Comod}_{\mathcal{V}}(C)$ have (co)equalizers (of (co)reflexive pairs) accordingly. This happens because the legs of the triangles (3.26, 3.28) are respectively monadic and comonadic. Of course, this agrees with the assumptions of the above proposition, since $B\otimes$- and -$\otimes C$ are the monad and comonad which give rise to the (co)monadic categories of modules and comodules.

Proposition 3.4.5. *If $\mathcal{V}$ is a symmetric monoidal closed category with equalizers, then f^* has a right adjoint for any monoid arrow f. Dually, if $\mathcal{V}$ has coequalizers and $\mathcal{V}^{\mathrm{op}}$ is monoidal closed, then $g_!$ has a left adjoint for any comonoid arrow g.*

Proof. There is a natural bijection

$$\mathbf{Mod}_{\mathcal{V}}(A)(f^*M,N)\cong\mathbf{Mod}_{\mathcal{V}}(B)(M,\mathrm{Hom}_A(B,N))$$

for any B-module M, A-module N and $f:A\to B$ monoid morphism. Thus we have an adjunction

$$\begin{tikzcd}
\mathbf{Mod}_{\mathcal{V}}(B) \arrow[r, shift left, "f^*"] \arrow[r, phantom, "\perp"] & \mathbf{Mod}_{\mathcal{V}}(A). \arrow[l, shift left, "\mathrm{Hom}_A(B,-)"]
\end{tikzcd}$$

The B-action on $\mathrm{Hom}_A(B,N)$ is the unique map induced by universality of the bottom equalizer

$$\begin{tikzcd}
B\otimes\mathrm{Hom}_A(B,M) \arrow[r, tail] \arrow[d, dashed, "\exists!\lambda_{\mathrm{Hom}_A(B,M)}"'] & B\otimes[B,M] \arrow[r, shift left, "1\otimes t"] \arrow[r, shift right, "1\otimes k"'] \arrow[d, "u"'] & B\otimes[A,[B,M]] \arrow[d, "v"] \\
\mathrm{Hom}_A(B,M) \arrow[r, tail] & {[B,M]} \arrow[r, shift left, "t"] \arrow[r, shift right, "k"'] & {[A,[B,M]],}
\end{tikzcd}$$

where u and v are adjuncts to composites of multiplication of B and evaluation. The left adjoint of $g_!$ is constructed dually. ☐

Obviously, the above sufficient conditions for the existence of adjoints for the corestriction of scalars are much less common to appear than the ones for the restriction. After all, for most interesting monoidal categories $\mathcal{V}$, their opposite $\mathcal{V}^{\mathrm{op}}$ is not monoidal closed.

In particular, for $\mathcal{V}=\mathbf{Mod}_R$ where R is a commutative ring, the situation is as follows.

Proposition 3.4.6. *The functor f^* for any R-algebra morphism $f:A\to B$ has a pair of adjoints*

$$\begin{tikzcd}
\mathbf{Mod}_B \arrow[r, "f^*" description] & \mathbf{Mod}_A. \arrow[l, bend right=40, "B\otimes_A-"', ""{name=U, below}] \arrow[l, bend left=40, "\mathrm{Hom}_A(B,-)", ""{name=D, above}]
\end{tikzcd}$$

with $\perp$ between each curved arrow and f^*.

On the other hand, the functor $g_!$ has a right adjoint for any R-coalgebra morphism $g:C\to D$, and also a left adjoint in certain cases, e.g. if g is between two finitely presentable projective R-coalgebras.

Proof. The symmetric monoidal closed category $\mathbf{Mod}_R$ has all limits and colimits, therefore Propositions 3.4.4 and 3.4.5 for the restriction of scalars apply and the respective adjoints are constructed as above.

Regarding the corestriction of scalars, the functor $C \otimes -$ does not in general preserve equalizers in $\mathbf{Mod}_R$ for any R-coalgebra C (except for flat coalgebras). Also $\mathbf{Mod}_R^{\mathrm{op}}$ is not a monoidal closed category, thus the above propositions do not apply in this case. However, since $g_!$ is cocontinuous and $\mathbf{Comod}_C$ is a locally presentable category, Theorem 3.0.1 can be applied instead, to give the existence of a right adjoint for any $g_!$. In particular, when C is a flat coalgebra, we can construct this adjoint as above:

$$\begin{tikzcd}
\mathbf{Comod}_C \arrow[r, bend left, "g_!"] \arrow[r, phantom, "\perp"] & \mathbf{Comod}_D. \arrow[l, bend left, "-\square_D C"]
\end{tikzcd}$$

Moreover, $\mathbf{Comod}_C$ is complete, well-powered and has a cogenerator as shown in [**Wis75**]. We can then apply the special adjoint functor theorem to obtain a right adjoint only when $g_!$ preserves all limits. For example, if the coalgebras C and D have duals in $\mathbf{Mod}_R$, the functors $- \otimes C$ and $- \otimes D$ preserve limits. Hence in the commutative triangle (3.28), the comonadic legs create all limits that the comonads preserve, hence $g_!$ is continuous. $\square$

CHAPTER 4

Enrichment

This chapter begins by presenting the most basic definitions and structures related to enriched category theory, largely following the standard book on the subject by Kelly [**Kel05**].

Then, a brief introduction to enriched bimodules is given, intended to clarify certain essential concepts of Chapter 7. The theory of bimodules (or distributors or profunctors) has been widely studied, and the notion of a distributor was first introduced by Lawvere. Here we restrict to the parts relevant to what follows, hence more emphasis is given on one-sided modules. Appropriate references are [**Bén73, Bor94a, DS97**], and also [**GS13**] where a theory of modules not between enriched categories but between *enriched bicategories* is developed.

In the last section, we give the definition of an action of a monoidal category on an ordinary category and we demonstrate in detail how a $\mathcal{V}$-representation may give rise to a $\mathcal{V}$-enriched category. This forms one direction of a correspondence between categories with an action from $\mathcal{V}$ with a certain adjoint and tensored $\mathcal{V}$-categories, for $\mathcal{V}$ a right closed monoidal category. In fact, the adjoint gives the hom-objects and the action gives the tensor of the enriched category. The main references are [**GP97, JK02**], and for example in [**McC00b**] the structure of the 2-category of $\mathcal{V}$-*actegories* (*i.e.* $\mathcal{V}$-representations) $\mathcal{V}$-**Act** is explored.

4.1. Basic definitions

Suppose that $(\mathcal{V}, \otimes, I, a, l, r)$ is a monoidal category. A $\mathcal{V}$-*enriched category* $\mathcal{A}$ consists of a set ob$\mathcal{A}$ of objects, a hom-object $\mathcal{A}(A,B) \in \mathcal{V}$ for each pair of objects of $\mathcal{A}$, a composition law

$$M : \mathcal{A}(B,C) \otimes \mathcal{A}(A,B) \to \mathcal{A}(A,C) \tag{4.1}$$

for each triple of objects, and an identity element $j_A : I \to \mathcal{A}(A,A)$ for each object, subject to the associativity and unit axioms expressed by the commutativity of

$$\begin{tikzcd}
(\mathcal{A}(C,D) \otimes \mathcal{A}(B,C)) \otimes \mathcal{A}(A,B) \arrow[rr, "a"] \arrow[d, "M\otimes 1"'] & & \mathcal{A}(C,D) \otimes (\mathcal{A}(B,C) \otimes \mathcal{A}(A,B)) \arrow[d, "1\otimes M"] \\
\mathcal{A}(B,D) \otimes \mathcal{A}(A,B) \arrow[dr, "M"'] & & \mathcal{A}(C,D) \otimes \mathcal{A}(A,C) \arrow[dl, "M"] \\
& \mathcal{A}(A,D), &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A}(B,B)\otimes\mathcal{A}(A,B) \arrow[r, "M"] & \mathcal{A}(A,B) & \mathcal{A}(A,B)\otimes\mathcal{A}(A,A) \arrow[l, "M"'] \\
I\otimes\mathcal{A}(A,B) \arrow[u, "j_B\otimes 1"] \arrow[ur, "l"'] & & \mathcal{A}(A,B)\otimes I. \arrow[u, "1\otimes j_A"'] \arrow[ul, "r"]
\end{tikzcd}$$

For example, **Set**-enriched categories are ordinary small categories, **Ab**-categories are additive categories, $\mathbf{Vect}_k$-categories are k-linear categories and **Cat**-enriched categories are 2-categories. The latter gives a different perspective of 2-category theory from the one presented in Chapter 2. Thus, in order to deal with 2-categories we can either employ the theory of bicategories or the theory of enriched categories.

Notice how in all the examples above, the *base* $\mathcal{V}$ of the enrichment is in fact enriched over itself: **Set** is an ordinary category, **Ab** is an additive category, $\mathbf{Vect}_k$ is a k-linear category and **Cat** is a 2-category. This is due to the fact that the base monoidal categories are closed, and the internal hom functor in any monoidal closed category $\mathcal{V}$

$$[-,-]:\mathcal{V}^{\mathrm{op}}\times\mathcal{V}\longrightarrow\mathcal{V}$$

induces an enrichment of the category over itself: the hom-object for $A,B\in\mathcal{V}$ is $[A,B]$, the composition law $M:[B,C]\otimes[A,B]\to[A,C]$ corresponds under the adjunction $-\otimes X\dashv[X,-]$ to the composite

$$[B,C]\otimes[A,B]\otimes A\xrightarrow{1\otimes\mathrm{ev}}[B,C]\otimes B\xrightarrow{\mathrm{ev}}C$$

and the identity $I\to[A,A]$ corresponds to $I\otimes A\xrightarrow{l_A}A$. It is a straightforward verification that these data indeed exhibit $\mathcal{V}$ as a $\mathcal{V}$-category.

If $\mathcal{A}$ is a $\mathcal{V}$-category for a symmetric monoidal category $\mathcal{V}$, then $\mathcal{A}^{\mathrm{op}}$ is also a $\mathcal{V}$-category called the *opposite* $\mathcal{V}$*-category*, with the same objects $\mathrm{ob}\mathcal{A}^{\mathrm{op}}=\mathrm{ob}\mathcal{A}$, and hom-objects $\mathcal{A}^{\mathrm{op}}(A,B):=\mathcal{A}(B,A)$. The composition law $\mathcal{A}^{\mathrm{op}}(B,C)\otimes\mathcal{A}^{\mathrm{op}}(A,B)\to\mathcal{A}^{\mathrm{op}}(A,C)$ is

$$\mathcal{A}(B,C)\otimes\mathcal{A}(B,A)\xrightarrow{s}\mathcal{A}(B,A)\otimes\mathcal{A}(C,B)\xrightarrow{M}\mathcal{A}(C,A)$$

and the identity elements $I\to\mathcal{A}^{\mathrm{op}}(A,A)$ are the same as in $\mathcal{A}$.

For $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, a $\mathcal{V}$*-functor* $F:\mathcal{A}\to\mathcal{B}$ between them consists of a function $F:\mathrm{ob}\mathcal{A}\to\mathrm{ob}\mathcal{B}$ together with a map

$$F_{AB}:\mathcal{A}(A,B)\to\mathcal{B}(FA,FB) \tag{4.2}$$

for each pair of objects in $\mathcal{A}$, subject to the commutativity of

$$\begin{tikzcd}
\mathcal{A}(B,C)\otimes\mathcal{A}(A,B) \arrow[r, "M"] \arrow[d, "F_{BC}\otimes F_{AB}"'] & \mathcal{A}(A,C) \arrow[d, "F_{AC}"] & I \arrow[r, "j_A"] \arrow[dr, "j_{FA}"'] & \mathcal{A}(A,A) \arrow[d, "F_{AA}"] \\
\mathcal{B}(FB,FC)\otimes\mathcal{B}(FA,FB) \arrow[r, "M"'] & \mathcal{B}(FA,FC), & & \mathcal{B}(FA,FA)
\end{tikzcd} \tag{4.3}$$

expressing the compatibility of F with composition and identities.

The notion of an enriched functor in the context of the examples above becomes respectively an ordinary functor, an additive functor, a k-linear functor and a 2-functor. Clearly the composite of two composable $\mathcal{V}$-functors is again a $\mathcal{V}$-functor, and the composition is associative and unital with $\mathbf{1}_{\mathcal{A}}$ the identity $\mathcal{V}$-functor.

If $\mathcal{V}$ is symmetric monoidal closed, then for any $\mathcal{V}$-category $\mathcal{A}$ the assignment $(A, B) \mapsto \mathcal{A}(A, B)$ is in fact the object function of a $\mathcal{V}$-functor of two variables

$$\mathrm{Hom}_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V} \tag{4.4}$$

where $\mathcal{V}$ is regarded as a $\mathcal{V}$-category via the internal hom. Its partial functors are the covariant and the contravariant *representable* $\mathcal{V}$-functors $\mathrm{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \to \mathcal{V}$, $\mathrm{Hom}_{\mathcal{A}}(-, B) : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$. For example, the former sends $B \in \mathrm{ob}\mathcal{A}$ to $\mathcal{A}(A, B) \in \mathcal{V}$, and on hom-objects it consists of arrows

$$\mathrm{Hom}_{\mathcal{A}}(A, -)_{BC} : \mathcal{A}(B, C) \to [\mathcal{A}(A, B), \mathcal{A}(A, C)]$$

which correspond to the composition arrows under $(- \otimes A) \dashv [A, -]$.

For $\mathcal{V}$-functors $F, G : \mathcal{A} \to \mathcal{B}$, a $\mathcal{V}$-*natural transformation* $\tau : F \Rightarrow G$ consists of an $\mathrm{ob}\mathcal{A}$-indexed family of components $\tau_A : I \to \mathcal{B}(FA, GA)$ satisfying the $\mathcal{V}$-naturality condition expressed by the commutativity of

$$\begin{tikzcd}
& I \otimes \mathcal{A}(A,B) \arrow[rr, "\tau_B \otimes F_{AB}"] & & \mathcal{B}(FB,GB) \otimes \mathcal{B}(FA,FB) \arrow[dr, "M"] & \\
\mathcal{A}(A,B) \arrow[ur, "l^{-1}"] \arrow[dr, "r^{-1}"'] & & & & \mathcal{B}(FA,GB). \\
& \mathcal{A}(A,B) \otimes I \arrow[rr, "G_{AB} \otimes \tau_A"'] & & \mathcal{B}(GA,GB) \otimes \mathcal{B}(FA,GA) \arrow[ur, "M"'] &
\end{tikzcd}$$

It is not hard to see that $\mathcal{V}$-natural transformations compose both vertically and horizontally, in a very similar way to ordinary natural transformations. Thus (small) $\mathcal{V}$-categories, $\mathcal{V}$-functors and a $\mathcal{V}$-natural transformations constitute a 2-category, which is denoted by $\mathcal{V}$-**Cat**.

When $\mathcal{V}$ is a symmetric monoidal category, we can define a tensor product in $\mathcal{V}$-**Cat**: $\mathcal{A} \otimes \mathcal{B}$ has objects $\mathrm{ob}\mathcal{A} \times \mathrm{ob}\mathcal{B}$, hom-objects

$$(\mathcal{A} \otimes \mathcal{B})((A, B), (A', B')) := \mathcal{A}(A, A') \otimes \mathcal{B}(B, B'),$$

the composition law is the composite

$$\begin{tikzcd}
\mathcal{A}(A',A'') \otimes \mathcal{B}(B',B'') \otimes \mathcal{A}(A,A') \otimes \mathcal{B}(B,B'') \arrow[r, dashed] \arrow[d, "1\otimes s\otimes 1"'] & \mathcal{A}(A,A'') \otimes \mathcal{B}(B,B'') \\
\mathcal{A}(A',A'') \otimes \mathcal{A}(A,A') \otimes \mathcal{B}(B',B'') \otimes \mathcal{B}(B,B'') \arrow[ur, "M\otimes M"'] &
\end{tikzcd}$$

and the identity element is

$$I \xrightarrow{\sim} I \otimes I \xrightarrow{j_A \otimes j_B} \mathcal{A}(A, A) \otimes \mathcal{B}(B, B).$$

The axioms are satisfied so $\mathcal{A}\otimes\mathcal{B}$ is a $\mathcal{V}$-category. The tensor product of $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations can also be defined accordingly, so that we obtain a 2-functor $\otimes : \mathcal{V}\text{-}\mathbf{Cat}\times\mathcal{V}\text{-}\mathbf{Cat}\to\mathcal{V}\text{-}\mathbf{Cat}$. The unit $\mathcal{I}$ is the *unit $\mathcal{V}$-category* with one object $*$ and $\mathcal{I}(*,*)=I$. Hence with the appropriate constraints, $\mathcal{V}\text{-}\mathbf{Cat}$ is a *monoidal 2-category* (for a formal definition, see for example [**BN96**]). Also, it has a symmetry $s_{\mathcal{A},\mathcal{B}} : \mathcal{A}\otimes\mathcal{B}\cong\mathcal{B}\otimes\mathcal{A}$ which renders it a symmetric monoidal 2-category.

There is the so-called 'underlying category functor'

$$(-)_0 : \mathcal{V}\text{-}\mathbf{Cat}\to\mathbf{Cat}$$

which maps the $\mathcal{V}$-category $\mathcal{A}$ to the ordinary category $\mathcal{A}_0=\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{I},\mathcal{A})$, the *underlying category* of the enriched $\mathcal{A}$. Explicitly, $\mathcal{A}_0$ has the same objects as the $\mathcal{V}$-enriched $\mathcal{A}$, while a map $f:A\to B$ in $\mathcal{A}_0$ is an *element* $f:I\to\mathcal{A}(A,B)$ of $\mathcal{A}(A,B)$, *i.e.* $\mathcal{A}_0(A,B)=\mathcal{V}(I,\mathcal{A}(A,B))$ as sets. There are appropriate definitions for the underlying $\mathcal{V}$-functor and underlying $\mathcal{V}$-natural transformation. The amount of information lost in the passage from enriched categories to their underlying categories depends very much on the base $\mathcal{V}$. In particular, how much information about $\mathcal{A}$ is retained by the underlying $\mathcal{A}_0$ depends on 'how faithful' the functor $\mathcal{V}(I,-)$ is.

We saw earlier that if $\mathcal{A}$ is enriched in a symmetric monoidal closed category $\mathcal{V}$, there is a $\mathcal{V}$-functor $\mathrm{Hom}_{\mathcal{A}}$ as in (4.4) which gives the hom-objects of the enrichment. There is also an ordinary functor between the underlying categories

$$\begin{aligned}\mathcal{A}(-,-):\ \mathcal{A}_0^{\mathrm{op}}\times\mathcal{A}_0 &\longrightarrow \mathcal{V}\\ (A,B) &\longmapsto \mathcal{A}(A,B)\end{aligned}\tag{4.5}$$

sometimes called the *enriched hom-functor*, which maps a pair of arrows $(A'\xrightarrow{f}A, B\xrightarrow{g}B')$ in $\mathcal{A}_0^{\mathrm{op}}\times\mathcal{A}_0$ to the top arrow

$$\begin{tikzcd}
\mathcal{A}(A,B) \arrow[d, "r^{-1}"'] \arrow[rr, dashed, "\mathcal{A}(f,g)"] & & \mathcal{A}(A',B'). \\
\mathcal{A}(A,B)\otimes I \arrow[d, "1\otimes f"'] & & \mathcal{A}(B,B')\otimes\mathcal{A}(A',B) \arrow[u, "M"'] \\
\mathcal{A}(A,B)\otimes\mathcal{A}(A',A) \arrow[r, "M"'] & \mathcal{A}(A',B) \arrow[r, "l^{-1}"'] & I\otimes\mathcal{A}(A',B) \arrow[u, "g\otimes 1"']
\end{tikzcd}$$

This functor is evidently the composite

$$\mathcal{A}_0^{\mathrm{op}}\times\mathcal{A}_0\longrightarrow(\mathcal{A}^{\mathrm{op}}\otimes\mathcal{A})_0\xrightarrow{(\mathrm{Hom}_{\mathcal{A}})_0}\mathcal{V}_{(0)}$$

where the first arrow is a canonical functor relating the two underlying categories. Notice how this functor $\mathcal{A}(-,-)$, unlike $\mathrm{Hom}_{\mathcal{A}}$, can be defined for categories enriched in any monoidal category $\mathcal{V}$, without further conditions on it.

Speaking loosely, we say that an ordinary category $\mathcal{C}$ is enriched in a monoidal category $\mathcal{V}$ when we have a $\mathcal{V}$-enriched category $\mathcal{A}$ and an isomorphism $\mathcal{A}_0\cong\mathcal{C}$. Consequently, *to be enriched in $\mathcal{V}$* is not a property, but additional structure. Of course, a given ordinary category may be enriched in more than one monoidal categories: that is evident in view of the change of base described below. But also, a category $\mathcal{C}$ may be enriched in $\mathcal{V}$ in more than one way, so that there may be many different $\mathcal{V}$-categories with the same underlying ordinary category.

PROPOSITION 4.1.1. *Suppose $F : \mathcal{V} \to \mathcal{W}$ is a lax monoidal functor between two monoidal categories. There is an induced 2-functor*

$$\tilde{F} : \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{W}\text{-}\mathbf{Cat}$$

between the 2-categories of $\mathcal{V}$ and $\mathcal{W}$-enriched categories, which maps any $\mathcal{V}$-category $\mathcal{A}$ to a $\mathcal{W}$-category with the same objects as $\mathcal{A}$ and hom-objects $F\mathcal{A}(A,B)$.

PROOF. Given a $\mathcal{V}$-category $\mathcal{A}$, the $\mathcal{W}$-category $\tilde{F}\mathcal{A}$ has objects ob$(\tilde{F}\mathcal{A})$ =ob$\mathcal{A}$ and hom-objects $(\tilde{F}\mathcal{A})(A,B) = F(\mathcal{A}(A,B)) \in \mathcal{W}$. The composition and the identities are given by

$$\begin{tikzcd}
F\mathcal{A}(B,C) \otimes F\mathcal{A}(A,B) \arrow[r, dashed] \arrow[d, "\phi_{\mathcal{A}(B,C),\mathcal{A}(A,B)}"'] & F\mathcal{A}(A,C) \\
F(\mathcal{A}(B,C) \otimes \mathcal{A}(A,B)) \arrow[ur, "FM"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
I_{\mathcal{W}} \arrow[r, dashed] \arrow[d, "\phi_0"'] & F\mathcal{A}(A,A) \\
FI_{\mathcal{V}} \arrow[ur, "Fj_A"'] &
\end{tikzcd}$$

where ϕ, ϕ_0 are the structure maps of the lax monoidal functor F. It can be checked that the diagrams of associativity and identities commute, therefore $\tilde{F}\mathcal{A}$ is a $\mathcal{W}$-category.

If $K : \mathcal{A} \to \mathcal{B}$ is a $\mathcal{V}$-functor with maps $K_{AB} : \mathcal{A}(A,B) \to \mathcal{B}(KA,KB)$ in $\mathcal{V}$ for every pair of objects in $\mathcal{A}$, we can form a $\mathcal{W}$-functor

$$\tilde{F}K : \tilde{F}\mathcal{A} \to \tilde{F}\mathcal{B}$$

with the same function on objects, and for every pair of objects in $\tilde{F}\mathcal{A}$ a map

$$(\tilde{F}K)_{AB} : F\mathcal{A}(A,B) \xrightarrow{F(K_{AB})} F\mathcal{B}(KA,KB)$$

in $\mathcal{W}$, such that the axioms of a $\mathcal{W}$-functor are satisfied.

If $\tau : K \Rightarrow L$ is a $\mathcal{V}$-natural transformation between $\mathcal{V}$-functors $K, L : \mathcal{A} \to \mathcal{B}$, its image $\tilde{F}\tau : \tilde{F}K \Rightarrow \tilde{F}L$ consists of an ob$(\tilde{F}\mathcal{A})$-indexed family of components

$$\begin{tikzcd}
I_{\mathcal{W}} \arrow[r, dashed, "\tilde{F}\tau_A"] \arrow[d, "\phi_0"'] & F\mathcal{B}(KA,LA) \\
FI_{\mathcal{V}} \arrow[ur, "F\tau_A"'] &
\end{tikzcd}$$

in $\mathcal{W}$, which satisfy the $\mathcal{W}$-naturality condition in a straightforward way. ☐

Another standard example of enrichment is the usual functor category between two $\mathcal{V}$-categories, which under specific assumptions on $\mathcal{V}$ obtains a $\mathcal{V}$-enriched structure itself. Explicitly, when $\mathcal{V}$ is a symmetric monoidal closed category with all limits, we can define the *enriched functor category* $[\mathcal{A}, \mathcal{B}]$ with objects $\mathcal{V}$-functors

$\mathcal{A}\to\mathcal{B}$, and hom-object $[\mathcal{A},\mathcal{B}](F,G)$ for any two $\mathcal{V}$-functors F, G the following end

$$\textstyle\int_{A\in\mathcal{A}}\mathcal{B}(FA,GA)\rightarrowtail\prod_{A\in\mathcal{A}}\mathcal{B}(FA,GA)\rightrightarrows\prod_{A,A'\in\mathcal{A}}[\mathcal{A}(A,A'),\mathcal{B}(FA,GA')]$$

constructed in detail in [**Kel05**, 2.1]. It is not hard to define a composition law and identity elements for the functor category, and the axioms which make $[\mathcal{A},\mathcal{B}]$ into $\mathcal{V}$-category follow from the corresponding axioms for $\mathcal{B}$.

It can be deduced that, when $\mathcal{V}$ has the above mentioned properties, the functor

$$-\otimes\mathcal{A}:\mathcal{V}\text{-}\mathbf{Cat}\to\mathcal{V}\text{-}\mathbf{Cat}$$

in the monoidal category $\mathcal{V}$-**Cat** has $[\mathcal{A},-]$ as its right adjoint, with a (2-)natural isomorphism

$$\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A}\otimes\mathcal{B},\mathcal{C})\cong\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A},[\mathcal{B},\mathcal{C}])$$

for any $\mathcal{V}$-categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. Therefore the symmetric monoidal 2-category $\mathcal{V}$-**Cat** has also a closed structure.

4.2. $\mathcal{V}$-enriched bimodules and modules

As mentioned in Examples 2.1.2 of bicategories, there is a bicategory of bimodules **BMod** where 1-cells are abelian groups M which are left R-modules and right S-modules for rings R,S, such that the two actions are compatible. More explicitly, these actions yield group homomorphisms

$$R\otimes M\xrightarrow{\cdot}M,\quad M\otimes S\xrightarrow{\cdot}M$$

such that $(r\cdot m)\cdot s=r\cdot(m\cdot s)$ for all $r\in R$, $s\in S$ and $m\in M$. Morphisms between them are group homomorphisms $f:M\to N$ which respect the R and S-actions. These data define a category of (R,S)-bimodules and bimodule maps between them, which is furthermore an additive category, *i.e.* each ${}_R\mathrm{Hom}_S(M,N)$ is an abelian group.

There is another equivalent formulation of these definitions, which make it easier to obtain a generalization to $\mathcal{V}$-enriched modules. Recall that a ring is essentially the same as an **Ab**-category with only one object, in the sense that the underlying additive group of the ring is the single hom-object and the multiplication of the ring is composition law. Then an (R,S)-bimodule can be regarded as an additive functor

$$S^{\mathrm{op}}\otimes R\to\mathbf{Ab}$$

where the opposite ring S^{op} has reversed multiplication. Equivalently, this amounts to an additive functor

$$R\to[S^{\mathrm{op}},\mathbf{Ab}].$$

In these terms, a bimodule map is an additive natural transformation between the respective additive functors.

The next step, since **Rng**=**Mon**(**Ab**), would be to consider bimodules for monoids in an arbitrary monoidal category $\mathcal{V}$. The definitions that arise are precisely the ones which were employed in Section 3.4 in order to study the existence of adjoints for the restriction of scalars. By analogy with ring bimodules, an (A, B)-bimodule M for monoids A and B in a symmetric monoidal closed $\mathcal{V}$ can also be expressed as a $\mathcal{V}$-functor

$$M : B^{\mathrm{op}} \otimes A \to \mathcal{V}$$

where the monoids A and B are viewed as one-object $\mathcal{V}$-categories, in the same way as rings were viewed as one-object additive categories.

Even more generally, we can consider left $\mathcal{A}$-/right $\mathcal{B}$-bimodules for general $\mathcal{V}$-categories $\mathcal{A}$, $\mathcal{B}$. Hence, define a *$\mathcal{V}$-bimodule* M to be a $\mathcal{V}$-functor

$$\mathcal{B}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{V} \tag{4.6}$$

for $\mathcal{V}$ a symmetric monoidal closed category, where the opposite category $\mathcal{B}^{\mathrm{op}}$ is $\mathcal{V}$-enriched by symmetry of $\mathcal{V}$, the tensor product in $\mathcal{V}$-**Cat** was defined in the previous section and $\mathcal{V}$ is enriched in itself via the internal hom. Equivalently, if $\mathcal{V}$ is moreover complete, a $(\mathcal{A}, \mathcal{B})$-bimodule is a $\mathcal{V}$-functor $\mathcal{A} \to [\mathcal{B}^{\mathrm{op}}, \mathcal{V}]$ using the monoidal closed structure of $\mathcal{V}$-**Cat**. Bimodules (enriched in **Set**) are also called *profunctors* or *distributors*. Maps of enriched bimodules are evidently defined as $\mathcal{V}$-natural transformations and are called *$\mathcal{V}$-bimodule maps*.

There is an alternative, more intuitive formulation of the definition of a $\mathcal{V}$-module (see [**Law73**]) which is closer to the initial notion of ordinary bimodules. More specifically, an $(\mathcal{A}, \mathcal{B})$-bimodule M is given by a family of objects $M(B, A) \in \mathcal{V}$ for all $(B, A) \in \mathrm{ob}\mathcal{A} \times \mathrm{ob}\mathcal{B}$, together with arrows

$$\mathcal{A}(A, A') \otimes M(B, A) \to M(A', B)$$
$$M(B, A) \otimes \mathcal{B}(B', B) \to M(A, B')$$

in $\mathcal{V}$, which satisfy usual axioms and are compatible with each other. A detailed presentation of the diagrams involved can be found for example in [**Car95**] and [**GS13**], and the equivalence between these two definitions of $\mathcal{V}$-bimodules is easily verified.

Regarding the maps between them, a $\mathcal{V}$-bimodule map $\alpha : M \to M'$ between two left $\mathcal{A}$-/right $\mathcal{B}$ bimodules consists of a family of arrows

$$\alpha_{A,B} : M(B, A) \to M'(B, A)$$

in $\mathcal{V}$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, which respect the $\mathcal{A}$ and $\mathcal{B}$-actions. These can be composed in an evident way, thus we have a category of $(\mathcal{A}, \mathcal{B})$-bimodules for any $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{V}$-$\mathbf{BMod}(\mathcal{A}, \mathcal{B})$ or $\mathcal{V}$-${}_{\mathcal{A}}\mathbf{Mod}_{\mathcal{B}}$. Notice that for the second characterization of $\mathcal{V}$-bimodules, we do not need any extra assumptions on the monoidal category $\mathcal{V}$.

Back to the example of ordinary bimodules, an important feature is the fact that there is a ‘composition’ operation, by taking the tensor product of bimodules over a ring. More precisely, given an (R, S)-bimodule M and a (S, T)-bimodule N, their tensor product $M \otimes_S N$ obtains a structure of a (R, T)-bimodule. Having in mind that bimodules constitute the 1-cells in the bicategory **BMod**, if we denote them as $M : R \rightarrow\!\!\!\!\!\!\mapsto S$ and $N : S \rightarrow\!\!\!\!\!\!\mapsto T$ so that they are also distinguished from ring homomorphisms, this process can be written

$$NM = M \otimes_S N : \ R \overset{M}{\relbar\joinrel\mapstochar\joinrel\rightarrow} S \overset{N}{\relbar\joinrel\mapstochar\joinrel\rightarrow} T \ ,$$

and by the canonical isomorphisms

$$(M \otimes_S N) \otimes_T L \cong M \otimes_S (N \otimes_T L), \quad M \otimes_S S \cong M$$

for a (T, V)-bimodule L, it is clear that this tensor product satisfies associativity and identity laws up to isomorphism.

In order to generalize the composition of ordinary bimodules to the enriched case, we will use the expression of the tensor product of modules over a ring as a coequalizer. For bimodules in a general monoidal category $\mathcal{V}$, this is expressed precisely by the construction (3.29) of the tensor product of a right B-bimodule and a left B-bimodule over a monoid B.

For this operation to be accordingly defined in the level of enriched bimodules, and for the collection of $\mathcal{V}$-bimodules and bimodule morphisms between two $\mathcal{V}$-categories to obtain the structure of a $\mathcal{V}$-enriched category itself, the base category $\mathcal{V}$ is requested to be a complete and cocomplete symmetric monoidal closed category. Based on the above idea, for two $\mathcal{V}$-bimodules $M : \mathcal{A} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{B}$ and $N : \mathcal{B} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{C}$ we define their composite $N \circ M : \mathcal{A} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{C}$ by specifying its components by the following coequalizer:

$$\begin{tikzcd}
\sum_{B,B' \in \mathcal{B}} M(B',A) \otimes \mathcal{B}(B,B') \otimes N(C,B) \arrow[r, shift left] \arrow[r, shift right] & \sum_{B \in \mathcal{B}} M(B,A) \otimes N(C,B) \arrow[r, two heads] & (N \circ M)(C,A).
\end{tikzcd}$$

The parallel arrows come from the $\mathcal{B}$-actions on M and N. This definition in fact exhibits the composite as the coend

$$N \circ M = \int_{B \in \mathcal{B}} M(B, -) \otimes N(-, B),$$

which inherits a left $\mathcal{A}$-action and a right $\mathcal{C}$-action, and we also write $(N \circ M)(C, A) = M(B, A) \otimes_{\mathcal{B}} N(C, B)$. This operation can be verified to be associative and unitary up to isomorphism by the associativity, left and right unit constraints of the monoidal category $\mathcal{V}$. So the above data indeed define a bicategory $\mathcal{V}$-**BMod** (or $\mathcal{V}$-**Dist** or $\mathcal{V}$-**Prof**) with objects $\mathcal{V}$-categories, 1-cells $\mathcal{V}$-bimodules and 2-cells $\mathcal{V}$-bimodule maps.

Bimodules can also be thought of as generalized $\mathcal{V}$-functors between $\mathcal{V}$-categories, considered as '$\mathcal{V}$-valued relations' between them (as in Lawvere's [**Law73**]). In particular, every $\mathcal{V}$-functor $F : \mathcal{A} \to \mathcal{B}$ gives rise to bimodules $F_* : \mathcal{A} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{B}$ and $F^* : \mathcal{B} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{A}$ defined by

$$F_*(B, A) = \mathcal{B}(B, FA), \quad F^*(A, B) = \mathcal{B}(FA, B). \tag{4.7}$$

This structure implies that $\mathcal{V}$-**BMod** fits in the context of the final Section 8.2.

For the purposes of this thesis, we are more interested in the categories of one-sided modules, *i.e.* left $\mathcal{A}$-modules or right $\mathcal{B}$-modules for $\mathcal{V}$-categories $\mathcal{A}$ or $\mathcal{B}$. We follow the second formulation of the definition of $\mathcal{V}$-modules, which does not require extra conditions on $\mathcal{V}$.

DEFINITION 4.2.1. A *left $\mathcal{A}$-module* Ψ, also written as $\Psi : \mathcal{A} \twoheadrightarrow \mathcal{I}$, is given by objects ΨA in $\mathcal{V}$ for each $A \in \mathcal{A}$ and arrows

$$\mu : \mathcal{A}(A, A') \otimes \Psi A \to \Psi A'$$

in $\mathcal{V}$ for each $A, A' \in \mathcal{A}$, subject to the commutativity of

$$\begin{tikzcd}
\mathcal{A}(A', A'') \otimes \mathcal{A}(A, A') \otimes \Psi A \arrow[r, "1\otimes\mu"] \arrow[d, "M\otimes 1"'] & \mathcal{A}(A', A'') \otimes \Psi A' \arrow[d, "\mu"] \\
\mathcal{A}(A, A'') \otimes \Psi A \arrow[r, "\mu"'] & \Psi A'',
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{A}(A, A) \otimes \Psi A \arrow[dr, "\mu"] & \\
\Psi A \arrow[ur, "j_A"] \arrow[rr, "1_{\Psi A}"'] & & \Psi A.
\end{tikzcd}$$

The arrows M and j_A are the composition and identity element in $\mathcal{V}$, and the associativity and identity constraints are suppressed. If $\Xi : \mathcal{A} \twoheadrightarrow \mathcal{I}$ is another left $\mathcal{A}$-module, then a *left module morphism* $\alpha : \Psi \to \Xi$ is given by an ob$\mathcal{A}$-indexed family

$$\alpha_A : \Psi A \to \Xi A$$

of arrows in $\mathcal{V}$, satisfying the commutativity of

$$\begin{tikzcd}
\mathcal{A}(A, A') \otimes \Psi A \arrow[r, "1\otimes\alpha"] \arrow[d, "\mu"'] & \mathcal{A}(A, A') \otimes \Xi A \arrow[d, "\mu"] \\
\Psi A' \arrow[r, "\alpha"'] & \Xi A'
\end{tikzcd}$$

for all $A, A' \in \mathcal{A}$.

The category of left $\mathcal{A}$-modules is denoted by $\mathcal{V}$-**Mod**(A) or $\mathcal{V}$-${}_{\mathcal{A}}$**Mod**. Dually, we can define the category of *right $\mathcal{B}$-modules* $\mathcal{V}$-**Mod**$_{\mathcal{B}}$, with objects $\mathcal{I} \twoheadrightarrow \mathcal{B}$.

We could define a left $\mathcal{A}$-module Ψ to be a $\mathcal{V}$-functor $\Psi : \mathcal{A} \to \mathcal{V}$ and a right $\mathcal{B}$-module Ξ to be a $\mathcal{V}$-functor $\Xi : \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$. This agrees with (4.6), since of course $\mathcal{A} \otimes \mathcal{I} \cong \mathcal{A}$ and $\mathcal{B}^{\mathrm{op}} \otimes \mathcal{I} \cong \mathcal{B}^{\mathrm{op}}$ for the unit $\mathcal{V}$-category $\mathcal{I}$, but that would require extra structure on $\mathcal{V}$ as clarified earlier. We would then be able to identify the categories $\mathcal{V}$-${}_{\mathcal{A}}$**Mod** and $\mathcal{V}$-**Mod**$_{\mathcal{B}}$ with the presheaf categories $[\mathcal{A}, \mathcal{V}]$ and $[\mathcal{B}^{\mathrm{op}}, \mathcal{V}]$ of $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations. Via this presentation, many useful properties are inherited from $\mathcal{V}$, such as completeness, cocompleteness (obtained pointwise) and local presentability: for any locally λ-presentable category $\mathcal{C}$ and small category $\mathcal{A}$, the functor category $\mathcal{C}^{\mathcal{A}} = [\mathcal{A}, \mathcal{C}]$ is locally λ-presentable itself by [**AR94**, 1.54].

Notice that the above concepts are evidently associated with the general notion of a module (or bimodule) for a monad in a bicategory, as described in Section 2.2. This relation will be illustrated at the last sections of Chapter 7, in the formal context of the bicategory of $\mathcal{V}$-matrices $\mathcal{V}$-**Mat**.

4.3. Actions of monoidal categories and enrichment

We now recall some parts of the general theory of actions of monoidal categories, leading to specific enrichments. We largely follow [**JK02**] by Janelidze and Kelly, adding some details.

An *action* of a monoidal category $\mathcal{V} = (\mathcal{V}, \otimes, I, a, l, r)$ on an ordinary category $\mathcal{D}$ is given by a functor

$$\begin{tikzcd} * : \mathcal{V} \times \mathcal{D} \arrow[r] & \mathcal{D} \\ (X, D) \arrow[r, mapsto] & X * D \end{tikzcd}$$

with a natural isomorphism with components $\chi_{XYD} : (X \otimes Y) * D \xrightarrow{\sim} X * (Y * D)$ and a natural isomorphism with components $\nu_D : I * D \xrightarrow{\sim} D$, satisfying the commutativity of the diagrams

$$\begin{tikzcd} ((X \otimes Y) \otimes Z) * D \arrow[r, "\chi"] \arrow[d, "a*1"'] & (X \otimes Y) * (Z * D) \arrow[r, "\chi"] & X * (Y * (Z * D)) \\ (X \otimes (Y \otimes Z)) * D \arrow[rr, "\chi"'] & & X * ((Y \otimes Z) * D), \arrow[u, "1*\chi"'] \end{tikzcd} \tag{4.8}$$

$$\begin{tikzcd} (I \otimes X) * D \arrow[rr, "\chi"] \arrow[dr, "l*1"'] & & I * (X * D) \arrow[dl, "\nu"] \\ & X * D, & \end{tikzcd}$$

$$\begin{tikzcd} (X \otimes I) * D \arrow[rr, "\chi"] \arrow[dr, "r*1"'] & & X * (I * D) \arrow[dl, "1*\nu"] \\ & X * D. & \end{tikzcd}$$

Notice that if $*$ is an action, then the opposite functor $*^{\mathrm{op}} : \mathcal{V}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}} \longrightarrow \mathcal{D}^{\mathrm{op}}$ is still an action: the corresponding natural isomorphisms are χ^{-1} and ν^{-1} and the action axioms follow from those for $*$.

For example, any monoidal category $\mathcal{V}$ has a canonical action on itself, by taking

$$* = \otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$$

and $\chi = a$, $\nu = l$ the monoidal constraints. This is sometimes called the *regular representation* of $\mathcal{V}$.

Remark 4.3.1. (*i*) In Bénabou's [**Bén67**], a very inspiring characterization of actions is provided, connecting the notion with a bicategorical construction. More specifically, it is asserted that a (left) action of a monoidal category $\mathcal{V}$ (*multiplicative* category in the terminology therein) on any category $\mathcal{A}$ can be identified with a bicategory $\mathcal{K}$ with only two objects $\{0, 1\}$ and hom-categories

$$\mathcal{K}(0,0) = \mathbf{1},\ \mathcal{K}(1,0) = \emptyset,\ \mathcal{K}(0,1) = \mathcal{A},\ \mathcal{K}(1,1) = \mathcal{V}.$$

The horizontal composition for the possible combinations of the objects $0, 1$ gives the tensor product $\otimes = \circ_{1,1,1}$ of $\mathcal{V}$ and the action $* = \circ_{0,1,1}$ on $\mathcal{A}$, the associativity and identity constraints give the monoidal constraints and the action structure transformations, whereas the coherence conditions correspond to the appropriate axioms.

In particular, the canonical action of any monoidal category $\mathcal{V}$ on itself gives rise to a bicategory $\mathcal{MV}$ with two objects as above, and hom-categories $\mathcal{MV}(0,0) = \mathbf{1}$,

$\mathcal{MV}(1,0)=\emptyset$, $\mathcal{MV}(0,1)=\mathcal{MV}(1,1)=\mathcal{V}$. This bicategory will be used for a certain description of global categories of modules and comodules in Chapter 6.

(*ii*) A *pseudomonoid* in a monoidal category is an object with multiplication and unit as defined in Section 3.3, for which the diagrams (3.13) commute up to coherent isomorphism. For example, a pseudomonoid in the cartesian monoidal category $(\mathbf{Cat},\times,\mathbf{1})$ is precisely a monoidal category $\mathcal{V}$, with multiplication being the tensor product and unit picking the unit object.

Furthermore, a *pseudomodule* for a pseudomonoid in a monoidal category is again defined as in Section 3.4, where the diagrams (3.20) commute up to isomorphism, satisfying coherence axioms. From this point of view, the action of a monoidal category described above is a *pseudoaction* of a pseudomonoid on an object of the monoidal **Cat**, *i.e.* $\mathcal{D}$ is a $\mathcal{V}$-pseudomodule. More on this viewpoint will be discussed in the final chapter.

Another example of an action which will be used repeatedly is the following.

Lemma 4.3.2. *Suppose $\mathcal{V}$ is a symmetric monoidal closed category. The internal hom*

$$[-,-]:\mathcal{V}^{\mathrm{op}}\times\mathcal{V}\longrightarrow\mathcal{V}$$

constitutes an action of the monoidal category $\mathcal{V}^{\mathrm{op}}$ on the category $\mathcal{V}$, via the standard natural isomorphisms

$$\chi_{XYZ}:[X\otimes Y,D]\xrightarrow{\sim}[X,[Y,Z]]$$
$$\nu_D:[I,D]\xrightarrow{\sim}D.$$

Moreover, the induced functor

$$\mathbf{Mon}[-,-]:\mathbf{Comon}(\mathcal{V})^{\mathrm{op}}\times\mathbf{Mon}(\mathcal{V})\longrightarrow\mathbf{Mon}(\mathcal{V})$$

is an action of the monoidal category $\mathbf{Comon}(\mathcal{V})^{\mathrm{op}}$ on the category $\mathbf{Mon}(\mathcal{V})$.

Proof. The isomorphisms χ_{XYZ},ν_D can be verified to satisfy the axioms (4.8) using the transpose diagrams under the adjunction $(-\otimes Y)\dashv[Y,-]$. Moreover, the functor $\mathbf{Mon}[-,-]$ induced between the categories of comonoids and monoids as in (3.17) is just a restriction of $[-,-]$. Hence the natural isomorphisms in $\mathbf{Mon}(\mathcal{V})$, reflected by the conservative forgetful functor $S:\mathbf{Mon}(\mathcal{V})\to\mathcal{V}$, render $\mathbf{Mon}[-,-]$ into an action too. Recall that $\mathbf{Comon}(\mathcal{V})$ and its opposite are monoidal since $\mathcal{V}$ is symmetric. □

For our purposes, it is a very important fact that given a category $\mathcal{D}$ along with an action of a monoidal category $\mathcal{V}$ with a parametrized adjoint, we obtain a $\mathcal{V}$-enriched category. In fact, this follows from a much stronger result of [**GP97**] regarding categories enriched in bicategories, as mentioned in the introduction.

Theorem 4.3.3. *Suppose that $\mathcal{V}$ is a monoidal category which acts on a category $\mathcal{D}$ via a functor $*:\mathcal{V}\times\mathcal{D}\to\mathcal{D}$ such that $-*D$ has a right adjoint $F(D,-)$ for* *every $D \in \mathcal{D}$, with a natural isomorphism*

$$\mathcal{D}(X * D, E) \cong \mathcal{V}(X, F(D,E)). \tag{4.9}$$

Then we can enrich $\mathcal{D}$ in $\mathcal{V}$, in the sense that there is a $\mathcal{V}$-category with the same objects as $\mathcal{D}$, hom-objects $F(A,B)$ for $A, B \in \mathrm{ob}\mathcal{D}$ and underlying category $\mathcal{D}$.

Proof. Suppose we have an adjunction

$$\begin{tikzcd} \mathcal{V} \arrow[r, bend left, "-*D"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "{F(D,-)}"] \end{tikzcd} \tag{4.10}$$

for every object D in $\mathcal{D}$, where $*$ is the action of $\mathcal{V}$ on $\mathcal{D}$. This implies that there is a unique way of defining a functor of two variables

$$F : \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathcal{V}$$

such that the isomorphism (4.9) is natural in all three variables, *i.e.* F is the parametrized adjoint of $(- * -)$. We are going to show in detail how these data induce an enrichment of $\mathcal{D}$ in $\mathcal{V}$, with enriched hom the functor F.

The composition law is the arrow $M : F(B,C) \otimes F(A,B) \to F(A,C)$ which corresponds uniquely under the adjunction $- * D \dashv F(D,-)$ to the composite

$$\begin{tikzcd} (F(B,C) \otimes F(A,B)) * A \arrow[r, dashed] \arrow[d, "\chi_{F(B,C),F(A,B),A}"'] & C \\ F(B,C) * (F(A,B) * A) \arrow[r, "1*\varepsilon^A_B"'] & F(B,C) * B \arrow[u, "\varepsilon^B_C"'] \end{tikzcd} \tag{4.11}$$

where ε is the counit of the adjunction (4.10). The identity element is the morphism $j_A : I \to F(A,A)$ which corresponds uniquely to the isomorphism

$$I * A \xrightarrow{\nu_A} A. \tag{4.12}$$

The associativity axiom diagram translates under the adjunction to the following diagram in $\mathcal{D}$

$$\begin{tikzcd} ((F(C,D) \otimes F(B,C)) \otimes F(A,B)) * A \arrow[rr, "a*1"] \arrow[d, "\chi"'] & & (F(C,D) \otimes (F(B,C) \otimes F(A,B))) * A \arrow[d, "\chi"] \\ (F(C,D) \otimes F(B,C)) * (F(A,B) * A) \arrow[d, "1*\varepsilon^A_B"'] \arrow[rrd, dotted, "\chi"] & & F(C,D) * ((F(B,C) \otimes F(A,B)) * A) \arrow[d, "1*\chi"] \\ (F(C,D) \otimes F(B,C)) * B \arrow[d, "\chi"'] & & F(C,D) * (F(B,C) * (F(A,B) * A)) \arrow[d, "1*(1*\varepsilon^A_B)"] \\ F(C,D) * (F(B,C) * B) \arrow[rr, dotted, "="] \arrow[d, "1*\varepsilon^B_C"'] & & F(C,D) * (F(B,C) * B) \arrow[d, "1*\varepsilon^B_C"] \\ F(C,D) * C \arrow[rd, "\varepsilon^C_D"'] & & F(C,D) * C \arrow[ld, "\varepsilon^C_D"] \\ & D & \end{tikzcd}$$

which commutes due to naturality of χ and ε. The identity axioms correspond to the diagrams

$$\begin{tikzcd}
 & (F(B,B)\otimes F(A,B))*A \arrow[rr, "\chi"] & & F(B,B)*(F(A,B)*A) \arrow[d, "1*\varepsilon^A_B"] \\
(I\otimes F(A,B))*A \arrow[ur, "(j_B\otimes 1)*1"] \arrow[r, "\chi"] \arrow[dr, swap, "l*1"] & I*(F(A,B)*A) \arrow[r, swap, "1*\varepsilon^A_B"] \arrow[urr, swap, "j_B*1"] \arrow[d, "\nu"] & I*B \arrow[r, swap, "j_B*1"] \arrow[dr, swap, "\nu"] & F(B,B)*B \arrow[d, "\varepsilon^B_B"] \\
 & F(A,B)*A \arrow[rr, swap, "\varepsilon^A_B"] & & B,
\end{tikzcd}$$

$$\begin{tikzcd}
 & (F(A,B)\otimes F(A,A))*A \arrow[rr, swap, "\chi"] & & F(A,B)*(F(A,A)*A) \arrow[d, "1*\varepsilon^A_A"] \\
(F(A,B)\otimes I)*A \arrow[ur, "(1\otimes j_A)*1"] \arrow[r, "\chi"] \arrow[dr, swap, "r*1"] & F(A,B)*(I*A) \arrow[rr, swap, "1*\nu"] \arrow[urr, swap, "1*(j_A*1)"] \arrow[d, swap, "1*\nu"] & & F(A,B)*A \arrow[d, "\varepsilon^A_B"] \\
 & F(A,B)*A \arrow[rr, swap, "\varepsilon^A_B"] & & B
\end{tikzcd}$$

in $\mathcal{D}$, which commute again for evident reasons.

Therefore we obtain a $\mathcal{V}$-enriched category $\mathcal{A}$, with objects ob$\mathcal{A}$ = ob$\mathcal{D}$ and hom-objects $\mathcal{A}(A,B) = F(A,B)$. The underlying category of $\mathcal{A}$ is precisely $\mathcal{D}$:

$$\mathcal{A}_o(A,B) = \mathcal{V}(I,\mathcal{A}(A,B)) = \mathcal{V}(I,F(A,B)) \cong \mathcal{D}(I*A,B) \cong \mathcal{D}(A,B)$$

$$\Rightarrow \mathcal{A}_o \cong \mathcal{D}$$

using the isomorphisms (4.9) and ν_A. □

As a straightforward application, we recover the well-known fact that the internal hom in a monoidal closed category $\mathcal{V}$ induces an enrichment of $\mathcal{V}$ in itself with hom-objects $[A,B]$, as mentioned in Section 4.1. This is the case, since the canonical action $* = \otimes$ has as parametrized adjoint the functor $[-,-]$.

As shown in detail in [**JK02**], when $\mathcal{V}$ is a monoidal closed category the natural isomorphism (4.9) lifts to a $\mathcal{V}$-natural isomorphism

$$\mathcal{A}(X*D,E) \cong [X,\mathcal{A}(D,E)].$$

The existence of a $\mathcal{V}$-enriched representation of $[X,\mathcal{A}(D,-)]$ is expressed by saying that the $\mathcal{V}$-category $\mathcal{A}$ is *tensored*, with $X*D$ as the tensor product of X and D. Furthermore, the case when not only the functor $(-*D)$ but also $(X*-)$ has a right adjoint $G(X,-)$ is addressed when $\mathcal{V}$ is moreover symmetric. Together with the natural isomorphism (4.9) we get

$$\mathcal{D}(D,G(X,E)) \cong \mathcal{D}(X*D,E) \cong \mathcal{V}(X,F(D,E)).$$

The bottomline is that the above assumptions result in the existence of a tensored and *cotensored* $\mathcal{V}$-enriched category, with underlying category $\mathcal{D}$, tensor product $X*D$ and cotensor product $G(X,E)$.

As a special case of the above theorem, suppose that there is an action of a monoidal category $\mathcal{V}$ on the opposite of a category $\mathcal{D} = \mathcal{B}^{\mathrm{op}}$ via a bifunctor

$$* : \mathcal{V} \times \mathcal{B}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}.$$

If we have an adjunction as in (4.10), the parametrized adjoint $F : \mathcal{B} \times \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$ of $*$ can be denoted as

$$P : \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \longrightarrow \mathcal{V}$$

by switching the entries of a pair in the cartesian product. The natural isomorphism (4.9) then becomes

$$\mathcal{B}(A, X * B) \cong \mathcal{V}(X, P(A, B))$$

for $X \in \mathcal{V}$ and $A, B \in \mathcal{B}$.

COROLLARY 4.3.4. *If* $* : \mathcal{V} \times \mathcal{B}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ *is an action of the monoidal closed* $\mathcal{V}$ *on the category* $\mathcal{B}^{\mathrm{op}}$ *along with an adjunction* $(- * B) \dashv P(-, B)$ *for each* $B \in \mathcal{B}$*, then* $\mathcal{B}^{\mathrm{op}}$ *is tensored* $\mathcal{V}$*-enriched with hom-objects* $\mathcal{B}^{\mathrm{op}}(A, B) = P(B, A)$.

Moreover, if $\mathcal{V}$ is symmetric then the opposite of a $\mathcal{V}$-enriched category is still enriched in $\mathcal{V}$. Hence in the situation above, there is an induced enrichment of $\mathcal{B} = (\mathcal{B}^{\mathrm{op}})^{\mathrm{op}}$ in $\mathcal{V}$ with the same objects and hom-objects $\mathcal{B}(A, B) = \mathcal{B}^{\mathrm{op}}(B, A)$.

COROLLARY 4.3.5. *Let* $\mathcal{V}$ *be a symmetric monoidal closed category acting on* $\mathcal{B}^{\mathrm{op}}$*. If for each object* B*, the action functor* $- * B : \mathcal{V} \to \mathcal{B}^{\mathrm{op}}$ *has a right adjoint* $P(-, B) : \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$*, then the parametrized adjoint*

$$P(-, -) : \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \longrightarrow \mathcal{V}$$

of the action provides the hom-objects of a cotensored $\mathcal{V}$*-enriched category with underlying ordinary category* $\mathcal{B}$ *and* $X * B$ *the cotensor product of* X *and* B*. If furthermore* $X * - : \mathcal{B}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ *has a right adjoint, the* $\mathcal{V}$*-category is also tensored.*

We are interested in these variations of Theorem 4.3.3, because our examples in the following chapters fall into these precise formulations.

CHAPTER 5

Fibrations and Opfibrations

This chapter begins with a detailed review of the basic concepts of the theory of fibrations and opfibrations, which plays a central role in the development of this thesis. Our presentation follows mainly [**Bor94b, Jac99, Joh02b**]. The notion of a fibred category, which arose from the work of Grothendieck in algebraic geometry, successfully captures the concept of a category varying over (or indexed by) a different category. There has also been a connection of fibrations with foundations for category theory, investigated in [**Bén85**].

Inside the total category of a fibration, the cartesian morphisms which are universally characterized incorporate a coherent structure: that of an indexed category, *i.e.* a certain pseudofunctor. This is best understood via the Grothendieck construction (see Theorem 5.2.1) originally in [**Gro61**], which demonstrates the essential equivalence between these two concepts. In fact, the coherent structure maps of an indexed category, whose existence is only implicit in fibrations, show that an indexed category has a structure whereas a fibration has a property (which determines such structure when a cleavage is chosen). Despite their correspondence, fibrations are technically often more convenient than indexed categories.

In the last section, we turn our attention to the fibrewise limits and adjunctions between fibred categories. Following the terminology and results of [**Her94, Jac99**], we slightly extend the existing theory by examining under which assumptions a fibred 1-cell between fibrations over different bases has a (fibred) adjoint. Hermida in his thesis [**Her93**] had already established the factorization of general fibred adjunctions in terms of cartesian fibred adjunctions and fibred adjunctions, suggesting an important characterization of fibred completeness. However, we follow a different approach to related problems.

5.1. Basic definitions

Consider a functor $P : \mathcal{A} \to \mathbb{X}$. A morphism $\phi : A \to B$ in $\mathcal{A}$ over a morphism $f = P(\phi) : X \to Y$ in $\mathbb{X}$ is called $(P\text{-})$*cartesian* if and only if, for all $g : X' \to X$ in $\mathbb{X}$ and $\theta : A' \to B$ in $\mathcal{A}$ with $P\theta = f \circ g$, there exists a unique arrow $\psi : A' \to A$ such that $P\psi = g$ and $\theta = \phi \circ \psi$:

$$\begin{tikzcd}
A' \arrow[rrd, two heads, "\theta"] \arrow[rd, dashed, "\exists!\psi"'] \arrow[dd, dotted] & & & \\
& A \arrow[r, two heads, "\phi"'] \arrow[dd, dotted] & B \arrow[dd, dotted] & \text{in } \mathcal{A} \\
X' \arrow[rrd, two heads, "f\circ g=P\theta"] \arrow[rd, "g"'] & & & \\
& X \arrow[r, two heads, "f=P\phi"'] & Y & \text{in } \mathbb{X}
\end{tikzcd}$$

For $X \in \mathrm{ob}\mathbb{X}$, the *fibre of P over X* written $\mathcal{A}_X$, is the subcategory of $\mathcal{A}$ which consists of objects A such that $P(A) = X$ and morphisms ϕ with $P(\phi) = 1_X$, called $(P-)$*vertical* morphisms.

The functor $P : \mathcal{A} \to \mathbb{X}$ is called a *fibration* if and only if, for all $f : X \to Y$ in $\mathbb{X}$ and $B \in \mathcal{A}_Y$, there is a cartesian morphism ϕ with codomain B above f, *i.e.* $\phi : A \to B$ with $P(\phi) = f$. We call such an ϕ a *cartesian lifting* of B along f. The category $\mathbb{X}$ is then called the *base* of the fibration, and $\mathcal{A}$ its *total* category.

Dually to the above, suppose we have a functor $U : \mathcal{C} \to \mathbb{X}$. A morphism $\beta : C \to D$ is *cocartesian* over a morphism $U\beta = f : X \to Y$ in $\mathbb{X}$ if, for all $g : Y \to Y'$ in $\mathbb{X}$ and all $\gamma : C \to D'$ with $U\gamma = g \circ f$, there exists a unique morphism $\delta : D \to D'$ such that $U\delta = g$ and $\gamma = \delta \circ \beta$:

$$\begin{tikzcd}
 & & D' \arrow[dd, dotted] & \\
C \arrow[r, "\beta"'] \arrow[urr, "\gamma"] \arrow[dd, dotted] & D \arrow[ur, dashed, "\exists!\delta"'] \arrow[dd, dotted] & & \text{in } \mathcal{C} \\
 & & Y' & \\
X \arrow[r, "f=U\beta"'] \arrow[urr, "g\circ f=U\gamma"] & Y \arrow[ur, "g"'] & & \text{in } \mathbb{X}
\end{tikzcd}$$

The functor $U : \mathcal{C} \to \mathbb{X}$ is an *opfibration* if U^{op} is a fibration, *i.e.* for every $C \in \mathcal{C}_X$ and $f : X \to Y$ in $\mathbb{X}$, there is a cocartesian morphism with domain C above f, called the *cocartesian lifting* of C along f. If U is both a fibration and an opfibration, it is called a *bifibration.*

Remark. What was above called 'cartesian' is sometimes called 'hypercartesian' instead. In that case, a cartesian morphism $\phi : A \to B$ would satisfy the property that for any $\theta : A' \to B$ with $P(\phi) = P(\theta)$, there is a unique vertical arrow $\psi : A' \to A$ with $\phi \circ \psi = \theta$:

$$\begin{tikzcd}
A' \arrow[d, dashed, "\exists!\psi"'] \arrow[dr, "\theta"] & & \\
A \arrow[r, "\phi"'] & B & \text{in } \mathcal{A}.
\end{tikzcd}$$

If we were to use this definition, we would have to add the requirement that cartesian arrows are closed under composition, in order to define a fibration.

Examples. (1) Every category $\mathcal{C}$ gives rise to the *family fibration*

$$f(\mathcal{C}) : Fam(\mathcal{C}) \longrightarrow \mathbf{Set}$$

over the category of sets. The category $Fam(\mathcal{C})$ has objects indexed families of objects in $\mathcal{C}$, $\{X_i\}_{i\in I}$ for a set I, and morphisms

$$(\{f_i\}_{i\in I}, u) : \{X_i\}_{i\in I} \longrightarrow \{Y_j\}_{i\in J}$$

are pairs which consist of a function $u : I \to J$ and a family of morphisms $f_i : X_i \to Y_{u(i)}$ in $\mathcal{C}$ for all i's. The functor $f(\mathcal{C})$ just takes a family of objects to its indexing set and a morphism to its function part. The cartesian arrows are pairs for which every f_i is an isomorphism, so a cartesian lifting of $\{Y_j\}_{j\in J}$ above $u : I \to J$ is

$$\begin{tikzcd}
\{Y_{u(i)}\}_{i\in I} \arrow[r, "(1,u)"] \arrow[d, dotted, "f(\mathcal{C})"'] & \{Y_j\}_{j\in J} \arrow[d, dotted, "f(\mathcal{C})"] & \text{in } Fam(\mathcal{C}) \\
I \arrow[r, "u"'] & J & \text{in } \mathbf{Set}.
\end{tikzcd}$$

(2) Consider the 'codomain' functor for any category $\mathcal{A}$

$$cod : \mathcal{A}^{\mathbf{2}} \longrightarrow \mathcal{A} \tag{5.1}$$

where $\mathcal{A}^{\mathbf{2}} = [\mathbf{2}, \mathcal{A}]$ is the *category of arrows* of $\mathcal{A}$, *i.e.* the functor category from $\mathbf{2} = (0 \to 1)$ with two objects and one non-identity arrow to $\mathcal{A}$. This functor takes a morphism $f : A \to B$ in $\mathcal{A}$ to its codomain B, and a commutative square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}$$

which expresses an arrow from f to g, to $k : B \to D$. Now, a *cod*-cartesian arrow of $C \xrightarrow{g} D$ above $k : B \to D$ is the pullback square

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "k"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}$$

therefore if $\mathcal{A}$ has pullbacks, *cod* is a fibration. Since this allows one to consider $\mathcal{A}$ as fibred over itself and this is central for developing category theory over $\mathcal{A}$, we call *cod* the *fundamental fibration* of $\mathcal{A}$. The fibre over an object A is simply the slice category $\mathcal{A}/A$. Dually we have the 'domain opfibration' with pushouts as cocartesian morphisms.

As an immediate consequence of the definition of cartesian and cocartesian morphisms, we have that if g and f are composable (co)cartesian arrows, their composite $g \circ f$ is again a (co)cartesian arrow. Also if $g \circ f$ and g are (co)cartesian arrows, then so is f. For example, for the fundamental fibration this is the standard result that if A and B as in

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "A"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "B"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

are pullbacks, then the pasted square is a pullback. Moreover, if the outer square and B are pullbacks, then so is A.

If $P : \mathcal{A} \to \mathbb{X}$ is a fibration, assuming the axiom of choice we may select a cartesian arrow over each $f : X \to Y$ in $\mathbb{X}$ and $B \in \mathcal{A}_Y$, denoted by

$$\mathrm{Cart}(f, B) : f^*(B) \longrightarrow B.$$

Such a choice of cartesian liftings is called a *cleavage* for P, which is then called a *cloven* fibration. Any fibration can be turned into a cloven one, using the axiom of choice to obtain a cleavage. Thus, henceforth we can assume that the fibrations we deal with are cloven. Dually, if U is an opfibration, for any $C \in \mathcal{C}_X$ and $f : X \to Y$ in $\mathbb{X}$ we can choose a cocartesian lifting of C along f

$$\mathrm{Cocart}(f, C) : C \longrightarrow f_!(C).$$

As a result of the above definitions, any arrow θ in the total category above f in a cloven fibration $P : \mathcal{A} \to \mathbb{X}$ factorizes uniquely into a vertical morphism followed by a cartesian one. Dually, any arrow γ in the total category above f in the base category of a cloven opfibration $U : \mathcal{C} \to \mathbb{X}$ factorizes uniquely into a cocartesian arrow followed by a vertical one:

$$\begin{tikzcd}
A \arrow[r, "\theta"] \arrow[d, dashed, "\psi"'] & B \arrow[dd, dotted, "P"] & \\
f^*B \arrow[ur, "\mathrm{Cart}(f,B)"'] \arrow[d, dotted, "P"'] & & \text{in } \mathcal{A} \\
X \arrow[r, "f=P\theta"'] & Y & \text{in } \mathbb{X},
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "\gamma"] \arrow[dr, "\mathrm{Cocart}(f,C)"'] \arrow[dd, dotted, "U"'] & D & \\
& f_!C \arrow[u, dashed, "\delta"'] \arrow[d, dotted, "U"] & \text{in } \mathcal{C} \\
X \arrow[r, "f=U\gamma"'] & Y & \text{in } \mathbb{X}.
\end{tikzcd}$$

Remark. Cartesian liftings of $B \in \mathcal{A}_Y$ along $f : X \to Y$ in $\mathbb{X}$ are unique up to vertical isomorphism:

$$\begin{tikzcd}
A' \arrow[dr, "\psi=\mathrm{Cart}(f,B)"] \arrow[d, dashed, bend right, "\alpha"'] & & \\
A \arrow[r, "\phi=\mathrm{Cart}(f,B)"'] \arrow[u, dashed, bend right, "\beta"'] \arrow[d, dotted, "P"'] & B \arrow[dd, dotted, "P"] & \text{in } \mathcal{A} \\
X \arrow[d, "1_X"'] \arrow[dr, "f"] & & \\
X \arrow[r, "f=P(\phi)"'] & Y & \text{in } \mathbb{X}
\end{tikzcd}$$

If ϕ and ψ are both cartesian morphisms, there exist unique $\alpha : A' \to A$ and unique $\beta : A \to A'$ vertical arrows such that $\phi \circ \alpha = \psi$ and $\psi \circ \beta = \phi$ respectively. It follows that $\alpha \circ \beta = 1_A$ and $\beta \circ \alpha = 1_{A'}$. Dually, cocartesian arrows are unique up to vertical isomorphism.

A cleavage for a fibration $P : \mathcal{A} \to \mathbb{X}$ induces, for every morphism $f : X \to Y$ in $\mathbb{X}$, a so-called *reindexing functor* between the fibre categories

$$f^* : \mathcal{A}_Y \longrightarrow \mathcal{A}_X. \tag{5.2}$$

This maps each $B \in \mathcal{A}_Y$ to $f^*(B)$, the domain of the cartesian lifting along f given by the cleavage, and each $\phi : B \to B'$ in the fibre $\mathcal{A}_Y$ to $f^*(\phi) : f^*(B) \to f^*(B')$, the unique vertical arrow making the diagram

$$\begin{tikzcd}
f^*(B) \arrow[r, "\mathrm{Cart}(f,B)"] \arrow[d, dashed, "f^*(\phi)"'] & B \arrow[d, "\phi"] \\
f^*(B') \arrow[r, "\mathrm{Cart}(f,B')"'] & B'
\end{tikzcd} \tag{5.3}$$

commute. Explicitly, since the composite $\phi \circ \mathrm{Cart}(f, B)$ has codomain B' in the total category $\mathcal{A}$, it uniquely factorizes through the chosen cartesian lifting of B' along f by universal property of cartesian arrows.

The uniqueness of the factorization through a chosen cartesian arrow implies immediately that $f^*(\psi) \circ f^*(\phi) = f^*(\psi \circ \phi)$, *i.e.* that f^* is a functor:

$$\begin{tikzcd}
f^*(B) \arrow[r, "\mathrm{Cart}(f,B)"] \arrow[d, dashed, "f^*(\phi)"] \arrow[dd, dashed, bend right=60, "f^*(\psi\circ\phi)"'] & B \arrow[d, "\phi"] & \\
f^*(B') \arrow[r, "\mathrm{Cart}(f,B')"'] \arrow[d, dashed, "f^*(\psi)"] & B' \arrow[d, "\psi"] & \\
f^*(B'') \arrow[r, "\mathrm{Cart}(f,B'')"'] \arrow[d, dotted, "P"'] & B'' \arrow[d, dotted, "P"] & \text{in } \mathcal{A} \\
X \arrow[r, "f"'] & Y & \text{in } \mathbb{X}
\end{tikzcd}$$

Dually, if $U : \mathcal{C} \to \mathbb{X}$ is a cloven opfibration, for every $f : X \to Y$ in $\mathbb{X}$ we get a reindexing functor between the fibres

$$f_! : \mathcal{C}_X \longrightarrow \mathcal{C}_Y$$

mapping each object $C \in \mathcal{C}_X$ to the codomain $f_!(C)$ of the chosen cocartesian lifting along f, and vertical morphisms $\gamma : C \to C'$ to the unique $f_!(\gamma)$ defined dually to (5.3).

Notice that the opfibration P^{op} for a fibration $P : \mathcal{A} \to \mathbb{X}$ has cocartesian liftings

$$\begin{tikzcd}
A \arrow[r, "\mathrm{Cocart}(f,A)"] \arrow[d, dotted, "P^{\mathrm{op}}"'] & f_!A \arrow[d, dotted, "P^{\mathrm{op}}"] & \text{in } \mathcal{A}^{\mathrm{op}} & & f^*A \arrow[r, "\mathrm{Cart}(f,A)"] \arrow[d, dotted, "P"'] & A \arrow[d, dotted, "P"] & \text{in } \mathcal{A} \\
X \arrow[r, "f"'] & X' & \text{in } \mathbb{X}^{\mathrm{op}} \arrow[u, phantom, "\equiv"] & & X' \arrow[r, "f"'] & X & \text{in } \mathbb{X}
\end{tikzcd}$$

and reindexing functors $f_! \equiv (f^*)^{\mathrm{op}} : \mathcal{A}_X^{\mathrm{op}} \longrightarrow \mathcal{A}_{X'}^{\mathrm{op}}$.

Remark 5.1.1. Due to the unique factorization of an arrow in a fibration and an opfibration through cartesian and cocartesian liftings respectively, we can deduce that a fibration $P : \mathcal{A} \to \mathbb{X}$ is also an opfibration (consequently a bifibration) if and only if, for every $f : X \to Y$ the reindexing $f^* : \mathcal{A}_Y \to \mathcal{A}_X$ has a left adjoint, namely $f_! : \mathcal{A}_X \to \mathcal{A}_Y$ (e.g. [**Her93**, Proposition 1.2.7]).

In general, for composable maps $f : X \to Y$ and $g : Y \to Z$ in the base category $\mathbb{X}$ of a fibration, it is not true that $f^* \circ g^* = (g \circ f)^*$. However, these functors are canonically isomorphic, as demonstrated by the following diagram:

$$\begin{tikzcd}
f^*g^*A \arrow[dr, "\mathrm{Cart}(f,g^*A)"] \arrow[dd, dashed, "\delta_A"'] & & & \\
& g^*A \arrow[dr, "\mathrm{Cart}(g,A)"] \arrow[ddd, dotted, no head] & & \\
(g \circ f)^*A \arrow[rr, "\mathrm{Cart}(g\circ f,A)"'] \arrow[d, dotted, no head] & & A \arrow[ddd, dotted, no head] & \text{in } \mathcal{A} \\
X \arrow[dr, "f"] \arrow[dd, "1_X"'] & & & \\
& Y \arrow[dr, "g"] & & \\
X \arrow[rr, "g\circ f"'] & & Z & \text{in } \mathbb{X}.
\end{tikzcd}$$

Since the composition of two cartesian arrows is again cartesian, δ_A is the unique vertical isomorphism which makes the above diagram commute. Thus we obtain a natural isomorphism

$$\delta^{f,g} : f^* \circ g^* \xrightarrow{\sim} (g \circ f)^* \tag{5.4}$$

with components vertical isomorphisms $\delta^{f,g}_A : f^*g^*A \cong (g \circ f)^*A$ for any $A \in \mathcal{A}$.

Moreover, the identity morphism $1_A : A \to A$ for an object A above X is cartesian over $1_X : X \to X$, and so there exists a unique vertical isomorphism $\gamma^X_A : A \cong (1_X)^*A$ making the top diagram

$$\begin{tikzcd}
A \arrow[d, dashed, "\gamma_A"'] \arrow[dr, "1_A"] & & \\
(1_X)^*A \arrow[r, "\mathrm{Cart}(1_X,A)"'] \arrow[d, dotted, no head] & A \arrow[d, dotted, no head] & \text{in } \mathcal{A} \\
X \arrow[r, "1_X"'] & X & \text{in } \mathbb{X}
\end{tikzcd}$$

commute. These morphisms are the components of a natural isomorphism

$$\gamma^X : 1_{\mathcal{A}_X} \xrightarrow{\sim} (1_X)^* \tag{5.5}$$

where $1_{\mathcal{A}_X}$ is the identity functor on the fibre $\mathcal{A}_X$. The natural transformations δ and γ will play an important part for the equivalence between fibrations and indexed categories described in the next section.

In a completely analogous way, for an opfibration $U : \mathcal{C} \to \mathbb{X}$ there is a natural isomorphism

$$q^{f,g} : (g \circ f)_! \xrightarrow{\sim} g_! \circ f_! \tag{5.6}$$

between the reindexing functors for composable arrows f and g, with components vertical isomorphisms $q^{f,g}_C : (g \circ f)_!C \cong g_!f_!C$ induced by universality of cocartesian arrows, and also a natural isomorpism

$$p^X : (1_X)_! \xrightarrow{\sim} 1_{\mathcal{C}_X}$$

with components vertical isomorphisms $p^X_C : (1_X)_!C \cong C$.

Nevertheless, a functorial choice of cartesian liftings is possible in specific situations. We usually assume, without loss of generality, that the cleavage is *normalized* in the sense that $\mathrm{Cart}(1_X, A) = 1_A$, in which case the isomorphisms γ_A are equalities. If also $\mathrm{Cart}(g \circ f, A) = \mathrm{Cart}(f, A) \circ \mathrm{Cart}(g, f^*(A))$, and so δ_A are equalities, the cleavage of the fibration is called a *splitting*, and a fibration endowed with a split cleavage is called a *split* fibration. Dually, we have the notion of a *split opfibration.*

We now turn to the appropriate notions of 1-cells and 2-cells for fibrations. A morphism of fibrations $(S, F) : P \to Q$ between $P : \mathcal{A} \to \mathbb{X}$ and $Q : \mathcal{B} \to \mathbb{Y}$ is given by a commutative square of functors and categories

$$\begin{tikzcd} \mathcal{A} \arrow[r, "S"] \arrow[d, "P"'] & \mathcal{B} \arrow[d, "Q"] \\ \mathbb{X} \arrow[r, "F"'] & \mathbb{Y} \end{tikzcd} \tag{5.7}$$

where S preserves cartesian arrows, meaning that if ϕ is P-cartesian, then $S\phi$ is Q-cartesian. The pair (S, F) is called a *fibred 1-cell.* In particular, when P and Q are fibrations over the same base category $\mathbb{X}$, we may consider fibred 1-cells of the form $(S, 1_{\mathbb{X}})$ displayed by commutative triangles

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "S"] \arrow[dr, "P"'] & & \mathcal{B} \arrow[dl, "Q"] \\ & \mathbb{X} & \end{tikzcd}$$

which are just cartesian functors S such that $Q \circ S = P$. Then S is called a *fibred functor.*

Dually, we have the notion of an *opfibred 1-cell* (K, F) and *opfibred functor* $(K, 1_{\mathbb{X}})$ between opfibrations over arbitrary bases or the same base respectively, where K preserves cocartesian arrows.

Remark 5.1.2. Any fibred or opfibred 1-cell determines a collection of functors $\{S_X : \mathcal{A}_X \to \mathcal{B}_{FX}\}$ between the fibre categories for all $X \in \mathrm{ob}\mathbb{X}$:

$$\begin{tikzcd} S_X : \mathcal{A}_X \arrow[r, "S|_X"] & \mathcal{B}_{FX} \\ A \arrow[r, mapsto, dotted] \arrow[d, "f"'] & SA \arrow[d, "Sf"] \\ A' \arrow[r, mapsto, dotted] & SA' \end{tikzcd} \tag{5.8}$$

This functor is well-defined, since $Q(SA) = F(PA) = FX$ by commmutativity of (5.7), so SA, SA' are in the fibre $\mathcal{B}_{FX}$. Also $Q(Sf) = F(Pf) = F(1_X) = 1_{FX}$ since F is a functor, so Sf is an arrow in $\mathcal{B}_{FX}$.

The following well-known proposition gives a way, given a fibration and a different functor to its base, to construct a new fibration over the domain of the given functor. A non-elementary proof (not as the one below) can be found in [**Gra66**].

Proposition 5.1.3 (Change of Base). *Given a fibration $Q : \mathcal{B} \to \mathbb{Y}$ and an arbitrary functor $F : \mathbb{X} \to \mathbb{Y}$, the pullback diagram*

$$\begin{tikzcd}
F^*(\mathcal{B}) \arrow[r, "\pi"] \arrow[d, "F^*Q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d, "Q"] \\
\mathbb{X} \arrow[r, "F"'] & \mathbb{Y}
\end{tikzcd}$$

exhibits $F^*Q : F^*(\mathcal{B}) \to \mathbb{X}$ as a fibration and (π, F) as a fibred 1-cell, i.e. *π preserves cartesian arrows.*

Proof. By construction of pullbacks in the complete category **Cat**, objects in $F^*(\mathcal{B})$ are pairs $(B, X) \in \mathrm{ob}\mathcal{B} \times \mathrm{ob}\mathbb{X}$ such that $QB = FX$, and morphisms are $(h, k) : (B, X) \to (B', X')$ with $h : B \to B'$ in $\mathcal{B}$, $k : X \to X'$ in $\mathbb{X}$ and $Qh = Fk$ in $\mathbb{Y}$. The functors π and F^*Q are the respective projections.

It can be easily verified, since Q is a fibration, that cartesian morphisms in $F^*(\mathcal{B})$ exist and are of the form

$$\begin{tikzcd}
((Ff)^*B, Z) \arrow[r, "(\mathrm{Cart}(Ff,B),f)"] \arrow[d, dotted, "F^*Q"'] & (B, X) \arrow[d, dotted, "F^*Q"] & \text{in } \mathbb{F}^*(\mathcal{B}) \\
Z \arrow[r, "f"] & X & \text{in } \mathbb{X}
\end{tikzcd}$$

where $\mathrm{Cart}(Ff, B)$ is the Q-cartesian lifting of B along Ff. The projection π obviously preserves cartesian arrows by the choice of cleavage. □

The same construction applies to opfibrations. We say that the fibration $P = F^*Q$ is obtained from Q *by change of base along* F. Notice also that for every object $X \in \mathrm{ob}\mathbb{X}$, we have an isomorphism $F^*(\mathcal{B})_X \cong \mathcal{B}_{FX}$ of the fibre categories which is given by S_X, the induced functor between the fibres

$$\begin{tikzcd}
F^*(\mathcal{B})_X \arrow[r, "S_X"] & \mathcal{B}_{FX} \\
(B, X) \arrow[r, mapsto, dotted] \arrow[d, "(f,1_X)"'] & B \arrow[d, "f"] \\
(B', X) \arrow[r, mapsto, dotted] & B'.
\end{tikzcd} \tag{5.9}$$

Going back to properties of fibred 1-cells, if we unravel the definition of a cartesian functor it is easy to deduce the following well-known result.

Lemma 5.1.4. *Suppose we have two fibrations $P : \mathcal{A} \to \mathbb{X}$, $Q : \mathcal{B} \to \mathbb{Y}$ and a fibred 1-cell (S, F) between them*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "S"] \arrow[d, "P"'] & \mathcal{B} \arrow[d, "Q"] \\
\mathbb{X} \arrow[r, "F"'] & \mathbb{Y}.
\end{tikzcd}$$

Then the reindexing functors commute up to isomorphism with the induced functors between the fibres. In other words, there is a natural isomorphism

$$\begin{tikzcd}
\mathcal{A}_Y \arrow[r, "S_Y"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\overset{\tau^f}{\cong}"] & \mathcal{B}_{FY} \arrow[d, "(Ff)^*"] \\
\mathcal{A}_X \arrow[r, "S_X"'] & \mathcal{B}_{FX}
\end{tikzcd} \tag{5.10}$$

for every $f : X \to Y$ in $\mathbb{X}$.

Proof. Consider a P-cartesian lifting $\mathrm{Cart}(f,A) : f^*A \to A$ of $A \in \mathcal{A}_Y$ along $f : X \to Y$ in $\mathbb{X}$. The functor S maps cartesian arrows to cartesian arrows, so the morphism

$$\begin{tikzcd}
S(f^*A) \arrow[r, "S\mathrm{Cart}(f{,}A)"] \arrow[d, dotted] & SA \arrow[d, dotted] & \text{in } \mathcal{B} \\
FX \arrow[r, "Ff"'] & FY & \text{in } \mathbb{Y}
\end{tikzcd}$$

is Q-cartesian above Ff with codomain SA. On the other hand, the canonical choice of a Q-cartesian lifting of SA along Ff is $\mathrm{Cart}(Ff, SA) : (Ff)^*(SA) \to SA$.

Since cartesian arrows are unique up to vertical isomorphism, there exists a unique vertical isomorphism $\tau^f_A : (Ff)^*(SA) \xrightarrow{\sim} S(f^*A)$ in the fibre $\mathcal{B}_{FX}$, such that the diagram

$$\begin{tikzcd}
(Ff)^*(SA) \arrow[d, "\tau^f_A"'] \arrow[dr, "\mathrm{Cart}(Ff{,}SA)"] & \\
S(f^*A) \arrow[r, "S\mathrm{Cart}(f{,}A)"'] & SA
\end{tikzcd}$$

commutes in the total category $\mathcal{B}$. The family of invertible arrows τ^f_A in fact determines a natural isomorphism τ^f as in (5.10). To establish naturality, for an arrow $m : A \to A'$ in the fibre $\mathcal{A}_Y$ we can form the following diagram:

$$\begin{tikzcd}
(Ff)^*(SA) \arrow[d, "\tau^f_A", "\cong"'] \arrow[dr, "\mathrm{Cart}(Ff{,}SA)"] \arrow[ddd, bend right=60, "(Ff)^*(Sm)"'] & & \\
S(f^*A) \arrow[r, "S\mathrm{Cart}(f{,}A)"'] \arrow[d, "S(f^*m)", "(**)"'] & SA \arrow[d, "Sm"] & \\
S(f^*A') \arrow[r, "S\mathrm{Cart}(f{,}A')"] \arrow[d, "{\tau^f_{A'}}^{-1}", "\cong"'] & SA' \arrow[dd, dotted] & \text{in } \mathcal{B} \\
(Ff)^*(SA') \arrow[ur, "\mathrm{Cart}(Ff{,}SA')"'] \arrow[d, dotted] & & \\
FX \arrow[r, "Ff"] & FY & \text{in } \mathbb{X}
\end{tikzcd}$$

The outer diagram commutes by definition of the mapping of $(Ff)^*$ on the arrow Sm, the right three inner diagrams commute for obvious reasons, hence the part (**) commutes as well, establishing naturality of τ^f. □

In particular, when S is a fibred functor between fibrations over the same base category $\mathbb{X}$, the isomorphism (5.10) is written

$$\begin{tikzcd}
\mathcal{A}_Y \arrow[r, "S_Y"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\overset{\tau^f}{\cong}"] & \mathcal{B}_Y \arrow[d, "f^*"] \\
\mathcal{A}_X \arrow[r, "S_X"'] & \mathcal{B}_X.
\end{tikzcd} \tag{5.11}$$

This lemma is relevant to the correspondence between fibrations and indexed categories, on the level of structure-preserving functors appropriate for these concepts. This will become clearer in the next section.

Now given two fibred 1-cells $(S, F), (T, G) : P \rightrightarrows Q$ between fibrations $P : \mathcal{A} \to \mathbb{X}$ and $Q : \mathcal{B} \to \mathbb{Y}$, a *fibred 2-cell* from (S, F) to (T, G) is a pair of natural transformations $(\alpha : S \Rightarrow T, \beta : F \Rightarrow G)$ with α above β, *i.e.* $Q(\alpha_A) = \beta_{PA}$ for all $A \in \mathcal{A}$. We can display a fibred 2-cell (α, β) between two fibred 1-cells as

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "S"{name=U}] \arrow[r, bend right, "T"'{name=D}] \arrow[d, "P"'] \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] & \mathcal{B} \arrow[d, "Q"] \\
\mathbb{X} \arrow[r, bend left, "F"{name=U2}] \arrow[r, bend right, "G"'{name=D2}] \arrow[from=U2, to=D2, phantom, "\Downarrow\beta"] & \mathbb{Y}.
\end{tikzcd} \tag{5.12}$$

In particular, when P and Q are fibrations over the same base category $\mathbb{X}$, we may consider fibred 2-cells of the form $(\alpha, 1_{1_{\mathbb{X}}}) : (S, 1_{\mathbb{X}}) \Rightarrow (T, 1_{\mathbb{X}})$ between the fibred functors S and T, displayed as

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, bend left, "S"{name=U}] \arrow[rr, bend right, "T"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \arrow[dr, "P"'] & & \mathcal{B} \arrow[dl, "Q"] \\
& \mathbb{X} &
\end{tikzcd}$$

which are in fact just natural transformations $\alpha : S \Rightarrow T$ such that $Q(\alpha_A) = 1_{PA}$, *i.e.* whose components are vertical arrows. A 2-cell like this is called a *fibred natural transformation.* Dually, we have the notion of an *opfibred 2-cell* and *opfibred natural transformation* between opfibred 1-cells and functors respectively.

In this way, we obtain a 2-category **Fib** of fibrations over arbitrary base categories, fibred 1-cells and fibred 2-cells, with the evident compositions coming from **Cat**. In particular, there is a 2-category **Fib**$(\mathbb{X})$ of fibrations over a fixed base category $\mathbb{X}$, fibred functors and fibred natural transformations. We also have the 2-categories **Fib**$_{\text{sp}}$ and **Fib**$(\mathbb{X})_{\text{sp}}$ of split fibrations and morphisms which preserve the splitting on the nose (*i.e.* up to equality and not only up to isomorphism). Dually, we have the 2-categories **OpFib** and **OpFib**($\mathbb{X}$) of opfibrations over arbitrary base categories and over a fixed base category $\mathbb{X}$ accordingly, as well as $\mathbf{OpFib}_{\mathrm{sp}}$ and $\mathbf{OpFib}(\mathbb{X})_{\mathrm{sp}}$ for the split cases.

As a matter of fact, **Fib** and **OpFib** are (non-full) sub-2-categories of the 2-category $\mathbf{Cat}^\mathbf{2} = [\mathbf{2}, \mathbf{Cat}]$ of 'arrows in **Cat**', with objects plain functors between categories, morphisms commutative squares of categories and functors as in (5.7) and 2-cells pairs of natural transformations as in (5.12). Also **Fib**($\mathbb{X}$) and **OpFib**($\mathbb{X}$) are sub-2-categories of the slice 2-category $\mathbf{Cat}/\mathbb{X}$.

These 2-categories form part of fibrations themselves: we already know that the functor $cod : \mathbf{Cat}^\mathbf{2} \to \mathbf{Cat}$ is the fundamental fibration (5.1), so consider its restriction to **Fib**. Proposition 5.1.3 implies that this functor

$$cod|_{\mathbf{Fib}} : \mathbf{Fib} \longrightarrow \mathbf{Cat}$$

which sends a fibration to its base is still a fibration, with cartesian morphisms pullback squares and fibre categories **Fib**($\mathbb{X}$) for a category $\mathbb{X} \in \mathbf{Cat}$. Also, the restricted functor

$$cod|_{\mathbf{OpFib}} : \mathbf{OpFib} \longrightarrow \mathbf{Cat}$$

is again a fibration, with fibres **OpFib**($\mathbb{X}$) for each category $\mathbb{X}$.

5.2. Indexed categories and the Grothendieck construction

Given a category $\mathbb{X}$, a $\mathbb{X}$-*indexed category* is a pseudofunctor

$$\mathscr{M} : \mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$$

which, by Definition 2.1.3, amounts to the following data: a category $\mathscr{M}X$ for every object $X \in \mathrm{ob}\mathbb{X}$ and a functor $\mathscr{M}f : \mathscr{M}Y \to \mathscr{M}X$ for each arrow $f : X \to Y$, together with natural isomorphisms $\delta_{f,g} : \mathscr{M}f \circ \mathscr{M}g \cong \mathscr{M}(g \circ f)$ for each composable pair of arrows and $\gamma_X : 1_{\mathscr{M}X} \cong \mathscr{M}(1_X)$ for each object in $\mathbb{X}$, satisfying associativity and identity laws (2.5, 2.6). The categories $\mathscr{M}X$ are usually called *fibres* and the functors $\mathscr{M}f$ are called *reindexing* and are sometimes denoted by f^*. The terminology already indicates the relation with fibrations.

If $\mathscr{M}$ and $\mathscr{H}$ are $\mathbb{X}$-indexed categories, a $\mathbb{X}$-*indexed functor* $\tau : \mathscr{M} \to \mathscr{H}$ is a pseudonatural transformation

$$\begin{tikzcd}
\mathbb{X}^{\mathrm{op}} \arrow[r, bend left, "\mathscr{M}"] \arrow[r, bend right, "\mathscr{H}"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathbf{Cat} .
\end{tikzcd}$$

By Definition 2.1.4, this means that for each object X of $\mathbb{X}$ there is a functor $\tau_X : \mathscr{M}X \to \mathscr{H}X$ and for each arrow $f : X \to Y$ there is a natural isomorphism

$$\begin{tikzcd}
\mathscr{M}X \arrow[r, "\mathscr{M}f"] \arrow[d, "\tau_X"'] \arrow[dr, phantom, "\overset{\tau_f}{\cong}"] & \mathscr{M}Y \arrow[d, "\tau_Y"] \\
\mathscr{H}X \arrow[r, "\mathscr{H}f"'] & \mathscr{H}Y
\end{tikzcd}$$

subject to the compatibility conditions with the $\delta_{f,g}$ and γ_X expressed by (2.9, 2.10).

If $\tau, \sigma : \mathscr{M} \to \mathscr{H}$ are $\mathbb{X}$-indexed functors, a $\mathbb{X}$-*indexed natural transformation* $m : \tau \to \sigma$ is a modification, which by Definition 2.1.5 consists of a family $m_X : \tau_X \Rightarrow \sigma_X$ of natural transformations for every object $X \in \mathrm{ob}\mathbb{X}$ subject to compatibility conditions with the coherence isomorphisms τ_f and σ_f expressed by (2.11).

Notice that in the above definitions, the ordinary category $\mathbb{X}$ is regarded as a 2-category with no non-trivial 2-cells. As discussed in Section 2.1, the above data form a 2-category $[\mathbb{X}^{\mathrm{op}}, \mathbf{Cat}]_{\mathrm{ps}}$ of $\mathbb{X}$-indexed categories, $\mathbb{X}$-indexed functors and $\mathbb{X}$-indexed natural transformations, also denoted as $\mathbf{ICat}(\mathbb{X})$.

The following establishes a correspondence between cloven fibration and indexed categories, due to Grothendieck, which amounts to an equivalence between the 2-categories $\mathbf{Fib}(\mathbb{X})$ and $\mathbf{ICat}(\mathbb{X})$ for a category $\mathbb{X}$.

THEOREM 5.2.1.

(i) Every cloven fibration $P : \mathcal{A} \to \mathbb{X}$ *gives rise to a* $\mathbb{X}$*-indexed category* $\mathscr{M}_P : \mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$.

(ii) [Grothendieck construction] *Every indexed category* $\mathscr{M} : \mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$ *gives rise to a cloven fibration* $P_{\mathscr{M}} : \mathfrak{G}\mathscr{M} \to \mathbb{X}$.

(iii) The above correspondences yield an equivalence of 2-categories

$$\mathbf{ICat}(\mathbb{X}) \simeq \mathbf{Fib}(\mathbb{X}) \tag{5.13}$$

so that $\mathscr{M}_{P_{\mathscr{M}}} \cong \mathscr{M}$ *and* $P_{\mathscr{M}_P} \cong P$.

PROOF. (i) Let $P : \mathcal{A} \to \mathbb{X}$ be a cloven fibration. We can define a pseudofunctor $\mathscr{M}_P : \mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$ as follows:

- Each object $X \in \mathbb{X}$ is mapped to the fibre category over this object, *i.e.* $\mathscr{M}_P(X) = \mathcal{A}_X$.
- Each morphism $f : X \to Y$ in $\mathbb{X}$ is mapped to the reindexing functor $\mathscr{M}_P(f) = f^* : \mathcal{A}_Y \to \mathcal{A}_X$ as in (5.2).
- Given $g : Y \to Z$ and $A \in \mathcal{A}_Z$, there is a natural isomorphism $\delta^{f,g} : \mathscr{M}_P(f) \circ \mathscr{M}_P(g) \xrightarrow{\sim} \mathscr{M}_P(g \circ f)$, explicitly described above (5.4).
- For any object $A \in \mathcal{A}_X$, there is a natural isomorphism $\gamma^X : 1_{\mathscr{M}_P(X)} \xrightarrow{\sim} \mathscr{M}_P(1_A)$ described in detail above (5.5).

It is straightforward to check that these natural isomorphisms δ and γ satisfy the coherence conditions for a pseudofunctor as described in the Definition 2.1.3.

(ii) Let $\mathbb{X}$ be a category and $\mathscr{M} : \mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$ an indexed category over $\mathbb{X}$. The *Grothendieck category* $\mathfrak{G}\mathscr{M}$ of $\mathscr{M}$ is defined as follows: objects are pairs (A, X) where $X \in \mathrm{ob}\mathbb{X}$ and $A \in \mathrm{ob}(\mathscr{M}X)$, and morphisms $(A, X) \to (B, Y)$ are pairs (ϕ, f) where $f : X \to Y$ is an arrow in $\mathbb{X}$ and $\phi : A \to (\mathscr{M}f)B$ is an arrow in $\mathscr{M}X$. This can also be written as

$$\begin{cases} A \xrightarrow{\phi} (\mathscr{M}f)B & \text{in } \mathscr{M}X \\ X \xrightarrow{f} Y & \text{in } \mathbb{X}. \end{cases} \tag{5.14}$$

The composite of two arrows in this category $(A, X) \xrightarrow{(\phi,f)} (B, Y) \xrightarrow{(\psi,g)} (C, Z)$ is $(\theta, g \circ f) : (A, X) \to (C, Z)$, where θ is the composite

$$A \xrightarrow{\phi} (\mathscr{M}f)B \xrightarrow{(\mathscr{M}f)\psi} (\mathscr{M}f \circ \mathscr{M}g)C \xrightarrow{(\delta_{f,g})_C} [\mathscr{M}(g \circ f)]C$$

for δ is the natural isomorphism as in (2.3) of the pseudofunctor $\mathscr{M}$. The coherence axiom (2.5) for $\delta_{g,f}$ ensures the associativity of this composition. Notice how we employ the components of the 2-cell $\delta_{f,g}$, since in this case it is actually a natural transformation (the codomain of the pseudofunctor is **Cat**).

Moreover, the identity arrow for each $(A, X) \in \mathfrak{G}\mathscr{M}$ is $(i, 1_X) : (A, X) \to (A, I)$, where i is the composite

$$A \xrightarrow{1_A} (1_{\mathscr{M}X})A \xrightarrow{(\gamma_X)_A} \mathscr{M}(1_X)A$$

where γ is the natural isomorphism as in (2.4). Again, the identity laws follow from the coherence conditions (2.6) of the pseudofunctor $\mathscr{M}$, and so $\mathfrak{G}\mathscr{M}$ is a category.

In fact, the projection functor

$$P_{\mathscr{M}} : \mathfrak{G}\mathscr{M} \longrightarrow \mathbb{X}$$

which maps each object (A, X) to X and each morphism (ϕ, f) to f is a cloven fibration: for each arrow $f : X \to Y$ of the base category $\mathbb{X}$ and an object (B, Y) over Y, we can choose the following top arrow

$$\begin{tikzcd}
((\mathscr{M}f)B, X) \arrow[r, "(1_{(\mathscr{M}f)B}, f)"] \arrow[d, dotted, "P_{\mathscr{M}}"'] & (B, Y) \arrow[d, dotted, "P_{\mathscr{M}}"] & \text{in } \mathfrak{G}\mathscr{M} \\
X \arrow[r, "f"'] & Y & \text{in } \mathbb{X}
\end{tikzcd}$$

to be the cartesian lifting $\mathrm{Cart}(f, (B, Y))$. Notice that the fibres $(\mathfrak{G}\mathscr{M})_X$ of the fibration $P_{\mathscr{M}}$ over $X \in \mathrm{ob}(\mathbb{X})$ are isomorphic to the categories $\mathscr{M}X$, due to the isomorphism $A \equiv (1_{\mathscr{M}X})A \cong \mathscr{M}(1_X)A$.

(iii) By Proposition 2.3.2, in order to exhibit an equivalence between two 2-categories, it is enough to construct a fully faithful and essentially surjective on objects 2-functor between them. Hence, we will demonstrate how the 'Grothendieck construction' mapping on objects $\mathscr{M} \mapsto (P_{\mathscr{M}} : \mathfrak{G}\mathscr{M} \to \mathbb{X})$ extends to a 2-functor

$$\mathfrak{P} : [\mathbb{X}^{\mathrm{op}}, \mathbf{Cat}]_{\mathrm{ps}} \longrightarrow \mathbf{Fib}(\mathbb{X})$$

with the following two properties:

- If $\mathscr{M}, \mathscr{H} : \mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$ are two $\mathbb{X}$-indexed categories, there is an isomorphism

$$\mathfrak{P}_{\mathscr{M},\mathscr{H}} : [\mathbb{X}^{\mathrm{op}}, \mathbf{Cat}]_{\mathrm{ps}}(\mathscr{M}, \mathscr{H}) \cong \mathbf{Fib}(\mathbb{X})(P_{\mathscr{M}}, P_{\mathscr{H}}) \tag{5.15}$$

between the category of pseudonatural transformations and modifications and the category of fibred functors and fibred natural transformations accordingly.

- Every fibration $P : \mathcal{A} \to \mathbb{X}$ is isomorphic to a fibration $P_{\mathscr{M}} : \mathfrak{G}\mathscr{M} \to \mathbb{X}$ arising from a pseudofunctor $\mathscr{M} : \mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$.

Consider a pseudonatural transformation $\tau : \mathscr{M} \Rightarrow \mathscr{H}$, consisting of functors $\tau_X : \mathscr{M}X \to \mathscr{H}X$ for all $X \in \text{ob}\mathbb{X}$ and natural isomorphisms $\tau^f : \tau_X \circ \mathscr{M}f \xrightarrow{\sim} \mathscr{H}f \circ \tau_Y$ for all arrows $f : X \to Y$. Then, define

$$\mathfrak{P}_{\mathscr{M},\mathscr{H}}(\tau) : \mathfrak{G}\mathscr{M} \longrightarrow \mathfrak{G}\mathscr{H}$$

to be the functor which maps an object $(A, X) \in \mathfrak{G}\mathscr{M}$ to $(\tau_X A, X) \in \mathfrak{G}\mathscr{H}$ and an arrow $(\phi, f) : (A, X) \to (B, Y)$ in $\mathfrak{G}\mathscr{M}$ like (5.14) to

$$\begin{cases} \tau_X A \xrightarrow{\psi} (\mathscr{H}f)(\tau_Y B) & \text{in } \mathscr{H}X \\ X \xrightarrow{f} Y & \text{in } \mathbb{X} \end{cases}$$

in $\mathfrak{G}\mathscr{H}$, where ψ is the composite

$$\tau_X A \xrightarrow{\tau_X(\phi)} (\tau_X \circ \mathscr{M}f)B \xrightarrow{\tau_B^f} (\mathscr{H}f \circ \tau_Y)B.$$

The fact that $\mathfrak{P}_{\mathscr{M},\mathscr{H}}(\tau)$ is a functor follows from the axioms of the pseudonatural transformation τ, and it can be easily shown that it preserves cartesian liftings, via the isomorphisms τ_B^f for all B. The triangle

$$\begin{tikzcd}
\mathfrak{G}\mathscr{M} \arrow[rr, "\mathfrak{P}_{\mathscr{M},\mathscr{L}}(\tau)"] \arrow[dr, "P_{\mathscr{M}}"'] & & \mathfrak{G}\mathscr{H} \arrow[dl, "P_{\mathscr{H}}"] \\
& \mathbb{X} &
\end{tikzcd}$$

commutes trivially, since the object of $\mathbb{X}$ which is projected by the fibrations remains unchanged, therefore $\mathfrak{P}_{\mathscr{M},\mathscr{H}}(\tau)$ is a fibred functor.

Now consider a modification $m : \tau \Rrightarrow \sigma$ between pseudonatural transformations $\tau, \sigma : \mathscr{M} \Rightarrow \mathscr{H}$, given by a family of natural transformations $m_X : \tau_X \Rightarrow \sigma_X$. We can then define a natural transformation

$$\mathfrak{P}_{\mathscr{M},\mathscr{H}}(m) : \mathfrak{P}_{\mathscr{M},\mathscr{H}}(\tau) \Rightarrow \mathfrak{P}_{\mathscr{M},\mathscr{H}}(\sigma)$$

by setting its components, for each (A, X) in $\mathfrak{G}\mathscr{M}$, to be $((m_X)_A, 1_X) : (\tau_X A, X) \to (\sigma_X A, X)$. The conditions which make

$$\begin{tikzcd}
\mathfrak{G}\mathscr{M} \arrow[rr, bend left, "\mathfrak{P}_{\mathscr{M},\mathscr{H}}(\tau)"] \arrow[rr, bend right, "\mathfrak{P}_{\mathscr{M},\mathscr{H}}(\sigma)"'] \arrow[rr, phantom, "\Downarrow \mathfrak{P}_{\mathscr{M},\mathscr{H}}(m)"] \arrow[dr, "P_{\mathscr{M}}"'] & & \mathfrak{G}\mathscr{H} \arrow[dl, "P_{\mathscr{H}}"] \\
& \mathbb{X} &
\end{tikzcd}$$

into a fibred natural transformation are satisfied by the coherence axioms for the modification m.

The above data define a 2-functor in a straightforward way, and moreover the functor $\mathfrak{P}_{\mathscr{M},\mathscr{H}}$ is an isomorphism of categories, since the mappings above are bijective. Therefore an isomorphism (5.15) is established.

For the second property, the goal is to show that every fibration $P : \mathcal{A} \to \mathbb{X}$ is specifically isomorphic to $P_{\mathscr{M}_P}$ in $\mathbf{Fib}(\mathbb{X})$. The latter fibration arises by applying the Grothendieck construction to the induced pseudofunctor $\mathscr{M}_P$ as constructed at part (i) of the proof. Indeed, there exists an invertible fibred functor

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "F"] \arrow[dr, "P"'] & & \mathfrak{G}\mathscr{M}_P \arrow[dl, "P_{\mathscr{M}_P}"] \\
& \mathbb{X} &
\end{tikzcd}$$

which maps an object A in $\mathcal{A}$ to the pair (A, PA) in the Grothendieck category $\mathfrak{G}\mathscr{M}_P$, and a morphism $\phi : A \to B$ to $(\theta, P\phi) : (A, PA) \to (B, PB)$, where θ is the unique vertical arrow of the P-factorization of ϕ:

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d, dashed, "\theta"'] & B \\
(P\phi)^*B \arrow[ur, "\mathrm{Cart}(P\phi{,}B)"'] &
\end{tikzcd}$$

Functoriality follows from uniqueness of cartesian liftings, and F is evidently bijective on objects and on arrows. Also, it preserves cartesian arrows and commutes with the fibrations P, $P_{\mathscr{M}_P}$ hence it is an isomorphism of fibrations over $\mathbb{X}$. □

Notice that in the Grothendieck construction above, we write the pairs in the opposite way from the standard notation. The same will apply for the form of objects and morphisms of all fibred categories studied later on.

The equivalence (5.13) clearly restricts to one between split fibrations over $\mathbb{X}$ and 'strict' $\mathbb{X}$-indexed categories, *i.e.* functors from $\mathbb{X}^{\mathrm{op}}$ to $\mathbf{Cat}$:

$$[\mathbb{X}^{\mathrm{op}}, \mathbf{Cat}] \simeq \mathbf{Fib}(\mathbb{X})_{\mathrm{sp}}$$

Dually, we have an analogous result relating opfibrations $U : \mathcal{C} \to \mathbb{X}$ and 'covariant indexed categories', *i.e.* pseudofunctors $\mathscr{F} : \mathbb{X} \to \mathbf{Cat}$.

THEOREM 5.2.2. *There is an equivalence of 2-categories*

$$[\mathbb{X}, \mathbf{Cat}]_{\mathrm{ps}} \simeq \mathbf{OpFib}(\mathbb{X}).$$

In particular, every opfibration $U : \mathcal{C} \to \mathbb{X}$ *is isomorphic to* $U_{\mathscr{F}} : \mathfrak{G}\mathscr{F} \to \mathbb{X}$ *arising from a pseudofunctor* $\mathscr{F} : \mathbb{X} \to \mathbf{Cat}$*, and there is an isomorphism of categories*

$$[\mathbb{X}, \mathbf{Cat}](\mathscr{F}, \mathscr{G}) \cong \mathbf{OpFib}(\mathbb{X})(U_{\mathscr{F}}, U_{\mathscr{G}})$$

for any two pseudofunctors $\mathscr{F}, \mathscr{G} : \mathbb{X} \to \mathbf{Cat}$.

The above theorems show how $\mathbb{X}$-indexed categories are 'essentially the same as' cloven fibrations over $\mathbb{X}$, and covariant indexed categories as opfibrations, hence we are able to freely pass from the one structure to the other depending on our needs. Via this process, we can also transfer properties and state them in the fibrational or indexed categories language at will.

As an example of how indexed and covariant indexed categories can be convenient means of studying fibrations and opfibrations, consider the following situation. If $K : \mathcal{C} \to \mathcal{D}$ is an opfibred functor between $U : \mathcal{C} \to \mathbb{X}$ and $V : \mathcal{D} \to \mathbb{X}$, by the dual of Lemma 5.1.4 there is a natural isomorphism

$$\begin{tikzcd} \mathcal{C}_X \arrow[r, "f_!"] \arrow[d, "K_X"'] \arrow[dr, phantom, "\cong"] & \mathcal{C}_Y \arrow[d, "K_Y"] \\ \mathcal{D}_X \arrow[r, "f_!"'] & \mathcal{D}_Y \end{tikzcd} \tag{5.16}$$

for any arrow $f : X \to Y$ in $\mathbb{X}$. We can also deduce this as follows. By Theorem 5.2.2, the opfibrations U, V correspond to pseudofunctors $\mathscr{F}, \mathscr{G} : \mathbb{X} \to \mathbf{Cat}$, in the sense that U is isomorphic to $U_{\mathscr{F}} : \mathfrak{G}\mathscr{F} \to \mathbb{X}$ and V is isomorphic to $U_{\mathscr{G}} : \mathfrak{G}\mathscr{G} \to \mathbb{Y}$. In particular $\mathcal{C}_X \cong \mathscr{F}X$, $\mathcal{D}_X \cong \mathscr{G}X$ and the reindexing functors are $\mathscr{F}f$ and $\mathscr{G}f$ respectively. Now, the opfibred functor K corresponds uniquely to an $\mathbb{X}$-indexed functor $\tau : \mathscr{F} \Rightarrow \mathscr{G}$, which is a pseudonatural transformation equipped with an (ordinary) natural isomorphism with components

$$\begin{tikzcd} \mathscr{F}X \arrow[r, "\mathscr{F}f"] \arrow[d, "\tau_X"'] \arrow[dr, phantom, "\overset{\tau_f}{\cong}"] & \mathscr{F}Y \arrow[d, "\tau_Y"] \\ \mathscr{G}X \arrow[r, "\mathscr{G}f"'] & \mathscr{G}Y \end{tikzcd}$$

for every $f : X \to Y$ in $\mathbb{X}$. This diagram corresponds uniquely to an isomorphism exactly like (5.16). This is evident after the realization that the functors K_X induced between the fibres as in Remark 5.1.2 are precisely τ_X.

As another example, suppose that $Q : \mathcal{B} \to \mathbb{Y}$ is a fibration which corresponds uniquely to the pseudofunctor $\mathscr{H} : \mathbb{Y}^{\text{op}} \to \mathbf{Cat}$. Then, if $F : \mathbb{X} \to \mathbb{Y}$ is a functor, the fibration F^*Q obtained from Q by change of base along F

$$\begin{tikzcd} F^*(\mathcal{B}) \arrow[r, "K"] \arrow[d, "F^*Q"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{B} \arrow[d, "Q"] \\ \mathbb{X} \arrow[r, "F"'] & \mathbb{Y} \end{tikzcd}$$

as in Proposition 5.1.3, corresponds to the composite pseudofunctor

$$\mathbb{X}^{\text{op}} \xrightarrow{F^{\text{op}}} \mathbb{Y}^{\text{op}} \xrightarrow{\mathscr{H}} \mathbf{Cat}.$$

This is evident by part (i) of the proof of the above theorem, since its mapping on objects is

$$\mathscr{H}(FX) \cong \mathcal{B}_{FX} \cong F^*(\mathcal{B})_X$$

by $\mathcal{B} \cong \mathfrak{G}\mathscr{H}$ and the isomorphism (5.9). On arrows, for $f : Z \to X$ in $\mathbb{X}$ and B above FX we have

$$(\mathscr{H}(Ff)B, X) \cong ((Ff)^*B, X) = f^*(B, X)$$

for f^* the reindexing functor of the fibration F^*Q.

The 2-categories of the form $\mathbf{ICat}(\mathbb{X})$ for each $\mathbb{X}$, sometimes also denoted as $\mathbf{Cat}_{\mathbb{X}}$, turn out to also be fibres of a fibration, like their equivalent $\mathbf{Fib}(\mathbb{X})$. Explicitly, there is a 2-category $\mathbf{ICat}$ with objects indexed categories $\mathscr{M} : \mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$ for arbitrary categories $\mathbb{X}$. A morphism from $\mathscr{M}$ to $\mathscr{H} : \mathbb{Y}^{\mathrm{op}} \to \mathbf{Cat}$ is given by a functor $F : \mathbb{X} \to \mathbb{Y}$ and an $\mathbb{X}$-indexed functor

$$\begin{tikzcd}
\mathbb{X}^{\mathrm{op}} \arrow[r, "\mathscr{M}"] \arrow[d, "F^{\mathrm{op}}"'] & \mathbf{Cat} \\
\mathbb{Y}^{\mathrm{op}} \arrow[ur, bend right, "\mathscr{H}"'] \arrow[u, phantom, shift left=6ex, "\Downarrow\tau"] &
\end{tikzcd}$$

and we write $(F,\tau) : \mathscr{M} \to \mathscr{H}$. Notice the direct relation with the indexed expression of pullbacks described above. A 2-cell $(F,\tau) \to (G,\sigma)$ is given by a natural transformation $\beta : F \Rightarrow G$ and a modification

$$\begin{tikzcd}
\mathbb{X}^{\mathrm{op}} \arrow[r, "F^{\mathrm{op}}"] \arrow[rr, bend left=40, "\mathscr{M}"] \arrow[r, bend right=50, "G^{\mathrm{op}}"'] \arrow[r, phantom, shift right=3ex, "\Downarrow\beta^{\mathrm{op}}"] & \mathbb{Y}^{\mathrm{op}} \arrow[r, "\mathscr{H}"] \arrow[u, phantom, "\Downarrow\tau"] & \mathbf{Cat} \arrow[r, phantom, "\overset{m}{\Rrightarrow}"] & \mathbb{X}^{\mathrm{op}} \arrow[r, bend left, "\mathscr{M}"] \arrow[r, bend right, "\mathscr{H}\circ G^{\mathrm{op}}"'] \arrow[r, phantom, "\Downarrow\sigma"] & \mathbf{Cat}.
\end{tikzcd}$$

Compositions and identities are defined using those in $\mathbf{Cat}$ and $\mathbf{ICat}(-)$. Hence, there is a (2-)functor

$$\mathit{base} : \mathbf{ICat} \longrightarrow \mathbf{Cat}$$

which maps an indexed category to its domain and a morphism to its first component. This is a split fibration, with fibres $\mathbf{ICat}(\mathbb{X})$ above $\mathbb{X}$ and reindexing functors precomposition with F^{op} for each $F : \mathbb{X} \to \mathbb{Y}$ in $\mathbf{Cat}$. For more details, we refer the reader to [**Her93**] or [**Jac99**].

Theorem 5.2.3. *There is a (2-)equivalence in the 2-category* $\mathbf{Fib}(\mathbf{Cat})$

$$\begin{tikzcd}
\mathbf{ICat} \arrow[rr, "\simeq"] \arrow[dr, "\mathit{base}"'] & & \mathbf{Fib} \arrow[dl, "\mathit{cod}"] \\
& \mathbf{Cat}. &
\end{tikzcd}$$

5.3. Fibred adjunctions and fibrewise limits

The notions of fibred and opfibred adjunction come from Definition 2.3.5 applied to the 2-categories **Fib** and **OpFib**.

Definition 5.3.1. Given fibrations $P : \mathcal{A} \to \mathbb{X}$ and $Q : \mathcal{B} \to \mathbb{Y}$, a *general fibred adjunction* is given by a pair of fibred 1-cells $(L,F) : P \to Q$ and $(R,G) : Q \to P$ together with fibred 2-cells $(\zeta,\eta) : (1_{\mathcal{A}}, 1_{\mathbb{X}}) \Rightarrow (RL, GF)$ and $(\xi,\varepsilon) : (LR, FG) \Rightarrow (1_{\mathcal{B}}, 1_{\mathbb{Y}})$ such that $L \dashv R$ via ζ,ξ and $F \dashv G$ via η,ε. This is displayed as

$$\begin{tikzcd}
\mathcal{A} \arrow[r, shift left, "L"] \arrow[d, "P"'] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, shift left, "R"] \arrow[d, "Q"] \\
\mathbb{X} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{Y} \arrow[l, shift left, "G"]
\end{tikzcd}$$

and we write $(L, F) \dashv (R, G) : Q \to P$. In particular, a *fibred adjunction* is an adjunction in the 2-category $\mathbf{Fib}(\mathbb{X})$, displayed as

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, shift left, "L"] \arrow[rr, phantom, "\perp"] \arrow[dr, "P"'] & & \mathcal{B} \arrow[ll, shift left, "R"] \arrow[dl, "Q"] \\
& \mathbb{X}. &
\end{tikzcd} \tag{5.17}$$

Notice that since (ζ, η) and (ξ, ε) are fibred 2-cells, by definition ζ is above η and ξ is above ε, which makes (P, Q) into a map of adjunctions (see Definition 2.3.6).

Dually, we have the notions of *general opfibred adjunction* and *opfibred adjunction* for adjunctions in the 2-categories $\mathbf{OpFib}$ and $\mathbf{OpFib}(\mathbb{X})$ respectively. Moreover, for the 2-categories $\mathbf{Fib}_{\mathrm{sp}}$, $\mathbf{Fib}(\mathbb{X})_{\mathrm{sp}}$, $\mathbf{OpFib}_{\mathrm{sp}}$ and $\mathbf{OpFib}(\mathbb{X})_{\mathrm{sp}}$ they are called *general split (op)fibred adjunction* and *split (op)fibred adjunction*. Then, the functors L and R are required to preserve the cleavages of the split (op)fibrations on the nose.

Since a basic aim in this section is to identify conditions under which (op)fibred functors and (op)fibred 1-cells have left or right adjoints, we recall the following well-known important fact (e.g. see [**Win90**, 4.5]).

Lemma 5.3.2. *Right adjoints in the 2-category* $\mathbf{Cat}/\mathbb{X}$ *preserve cartesian arrows and dually left adjoints in* $\mathbf{Cat}/\mathbb{X}$ *preserve cocartesian arrows. The same holds for adjoints in the 2-category* $\mathbf{Cat}^{\mathbf{2}}$.

This will prove very useful, since for example if a fibred functor has an ordinary right adjoint between the total categories which commutes with the fibrations, then the adjoint is necessarily fibred too.

It is clear that a fibred adjunction as in (5.17) induces fibrewise adjunctions

$$\begin{tikzcd}
\mathcal{A}_X \arrow[r, shift left, "L_X"] \arrow[r, phantom, "\perp"] & \mathcal{B}_X \arrow[l, shift left, "R_X"]
\end{tikzcd}$$

between the fibre categories for each X in $\mathbb{X}$. In the converse direction, we have the following result, see for example [**Bor94b**, 8.4.2] or [**Jac99**, 1.8.9].

Proposition 5.3.3. *Suppose* $S : Q \to P$ *is a fibred functor between fibrations* $Q : \mathcal{B} \to \mathbb{X}$ *and* $P : \mathcal{A} \to \mathbb{X}$. *Then* S *has a fibred left adjoint* L *if and only if for each* $X \in \mathbb{X}$ *we have* $L_X \dashv S_X$, *and the adjunct arrows*

$$\chi_A : (L_X \circ f^*)A \longrightarrow (f^* \circ L_Y)A \tag{5.18}$$

described below are isomorphisms for all $A \in \mathcal{A}_Y$ *and* $f : X \to Y$. *Similarly,* S *has a fibred right adjoint* R *iff* $S_X \dashv R_X$ *and* $(f^* \circ R_Y)B \cong (R_X \circ f^*)B$.

Remark. An equivalent formulation of the above, coming from the correspondent notion of indexed adjunctions (*i.e.* adjunction in the 2-category $\mathbf{ICat}(\mathbb{X})$), appears in [**Her93**]: a fibred adjunction $L \dashv R$ amounts to a family of adjunctions $\{L_X \dashv R_X : \mathcal{B}_X \to \mathcal{A}_X\}_{X \in \mathbb{X}}$ such that for every $f : Y \to X$, (f^{*P}, f^{*Q}) is a pseudo-map of adjunctions.

PROOF. Since S is cartesian, the image of a cartesian lifting

$$S_X(f^*L_YA) \xrightarrow{S\mathrm{Cart}(f,L_YA)} S_Y(L_YA)$$

is again a cartesian arrow above f in the total category $\mathcal{A}$, for any $A \in \mathcal{A}_Y$. Therefore the composite top arrow below factorizes uniquely through it via an isomorphism:

$$\begin{tikzcd}
f^*A \arrow[r, "\mathrm{Cart}(f{,}A)"] \arrow[d, dashed, "\exists!\pi_A"'] & A \arrow[d, "\eta^Y_A"] & \\
S_X(f^*L_YA) \arrow[r, "S\mathrm{Cart}(f{,}L_YA)"'] \arrow[d, dotted] & S_YL_YA \arrow[d, dotted] & \text{in } \mathcal{A} \\
X \arrow[r, "f"'] & Y & \text{in } \mathbb{X}
\end{tikzcd} \tag{5.19}$$

The arrow χ_A in (5.18) which we require to be an isomorphism is the one that corresponds under $L_X \dashv S_X$ to π_A:

$$\begin{array}{cc}
\pi_A : f^*A \longrightarrow S_Xf^*L_YA & \text{in } \mathcal{A}_X \\
\hline
\chi_A : L_Xf^*A \longrightarrow f^*L_YA & \text{in } \mathcal{B}_X
\end{array}$$

Then, these L_X assemble into a fibred left adjoint $L : \mathcal{A} \to \mathcal{B}$: on objects we define $LA := L_YA$ for $A \in \mathcal{A}_Y$, and on arrows we define $L(\phi)$ for

$$\begin{tikzcd}
C \arrow[r, "\phi"] \arrow[d, "\theta"'] & A \arrow[dd, dotted, bend left] & \\
f^*A \arrow[ur, "\mathrm{Cart}(f{,}A)"'] \arrow[d, dotted] & & \text{in } \mathcal{A} \\
X \arrow[r, "f"'] & Y & \text{in } \mathbb{X}
\end{tikzcd}$$

to be the composite

$$\begin{tikzcd}
L_XC \arrow[r, dashed, "L\phi"] \arrow[d, "L_X\theta"'] & L_YA \arrow[ddd, dotted, bend left] & \\
L_Xf^*A \arrow[d, "\chi_A"'] & & \text{in } \mathcal{B} \\
f^*L_YA \arrow[uur, "\mathrm{Cart}(f{,}L_YA)"'] \arrow[d, dotted] & & \\
X \arrow[r, "f"'] & Y & \text{in } \mathbb{X}.
\end{tikzcd} \tag{5.20}$$

Functoriality of L follows, and also we can directly verify that it is a cartesian functor. Using the fibrewise adjunctions we can also show that η and ε are natural with respect to all morphisms and not just those in the fibres. ☐

REMARK 5.3.4. There is an equivalent and perhaps more intuitive way of phrasing the condition that the transpose χ_A of π_A defined in (5.19) is an isomorphism, as in [**Jac99**] or [**KK13**]. We require that the *Beck-Chevalley condition* holds, *i.e.* the mate

$$\begin{tikzcd}
\mathcal{A}_Y \arrow[r, "L_Y"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\overset{\chi}{\Nearrow}"] & \mathcal{B}_Y \arrow[d, "f^*"] \\
\mathcal{A}_X \arrow[r, "L_X"'] & \mathcal{B}_X
\end{tikzcd}$$

of the canonical invertible 2-cell

$$\begin{tikzcd}
\mathcal{B}_Y \arrow[r, "S_Y"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\cong"] & \mathcal{A}_Y \arrow[d, "f^*"] \\
\mathcal{B}_X \arrow[r, "S_X"'] & \mathcal{A}_X
\end{tikzcd}$$

as in (5.11) which comes with the cartesian functor $S : \mathcal{B} \to \mathcal{A}$, is invertible as well. Using the mates correspondence of Proposition 2.3.7, we can explicitly compute the component χ_A as the composite

$$\begin{tikzcd}
L_X f^* A \arrow[r, "L_X f^* \eta_A"] \arrow[drr, dashed, "\chi_A"'] & L_X f^* S_Y L_Y A \arrow[r, "L_X \tau_{L_Y A}"] & L_X S_X f^* L_Y A \arrow[d, "\varepsilon_{f^* L_Y A}"] \\
 & & f^* L_Y A
\end{tikzcd}$$

by applying (2.17) for the adjunctions $L_Y \dashv S_Y$ and $L_X \dashv S_X$.

Similarly for the existence of a right fibred adjoint, the mate

$$\begin{tikzcd}
\mathcal{A}_Y \arrow[r, "R_Y"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\Swarrow_{\omega}"] & \mathcal{B}_Y \arrow[d, "f^*"] \\
\mathcal{A}_X \arrow[r, "R_X"'] & \mathcal{B}_X
\end{tikzcd}$$

under the fibrewise adjunctions $S_{(-)} \dashv R_{(-)}$ is requested to be an isomorphism.

Notice that in order to just define an ordinary left adjoint $L : \mathcal{A} \to \mathcal{B}$ of the fibred functor S between the total categories, the adjunction between the fibres and the components of the mate χ are sufficient, as can be seen from the defining diagram (5.20). The supplementary fact that χ should be an isomorphism ensures that this adjoint is also cartesian, therefore constitutes a fibred adjoint of K. On the other hand, for the existence of a right adjoint of S, the natural transformation ω being an isomorphism is required for the very construction of R, since the components ω_A initially go to the opposite direction than the one needed.

Similarly, there is a dual result concerning fibrewise adjunctions between opfibrations over a fixed base.

Proposition 5.3.5. *Suppose that $K : U \to V$ is an opfibred functor between opfibrations $U : \mathcal{C} \to \mathbb{X}$ and $V : \mathcal{D} \to \mathbb{X}$. It has a right opfibred adjoint $R : \mathcal{D} \to \mathcal{C}$ (respectively left opfibred adjoint L) if and only if it has a fibrewise adjoint $K_X \dashv R_X$*

(respectively $L_X \dashv K_X$) and the mate of the isomorphism $\sigma : K_Y \circ f_! \cong f_! \circ K_X$, given by the components

$$\begin{tikzcd}
f_! R_X D \arrow[r, "\eta_{f_! R_X D}"] \arrow[drr, dashed] & R_Y K_Y f_! R_X D \arrow[r, "R_Y \sigma_{R_X D}"] & R_Y f_! K_X R_X D \arrow[d, "R_Y f_! \varepsilon_D"] \\
& & R_Y f_! D
\end{tikzcd} \tag{5.21}$$

(respectively the mate $L_Y f_! \Rightarrow f_! L_Y$) for any $D \in \mathcal{D}_X$ is also invertible.

These results give rise to questions concerning adjunctions between fibrations over two different bases rather than the same as above. In this direction, Theorem 5.3.7 below is a generalization whose special case coincides with the above proposition. In what follows, we emphasize more on the existence of a total adjoint (induced only by its mapping on objects) and then we proceed to its full description. We initially look at opfibrations because of the nature of the examples that arise later.

Lemma 5.3.6. *Suppose $(K, F) : U \to V$ is an opfibred 1-cell given by the commutative square*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "K"] \arrow[d, "U"'] & \mathcal{D} \arrow[d, "V"] \\
\mathbb{X} \arrow[r, "F"'] & \mathbb{Y}
\end{tikzcd}$$

and there is an adjunction between the base categories

$$\begin{tikzcd}
\mathbb{X} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{Y} \arrow[l, shift left, "G"]
\end{tikzcd}. \tag{5.22}$$

with counit ε. If, for each $Y \in \mathbb{Y}$, the composite functor

$$\mathcal{C}_{GY} \xrightarrow{K_{GY}} \mathcal{D}_{FGY} \xrightarrow{(\varepsilon_Y)_!} \mathcal{D}_Y \tag{5.23}$$

has a right adjoint R_Y, then $K : \mathcal{C} \to \mathcal{D}$ between the total categories has a right adjoint, with $R_{(-)}$ its mapping on objects.

Proof. The adjunction $(\varepsilon_Y)_! K_{GY} \dashv R_Y$ comes with a natural isomorphism

$$\mathcal{D}_Y([(\varepsilon_Y)_! \circ K_{GY}](Z), D) \cong \mathcal{C}_{GY}(Z, R_Y(D)) \tag{5.24}$$

for any $Z \in \mathcal{C}_{GY}$, $D \in \mathcal{D}_Y$. We claim that this induces a bijective correspondence

$$\mathcal{D}(KC, D) \cong \mathcal{C}(C, R_Y D) \tag{5.25}$$

for any $C \in \mathcal{C}_X$ and $D \in \mathcal{D}_Y$, which is natural in C. In other words, there is a representation of the functor $\mathcal{D}(K-, D)$ with representing object $R_Y D$. Then, by adjunctions via representations, there is a unique way to define a functor

$$R : \mathcal{D} \longrightarrow \mathcal{C}$$

with object functions $R_{(-)}$ depending on the fibre of the objects, such that (5.25) is natural also in D thus gives an adjunction $K \dashv R$.

An element of the left hand side of (5.25) is an arrow $m : KC \to D$ in the total category $\mathcal{D}$, which can be encoded by

$$\begin{cases} f_!(KC) \xrightarrow{k} D & \text{in } \mathcal{D}_Y \\ FX \xrightarrow{f} Y & \text{in } \mathbb{Y} \end{cases} \tag{5.26}$$

where k is the unique vertical arrow of the factorization $m = k \circ \mathrm{Cocart}(f, KC)$.

An element of the right hand side of (5.25) is an arrow $n : C \to R_Y D$ in the total category $\mathcal{C}$, *i.e.*

$$\begin{cases} g_!C \xrightarrow{l} R_Y D & \text{in } \mathcal{C}_{GY} \\ X \xrightarrow{g} GY & \text{in } \mathbb{X} \end{cases}$$

where $n = l \circ \mathrm{Cocart}(g, C)$. By the natural isomorphism (5.24) and the adjunction (5.22), this pair of arrows corresponds bijectively to a pair

$$\begin{cases} [(\varepsilon_Y)_! \circ K_{GY}](g_!C) \xrightarrow{\hat{l}} D & \text{in } \mathcal{D}_Y \\ \qquad\qquad FX \xrightarrow{\tilde{g}} Y & \text{in } \mathbb{Y} \end{cases}$$

where $\hat{l}$ is the adjunct of l under $(\varepsilon_{Y!}K_{GY} \dashv R_Y)$ and $\tilde{g}$ is the adjunct of g under $F \dashv G$, hence it satisfies $\tilde{g} = Fg \circ \varepsilon_Y$.

In order for this pair to actually constitute an arrow $KC \to D$ in $\mathcal{D}$ as in (5.26), it is enough to show that

$$[(\varepsilon_Y)_! K_{GY}](g_!C) \cong \tilde{g}_!(KC)$$

in the fibre $\mathcal{D}_Y$. For that, observe that the diagram

$$\begin{tikzcd}
\mathcal{C}_X \arrow[r, "g_!"] \arrow[d, "K_X"'] & \mathcal{C}_{GY} \arrow[r, "K_{GY}"] & \mathcal{D}_{FGY} \arrow[d, "(\varepsilon_Y)_!"] \\
\mathcal{D}_{FX} \arrow[rru, dashed, "(Fg)_!"'] \arrow[rr, "\tilde{g}_!"'] & & \mathcal{D}_Y
\end{tikzcd} \tag{5.27}$$

commutes up to isomorphism: the left part is an isomorphism for any cocartesian functor K, dual to (5.10), and the right part is the isomorphism

$$q^{Fg,\varepsilon_Y} : \tilde{g}_! \xrightarrow{\sim} (Fg)_! \circ (\varepsilon_Y)_!$$

from the uniqueness of cartesian liftings, as in (5.6).

In other words, the bijective correspondence (5.25) is formally induced by a mapping $\mathcal{C}(C, R_Y D) \to \mathcal{D}(KC, D)$ explicitly given by

$$\begin{cases} g_!C \xrightarrow{l} RD & \text{in } \mathcal{C}_{GY} \\ X \xrightarrow{g} GY & \text{in } \mathbb{X} \end{cases} \mapsto \begin{cases} (\varepsilon Fg)!KC \overset{q}{\cong} \varepsilon_!(Fg)_!KC \overset{\varepsilon_!\sigma^g}{\cong} \varepsilon_!Kg_!C \xrightarrow{\theta(l)} D & \text{in } \mathcal{D}_Y \\ FX \xrightarrow{\tilde{g}} Y & \text{in } \mathbb{Y} \end{cases}$$

where θ is the natural bijection (5.24) and ε is short for ε_Y. Naturality in C can be checked, so a right adjoint R of K between the total categories can be defined. □

Since this result in essence generalizes Proposition 5.3.5, it is reasonable to explore the appropriate conditions in order for this right adjoint R to be cocartesian and thus to establish a general opfibred adjunction. Initially we are interested in adjusting Remark 5.3.4 on this case.

If we call σ^f the isomorphism induced by cocartesianness of the functor K employed in the above proof, for some $h : Y \to W$ in $\mathbb{Y}$ in particular we have a natural isomorphism

$$\begin{tikzcd}
\mathcal{C}_{GY} \arrow[r, "K_{GY}"] \arrow[d, "(Gh)_!"'] \arrow[dr, phantom, "\overset{\sigma^{Gh}}{\cong}"] & \mathcal{D}_{FGY} \arrow[d, "(FGh)_!"] \\
\mathcal{C}_{GW} \arrow[r, "K_{GW}"'] & \mathcal{D}_{FGW}.
\end{tikzcd}$$

Also, by sheer naturality of ε, we have an isomorphism

$$\nu : (\varepsilon_W)_!(FGh)_! \overset{q}{\cong} (\varepsilon_W \circ FGh)_! = (h \circ \varepsilon_Y)_! \overset{q}{\cong} h_!(\varepsilon_Y)_!.$$

We can now form an invertible composite 2-cell

$$\begin{tikzcd}
\mathcal{C}_{GY} \arrow[r, "K_{GY}"] \arrow[d, "(Gh)_!"'] \arrow[dr, phantom, "\overset{\sigma^{Gh}}{\cong}"] & \mathcal{D}_{FGY} \arrow[r, "(\varepsilon_Y)_!"] \arrow[d, "(FGh)_!"] \arrow[dr, phantom, "\overset{\nu}{\cong}"] & \mathcal{D}_Y \arrow[d, "h_!"] \\
\mathcal{C}_{GW} \arrow[r, "K_{GW}"'] & \mathcal{D}_{FGW} \arrow[r, "(\varepsilon_W)_!"'] & \mathcal{D}_W.
\end{tikzcd} \tag{5.28}$$

Its mate ω under the adjunctions $(\varepsilon_Y)_!K_{GY} \dashv R_Y$ and $(\varepsilon_W)_!K_{GW} \dashv R_W$ has components, by (2.16),

$$\begin{tikzcd}
(Gh)_!R_Y D \arrow[r, "\bar{\eta}^W"] \arrow[drr, dashed, "\omega_D"'] & \big(R_W((\varepsilon_W)_!K_{GW})\big)(Gh)_!R_Y D \arrow[r, "R_W(\sigma^{Gh} * \nu)"] & R_W\big(h_!(\varepsilon_E)_!K_{GY}\big)R_Y D \arrow[d, "R_W h_! \bar{\varepsilon}^Y"] \\
 & & (R_W h_!)D
\end{tikzcd} \tag{5.29}$$

where $\bar{\eta}$ and $\bar{\varepsilon}$ are the unit and counit of the adjunctions $\varepsilon_{(-)_!}K_{G(-)} \dashv R_{(-)}$. These arrows ω_D which generalize the composites (5.21), are essential for the explicit construction of R.

In a dual way to Proposition 5.3.3, R maps an arrow

$$\begin{tikzcd}
D \arrow[r, "k"] \arrow[dr, "\mathrm{Cocart}(h,D)"'] \arrow[dd, dotted, no head] & E & \\
 & h_!D \arrow[u, "\psi"'] \arrow[d, dotted, no head] & \text{in } \mathcal{D} \\
Y \arrow[r, "h"'] & W & \text{in } \mathbb{Y}
\end{tikzcd}$$

to the composite

$$\begin{tikzcd}
R_Y D \arrow[rr, dashed, "Rk"] \arrow[ddrr, "\mathrm{Cocart}(Gh{,}R_W D)"'] \arrow[ddd, dotted, no head] & & R_W E & \\
& & R_W(h_! D) \arrow[u, "R_W \psi"'] & \text{in } \mathcal{C} \\
& & (Gh)_! R_Y D \arrow[u, "\omega_D"'] \arrow[d, dotted, no head] & \\
GY \arrow[rr, "Gh"'] & & GW & \text{in } \mathbb{X}
\end{tikzcd}$$

where ω_D are the arrows (5.29). It is now not hard to see that by construction of R, the square of categories and functors

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "U"'] & \mathcal{D} \arrow[l, "R"'] \arrow[d, "V"] \\
\mathbb{X} & \mathbb{Y} \arrow[l, "G"]
\end{tikzcd}$$

commutes. Moreover, if (ζ, ξ) is the unit and counit of $K \dashv R$, the pairs (ζ, η) and (ξ, ε) are above each other. Consequently $(K, F) \dashv (R, G)$ is already an adjunction in $\mathbf{Cat^2}$. Finally, if we request that the ω_D's are isomorphisms, putting $k = \mathrm{Cocart}(g, D)$ in the mapping above exhibits the cocartesianness of R.

THEOREM 5.3.7. *Suppose $(K, F) : U \to V$ is an opfibred 1-cell and $F \dashv G$ is an adjunction between the bases of the fibrations, as in*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "K"] \arrow[d, "U"'] & \mathcal{D} \arrow[d, "V"] \\
\mathbb{X} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{Y}. \arrow[l, shift left, "G"]
\end{tikzcd}$$

If the composite (5.23) has a right adjoint for each $Y \in \mathbb{Y}$, then K has a right adjoint R between the total categories, with $(K, F) \dashv (R, G)$ in $\mathbf{Cat^2}$. If the mate

$$\begin{tikzcd}
\mathcal{D}_Y \arrow[r, "R_Y"] \arrow[d, "h_!"'] \arrow[dr, phantom, "\Swarrow \omega"] & \mathcal{C}_{GY} \arrow[d, "(Gh)_!"] \\
\mathcal{D}_W \arrow[r, "R_W"'] & \mathcal{C}_{GW}
\end{tikzcd}$$

of the composite invertible 2-cell (5.28) is moreover an isomorphism for any $h : Z \to W$ in $\mathbb{Y}$, then R is cocartesian and so

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "K"] \arrow[r, phantom, "\perp"] \arrow[d, "U"'] & \mathcal{D} \arrow[l, shift left, "R"] \arrow[d, "V"] \\
\mathbb{X} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{Y} \arrow[l, shift left, "G"]
\end{tikzcd}$$

is a general opfibred adjunction. Conversely, if $(K, F) \dashv (R, G)$ in **OpFib**, *then evidently $F \dashv G$, $K \dashv R$, R is cocartesian, and moreover for every $Y \in \mathbb{Y}$ there is an adjunction $(\varepsilon_Y)_! K_{GY} \dashv R_Y$ between the fibres.*

Proof. The first part is just Lemma 5.3.6 and the process that follows. For the converse, start with some $f : C \to R_Y D$ in $\mathcal{C}_{GY}$. There is a bijective correspondence

$$\begin{array}{cc} (C,GY) \xrightarrow{(f,1_{GY})} (R_Y D, GY) \equiv R(D,Y) & \text{in } \mathcal{C} \\ \hline K(C,GY) \equiv (K_{GY}C, FGY) \xrightarrow{(\bar{f},\varepsilon_Y)} (D,Y) & \text{in } \mathcal{D} \end{array}$$

since $K \dashv R$, but the latter morphism is uniquely determined by the vertical arrow $\bar{\bar{f}} : (\varepsilon_Y)_! K_{GY} C \to D$ in $\mathcal{D}_Y$ because of the factorization of any arrow through the cocartesian lifting. Hence the required fibrewise adjunction is established. □

Dually, we get the following version about adjunctions between fibrations.

Theorem 5.3.8. *Suppose $(S,G) : Q \to P$ is a fibred 1-cell between two fibrations and $F \dashv G$ is an adjunction between the bases, as shown in the diagram*

$$\begin{tikzcd} \mathcal{A} \arrow[d, "P"'] & \mathcal{B} \arrow[l, "S"'] \arrow[d, "Q"] \\ \mathbb{X} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{Y}. \arrow[l, shift left, "G"] \end{tikzcd}$$

If, for each $X \in \mathbb{X}$, the composite functor

$$\mathcal{B}_{FX} \xrightarrow{S_{FX}} \mathcal{A}_{GFX} \xrightarrow{\eta_X^*} \mathcal{A}_X$$

has a left adjoint L_X, then S has a left adjoint L between the total categories, with $(L,F) \dashv (S,G)$ in $\mathbf{Cat^2}$. Furthermore, if the mate

$$\begin{tikzcd} \mathcal{A}_Z \arrow[r, "L_Z"] \arrow[d, "f^*"'] & \mathcal{B}_{FZ} \arrow[d, "(Ff)^*"] \\ \mathcal{A}_X \arrow[r, "L_X"'] \arrow[ur, phantom, "\Nearrow"] & \mathcal{B}_{FX} \end{tikzcd}$$

of the composite isomorphism

$$\begin{tikzcd} \mathcal{B}_{FZ} \arrow[r, "S_{FZ}"] \arrow[d, "(Ff)^*"'] \arrow[dr, phantom, "\overset{\tau^{Ff}}{\cong}"] & \mathcal{A}_{GFZ} \arrow[r, "(\eta_Z)^*"] \arrow[d, "(GFf)^*"] \arrow[dr, phantom, "\overset{\kappa}{\cong}"] & \mathcal{A}_Z \arrow[d, "f^*"] \\ \mathcal{B}_{FX} \arrow[r, "S_{FX}"'] & \mathcal{A}_{GFX} \arrow[r, "(\eta_X)^*"'] & \mathcal{A}_X \end{tikzcd}$$

is invertible for any $f : X \to Z$ in $\mathbb{X}$, then

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] \arrow[d, "P"'] & \mathcal{B} \arrow[l, shift left, "S"] \arrow[d, "Q"] \\ \mathbb{X} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{Y} \arrow[l, shift left, "G"] \end{tikzcd}$$

is a general fibred adjunction. Conversely, if $(L,F) \dashv (S,G)$ is an adjunction in **Fib**, *we have adjunctions $L_X \dashv \eta_X^* S_{FX}$ for all $X \in \mathbb{X}$.*

In the above composite 2-cell, the 2-isomorphism τ^{Ff} comes from the cartesian functor S as in (5.10) and κ from naturality of η, the unit of the base adjunction.

We finish this section with some general results concerning fibrewise completeness and cocompleteness. In fact, Hermida's work on fibred adjunctions was mainly motivated by its applications on the existence of fibred limits and colimits. For us though, the establishment of general (op)fibred adjunctions serves different purposes.

For any small category $\mathcal{J}$, we say that a fibration $P : \mathcal{A} \to \mathbb{X}$ has *fibred $\mathcal{J}$-limits* (respectively *colimits*) if and only if the fibred functor $\hat{\Delta}_{\mathcal{J}} : \mathcal{A} \to \Delta^*([\mathcal{J}, \mathcal{A}])$ uniquely determined by the diagram below has a fibred right (respectively left) adjoint:

$$\begin{tikzcd}
\mathcal{A} \arrow[drr, bend left, "\tilde{\Delta}_{\mathcal{J}}"] \arrow[dr, dashed, "\hat{\Delta}_{\mathcal{J}}"] \arrow[ddr, bend right, "P"'] \arrow[ddr, phantom, "\star"] & & \\
& \Delta^*_{\mathcal{J}}[\mathcal{J}, \mathcal{A}] \arrow[r, "\pi"] \arrow[d, "{\Delta^*_{\mathcal{J}}[\mathcal{J},P]}"] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathcal{J}, \mathcal{A}]} \arrow[d, "{[\mathcal{J},P]}"] \\
& \mathbb{X} \arrow[r, "\Delta_{\mathcal{J}}"] & {[\mathcal{J}, \mathbb{X}]}
\end{tikzcd} \tag{5.30}$$

where $\Delta_{\mathcal{J}}$ and $\tilde{\Delta}_{\mathcal{J}}$ are the constant diagram functors. Notice that $[\mathcal{J}, P]$ is a fibration when P is, where cartesian morphisms are formed componentwise. We write $\big(\hat{\lim}_{\mathcal{J}} \dashv \hat{\Delta}_{\mathcal{J}} \dashv \hat{\operatorname{colim}}_{\mathcal{J}}\big)$ when the fibration P has fibred limits and colimits. Dually we can define *opfibred $\mathcal{J}$-colimits* and *limits* for an opfibration U.

Proposition 5.3.9. *A fibration $P : \mathcal{A} \to \mathbb{X}$ has all fibred $\mathcal{J}$-limits (colimits) if and only if every fibre has $\mathcal{J}$-limits (colimits) and the reindexing functors f^* preserve them, for any arrow f.*

Proof. By Proposition 5.3.3, the fibred functor $\hat{\Delta}_{\mathcal{J}}$ has a fibred right adjoint R if and only if there is an adjunction between the fibres $(\hat{\Delta}_{\mathcal{J}})_X \dashv R_X$ and we have isomorphisms $(R_X f^*)C \cong (f^* R_Y)C$ for any $f : X \to Y$ and $C \in \mathcal{C}_Y$. The first condition is equivalent to each fibre $\mathcal{A}_X$ being $\mathcal{J}$-complete, since

$$(\Delta^*_{\mathcal{J}}[\mathcal{J}, \mathcal{A}])_X \cong [\mathcal{J}, \mathcal{A}]_{\Delta_{\mathcal{J}} X} = [\mathcal{J}, \mathcal{A}_X]$$

by construction of the pullback fibration, and $(\hat{\Delta}_{\mathcal{J}})_X : \mathcal{A}_X \to [\mathcal{J}, \mathcal{A}_X]$ is the constant diagram functor. If we call this fibrewise adjoint $R_X = \lim_X$, the second condition becomes

$$(\lim_X \circ [\mathcal{J}, f^*])F \cong (f^* \circ \lim_Y)F$$

for any functor $F : \mathcal{J} \to \mathcal{A}_Y$, which means precisely that any f^* preserves limits between the fibre categories. Dual arguments apply for the existence of colimits. ☐

There is an equivalent definition of a fibred $\mathcal{J}$-complete fibration $P : \mathcal{A} \to \mathbb{X}$. In [**Bor94b**, 8.5.1], it is stated that P has all $\mathcal{J}$-limits when the (outer) fibred 1-cell $(\tilde{\Delta}_{\mathcal{J}}, \Delta_{\mathcal{J}})$ given by (5.30) has a fibred right adjoint. The difference relates to whether we require an adjunction between fibrations over the same bases or not, since the factorization through the pullback is a tool which permits the restriction of the problem from **Fib** to **Fib**($\mathbb{X}$). The following result illustrates the latter.

THEOREM 5.3.10. [**Her93**, 3.2.3] *Given* $P : \mathcal{A} \to \mathbb{X}$*,* $Q : \mathcal{B} \to \mathbb{Y}$*,* $F \dashv G : \mathbb{Y} \to \mathbb{X}$ *via* η*,* ε *and a fibred 1-cell* $(S, F) : P \to Q$ *as shown in the following diagram*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "S"] \arrow[d, "P"'] & \mathcal{B} \arrow[d, "Q"] \\
\mathbb{X} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{Y}, \arrow[l, shift left, "G"]
\end{tikzcd}$$

let $\hat{S} : P \to F^*Q$ *in* $\mathbf{Fib}(\mathbb{X})$ *be the unique mediating functor in*

$$\begin{tikzcd}
\mathcal{A} \arrow[rrd, bend left, "S"] \arrow[rd, dashed, "\hat{S}"] \arrow[rdd, bend right, "P"'] & & \\
& F^*\mathcal{B} \arrow[r, "\pi"] \arrow[d, "F^*Q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d, "Q"] \\
& \mathbb{X} \arrow[r, "F"'] & \mathbb{Y}.
\end{tikzcd}$$

Then, the following statements are equivalent:

i) There exists $R : \mathcal{B} \to \mathcal{A}$ *such that* $S \dashv R$ *in* **Cat** *and* $(S, F) \dashv (R, G)$ *in* **Fib**.

ii) There exists $\hat{R} : F^*Q \to P$ *such that* $\hat{S} \dashv \hat{R}$ *in* $\mathbf{Fib}(\mathbb{X})$.

This theorem uses the fact that change of base along F as in Proposition 5.1.3 yields a so-called *cartesian fibred adjunction* when F has a right adjoint, meaning (π, F) has an adjoint in **Fib**. Therefore, by performing change of base along a left adjoint functor, we can factorize a general fibred adjunction into a cartesian and 'vertical' fibred adjunction, hence 'reduce' a general fibred adjunction to a fibred adjunction. Dually, this can be done for a general opfibred adjunction accordingly.

Using the above theorem, we can deduce fibrewise completeness conditions from the total category of the fibration and vice versa.

COROLLARY 5.3.11. [**Her93**, 3.3.6] *Let* $\mathcal{J}$ *be a small category and* $P : \mathcal{A} \to \mathbb{X}$ *be a fibration such that the base category* $\mathbb{X}$ *has all* $\mathcal{J}$*-limits. Then the fibration* P *has all fibred* $\mathcal{J}$*-limits if and only if* $\mathcal{A}$ *has and* P *strictly preserves (chosen)* $\mathcal{J}$*-limits.*

The proof relies on Lemma 5.3.2 and essentially constructs a general fibred adjunction $(\tilde{\Delta}_{\mathcal{J}}, \Delta_{\mathcal{J}}) \dashv (\widetilde{\lim}_{\mathcal{J}}, \lim_{\mathcal{J}})$ for the outer diagram (5.30). Dually, we obtain fibred colimits for an opfibration with a cocomplete base, from colimits in the total category which are strictly preserved by the opfibration.

REMARK. In essence, Theorems 5.3.7 and 5.3.8 relate to very similar questions as Theorem 5.3.10, namely the assumptions under which we obtain general fibred and opfibred adjunctions (starting with an (op)fibred 1-cell). However, they actually respond to the exact opposite problems: Theorem 5.3.8 provides with a *left* adjoint between the total functors, whereas Theorem 5.3.10 reduces the existence of a *right* fibred 1-cell adjoint to a right fibred adjoint. This connection should perhaps be further explored. For example, we could use the new results to study fibred cocompleteness of fibrations and fibred completeness of opfibrations.

PART II

CHAPTER 6

Enrichment of Monoids and Modules

6.1. Universal measuring comonoid and enrichment

The notion of the universal measuring coalgebra was first introduced by Sweedler [**Swe69**] in the context of vector spaces over a field k. The question that motivated the definition of measuring coalgebras is under which conditions, for A, B k-algebras and C a k-coalgebra, the linear map $\rho \in \mathrm{Hom}_k(A, \mathrm{Hom}_k(C, B))$ corresponding under the usual tensor-hom adjunction to $\sigma \in \mathrm{Hom}_k(C \otimes_k A, B)$ in $\mathbf{Vect}_k$ is actually an algebra map.

More explicitly, the natural bijective correspondence defining the adjunction $(- \otimes_k C) \dashv \mathrm{Hom}_k(C, -)$ is given by the invertible mapping

$$\begin{tikzcd}
\mathbf{Vect}_k(A, \mathrm{Hom}_k(C,B)) \arrow[r] & \mathbf{Vect}_k(A \otimes C, B) \\
A \xrightarrow{\rho} \mathrm{Hom}_k(C,B) \arrow[r, mapsto] & A \otimes C \xrightarrow{\bar{\rho}} B \\
 & a \otimes c \mapsto [\rho(a)](c)
\end{tikzcd}$$

where of course $\mathbf{Vect}_k(-,-) = \mathrm{Hom}_k$. If C is a k-coalgebra and B a k-algebra, it is well-known that $\mathrm{Hom}_k(C, B)$ obtains the structure of a k-algebra via convolution, also by Remark 3.3.2. Hence if A is also a k-algebra, we may ask under which conditions on $\bar{\rho}$, the corresponding linear map ρ is a k-algebra homomorphism. This resulted in the following definition.

Definition. If A, B are k-algebras, C a k-coalgebra and $\sigma : C \otimes_k A \to B$ a linear map, we say that (σ, C) *measures* A to B when σ satisfies:

$$\sigma(c \otimes aa') = \sum_{(c)} \sigma(c_{(1)} \otimes a)\sigma(c_{(2)} \otimes a')$$
$$\sigma(c \otimes 1) = \epsilon(c)1$$

where the sum comes from the sigma notation for the comultiplication of C, and ϵ is the counit.

There is a category of *measuring coalgebras* and it has a terminal object $P(A, B)$, equivalently defined by the following one-to-one correspondences

$$\begin{aligned} \mathbf{Alg}_k(A, \mathrm{Hom}_k(C,B)) &\cong \{\sigma \in \mathrm{Hom}_k(C \otimes A, B) | \sigma \text{ measures}\} \qquad (6.1) \\ &\cong \mathbf{Coalg}_k(C, P(A,B)) \end{aligned}$$

where the first isomorphism comes from the definition of measuring, and the second expresses the universal property of $P(A, B)$. This object is called the *universal* *measuring coalgebra*, and in [**Swe69**, Theorem 7.0.4] is constructed as the sum of certain subcoalgebras of the cofree coalgebra on the vector space $\mathrm{Hom}_k(A,B)$.

As illustrated in Section 3.3, $\mathbf{Alg}_k$ and $\mathbf{Coalg}_k$ are the categories of monoids and comonoids and Hom_k is the internal hom in the symmetric monoidal closed category $\mathbf{Vect}_k$ of k-vector spaces and k-linear maps. The aim is to obtain a generalization of $P(A,B)$ in a broader setting, by identifying the appropriate assumptions on a monoidal category $\mathcal{V}$ in place of $\mathbf{Vect}_k$ which allow its existence.

Consider a symmetric monoidal closed category $\mathcal{V}$. The lax monoidal internal hom functor induces a functor between the categories of comonoids and monoids as in (3.17),

$$\begin{aligned}\mathbf{Mon}[-,-]:\ \mathbf{Comon}(\mathcal{V})^{\mathrm{op}}\times\mathbf{Mon}(\mathcal{V}) &\longrightarrow \mathbf{Mon}(\mathcal{V})\\ (\ C\ ,\ A\ ) &\longmapsto [C,A]\end{aligned}$$

which is in fact just the restriction of the internal hom on $\mathbf{Comon}(\mathcal{V})^{\mathrm{op}}\times\mathbf{Mon}(\mathcal{V})$. If we call this functor of two variables H, in order to generalize the isomorphism (6.1) it is enough to prove that the functor

$$H(-,B)^{\mathrm{op}}:\mathbf{Comon}(\mathcal{V})\longrightarrow\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$$

for a fixed monoid B has a right adjoint. Because of the useful properties of the categories of monoids and comonoids in admissible categories discussed in Section 3.3, and since $\mathbf{Vect}_k$ is itself an example of such a category, we continue in this direction.

PROPOSITION 6.1.1. *Suppose that $\mathcal{V}$ is a locally presentable symmetric monoidal closed category. There is an adjunction $H(-,B)^{\mathrm{op}}\dashv P(-,B)$ with a natural isomorphism*

$$\mathbf{Mon}(\mathcal{V})(A,[C,B])\cong\mathbf{Comon}(\mathcal{V})(C,P(A,B)) \tag{6.2}$$

for any monoids A, B and comonoid C.

PROOF. A monoidal category $\mathcal{V}$ with these properties belongs to the class of admissible categories, therefore Proposition 3.3.5 applies. As a result, the category of comonoids $\mathbf{Comon}(\mathcal{V})$ is a locally presentable category, and in particular cocomplete with a small dense subcategory. Moreover, there is a commutative diagram

$$\begin{tikzcd}\mathbf{Comon}(\mathcal{V})^{\mathrm{op}} \arrow[r, "H(-,B)"] \arrow[d, "U^{\mathrm{op}}"'] & \mathbf{Mon}(\mathcal{V}) \arrow[d, "S"] \\ \mathcal{V}^{\mathrm{op}} \arrow[r, "{[-,SB]}"'] & \mathcal{V}\end{tikzcd}$$

where the forgetful functors U, S are respectively comonadic and monadic. The bottom functor $[-,SB]$ is continuous as the right adjoint of $[-,SB]^{\mathrm{op}}$ as in (3.9), thus the diagram exhibits $H(-,B)$ as a continuous functor. Hence by Theorem 3.0.1, the cocontinuous $H(-,B)^{\mathrm{op}}$ has a right adjoint $P(-,B)$ with an isomorphism as in (6.2). Since this is natural in A and C, there is a unique way to define a functor of two variables

$$P(-,-):\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}\times\mathbf{Mon}(\mathcal{V})\longrightarrow\mathbf{Comon}(\mathcal{V}) \tag{6.3}$$

which is the parametrized adjoint of $H^{\mathrm{op}}(-,-)$ by 'adjunctions with a parameter' Theorem 3.0.2. ☐

The object $P(A,B)$ for monoids A, B is called the *universal measuring comonoid*, and the functor P is called the universal measuring comonoid functor or *Sweedler hom* in [**AJ13**]. Notice that in fact, a parametrized adjoint for H^{op} should have domain $\mathbf{Mon}(\mathcal{V})\times\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$, but it is more natural to work with an essentially identical functor which is contravariant on the first entry, just by switching the cartesian product in our notation.

In particular, for the admissible monoidal closed $\mathbf{Mod}_R$ for a commutative ring R, there is a natural isomorphism

$$\mathbf{Coalg}_R(C,P(A,B))\cong\mathbf{Alg}_R(A,\mathrm{Hom}_R(C,B)) \tag{6.4}$$

defining the *universal measuring coalgebra* $P(A,B)$. This is also given by [**Por08a**, Proposition 4].

REMARK 6.1.2. It is a well-known fact that the *dual* $C^*=\mathrm{Hom}_k(C,k)$ of a k-coalgebra, where k is viewed as an algebra over itself, has a natural structure of an algebra. On the other hand, if A is a k-algebra, its dual $A^*=\mathrm{Hom}_k(A,k)$ in general fails to be a coalgebra, unless for example it is finite dimensional as a k-vector space. This is due to the failure of the canonical linear map

$$V^*\otimes_k W^*\to(V\otimes_k W)^*$$

which gives the lax monoidal structure on Hom_k, to always be invertible. However, we can define the subspace

$$A^0=\{g\in A^*|\exists\text{ ideal }I\subset\mathrm{ker}g\text{ s.t. }(\mathrm{ker}g/I)\text{ f.d.}\}$$

of A^* which turns out to have the structure of a coalgebra. Then, the *dual algebra functor* $\mathrm{Hom}_k(-,k)=(-)^*$ is adjoint to $(-)^0$ via the classical isomorphism

$$\mathbf{Coalg}_k(C,A^0)\cong\mathbf{Alg}_k(A,C^*).$$

This is a special case of (6.4) for $R=k$, hence Proposition 6.1.1 in fact generalizes the dual algebra functor adjunction to $\mathbf{Mod}_R$, but also in a sense to a more general monoidal category $\mathcal{V}$, with $(-)^*\cong[-,I]$ and $(-)^0\cong P(-,I)$.

We now proceed to the statement and proof of a lemma which connects the adjunction (6.2) with the usual $(-\otimes C)\dashv[C,-]$ defining the internal hom.

LEMMA 6.1.3. *Suppose we have a monoid arrow $f:A\to[C,B]$ for A, B monoids, C a comonoid in a locally presentable symmetric monoidal closed category $\mathcal{V}$. If this arrow corresponds to $\bar{f}:A\otimes C\to B$ in $\mathcal{V}$ under $(-\otimes C)\dashv[C,-]$ and to $\hat{f}:C\to P(A,B)$ in* $\mathbf{Comon}(\mathcal{V})$ *under $H(-,B)^{\mathrm{op}}\dashv P(-,B)$, then the two* *transposes are connected via*

$$\bar{f} = (\varepsilon \otimes \hat{f}) \circ \mathrm{ev} \tag{6.5}$$

where ev *is the evaluation and* ε *the counit of the universal measuring comonoid adjunction.*

Proof. Consider the following diagram

$$\begin{tikzcd}
 & {[P(A,B),B] \otimes C} \arrow[r, "1\otimes\hat{f}"] \arrow[ddr, "{[\hat{f},1]\otimes 1}"] & {[P(A,B),B] \otimes P(A,B)} \arrow[dr, "\mathrm{ev}_B"] & \\
A \otimes C \arrow[ur, "\varepsilon_A\otimes 1"] \arrow[drr, "f\otimes 1"'] \arrow[rrr, dashed, "\bar{f}"'] & & & B \\
 & & {[C,B] \otimes C} \arrow[ur, "\mathrm{ev}_B"'] &
\end{tikzcd}$$

where the bottom composite defines $\bar{f}$. Notice that the counit ε in reality has components $H(P(A,B),B)^{\mathrm{op}} \to A$ in $\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$.

The left part of the diagram gives f from its transpose map $\hat{f}$ under $H(-,B)^{\mathrm{op}} \dashv P(-,B)$. The right part commutes by dinaturality as in (3.8) of the counit $\mathrm{ev}_D^E : [D,E] \otimes D \to E$ of the parametrized adjunction $(-\otimes-) \dashv [-,-]$. Therefore the diagram commutes and the relation (6.5) holds. □

We can now combine the existence of the universal measuring comonoid $P(A,B)$ with the theory of actions of monoidal categories in Section 4.3, in order to establish an enrichment of $\mathbf{Mon}(\mathcal{V})$ in the symmetric monoidal closed $\mathbf{Comon}(\mathcal{V})$. Recall that for any symmetric monoidal closed category $\mathcal{V}$, the internal hom

$$[-,-] : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \longrightarrow \mathcal{V}$$

is an action of the monoidal category $\mathcal{V}^{\mathrm{op}}$ on the category $\mathcal{V}$, as explained in Lemma 4.3.2. Furthermore, the restricted functor on the categories of comonoids and monoids $H = \mathbf{Mon}[-,-]$ is an action too, by the same lemma. Finally, the opposite functor of an action is still an action. Therefore, for the action

$$H^{\mathrm{op}} : \mathbf{Comon}(\mathcal{V}) \times \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \longrightarrow \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \tag{6.6}$$

of the symmetric monoidal closed category $\mathbf{Comon}(\mathcal{V})$ (see Proposition 3.3.6) on the ordinary category $\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$, Corollaries 4.3.4 and 4.3.5 apply.

Theorem 6.1.4. *Let* $\mathcal{V}$ *be a locally presentable symmetric monoidal closed category and* P *the Sweedler hom functor.*

(1) The opposite category of monoids $\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$ *is enriched in the category of comonoids* $\mathbf{Comon}(\mathcal{V})$*, with hom-objects*

$$\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}(A,B) = P(B,A)$$

where the $\mathbf{Comon}(\mathcal{V})$*-enriched category is denoted by the same name.*

(2) The category of monoids $\mathbf{Mon}(\mathcal{V})$ *is a tensored and cotensored* $\mathbf{Comon}(\mathcal{V})$*-enriched category, with hom-objects*

$$\mathbf{Mon}(\mathcal{V})(A,B) = P(A,B)$$

and cotensor products $[C,B]$ *for any comonoid* C *and monoid* B*.*

Proof. By Proposition 6.1.1, there is an adjunction

$$\begin{tikzcd}\mathbf{Comon}(\mathcal{V}) \arrow[r, shift left, "H(-,B)^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] & \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \arrow[l, shift left, "P(-,B)"]\end{tikzcd}$$

which defines the bifunctor P (6.3) as the parametrized adjoint of the bifunctor H^{op}. The latter is an action, thus an enrichment of the category acted on is induced, as well as of its opposite category $\mathbf{Mon}(\mathcal{V})$ since the monoidal category $\mathbf{Comon}(\mathcal{V})$ is symmetric.

In particular, since $\mathbf{Comon}(\mathcal{V})$ is closed, the action $[-,-]$ which induces the enrichment of $\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$ renders it a tensored $\mathbf{Comon}(\mathcal{V})$-category, hence its opposite enriched category is cotensored. On the other hand, $\mathbf{Mon}(\mathcal{V})$ is also a tensored $\mathbf{Comon}(\mathcal{V})$-category because the functor

$$H(C,-)^{\mathrm{op}} : \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \longrightarrow \mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$$

has a right adjoint for every comonoid C. This follows from the Adjoint Triangle Theorem (see [**Dub68**]) applied to the commutative diagram

$$\begin{tikzcd}\mathbf{Mon}(\mathcal{V}) \arrow[r, "H(C,-)"] \arrow[d, "S"'] & \mathbf{Mon}(\mathcal{V}) \arrow[d, "S"] \\ \mathcal{V} \arrow[r, "{[C,-]}"'] & \mathcal{V}.\end{tikzcd}$$

The forgetful S is monadic, the locally presentable $\mathbf{Mon}(\mathcal{V})$ has coequalizers and $[C,-]$ has a left adjoint $(-\otimes C)$. Therefore $H(C,-)$ has a left adjoint $C \triangleright -$ for all C's and so there is a unique way to define a bifunctor

$$\triangleright : \mathbf{Comon}(\mathcal{V}) \times \mathbf{Mon}(\mathcal{V}) \longrightarrow \mathbf{Mon}(\mathcal{V}). \tag{6.7}$$

In [**AJ13**], this functor is called the *Sweedler product.* □

6.2. Global categories of modules and comodules

In Section 3.4, the categories $\mathbf{Mod}_{\mathcal{V}}(A)$ and $\mathbf{Comod}_{\mathcal{V}}(C)$ of A-modules and C-comodules for a monoid A and a comonoid C in a monoidal category $\mathcal{V}$ were defined. The idea here is that there exist global categories of modules and comodules, which contain all these 'fixed (co)monoids' categories, with appropriate arrows between modules and comodules of actions and coactions from different sources. These global categories are central for the development of this thesis, and their construction is interrelated with the theory of fibrations and opfibrations.

Definition 6.2.1. The *global category of comodules* **Comod** is the category of all C-comodules X for any comonoid C, denoted by X_C. A morphism $k_g : X_C \to Y_D$ for X a C-comodule and Y a D-comodule consists of a comonoid morphism $g : C \to D$ and an arrow $k : X \to Y$ in $\mathcal{V}$ which makes the diagram

$$\begin{tikzcd}
X \arrow[r, "\delta"] \arrow[d, "k"'] & X \otimes C \arrow[r, "1\otimes g"] & X \otimes D \arrow[d, "k\otimes 1"] \\
Y \arrow[rr, "\delta"'] & & Y \otimes D
\end{tikzcd}$$

commute. Dually, the *global category of modules* **Mod** has as objects all A-modules M for any monoid A, and morphisms are $p_f : M_A \to N_B$ where $f : A \to B$ is a monoid morphism and $p : M \to N$ makes the dual diagram

$$\begin{tikzcd}
A \otimes M \arrow[rr, "\mu"] \arrow[d, "1\otimes p"'] & & M \arrow[d, "p"] \\
A \otimes N \arrow[r, "f\otimes 1"'] & B \otimes N \arrow[r, "\mu"'] & N
\end{tikzcd}$$

commute. Conventially, unless otherwise stated the modules considered will be left and the comodules considered will be right.

There are obvious forgetful functors

$$G : \mathbf{Mod} \longrightarrow \mathbf{Mon}(\mathcal{V}) \quad \text{and} \quad V : \mathbf{Comod} \longrightarrow \mathbf{Comon}(\mathcal{V}) \tag{6.8}$$

which simply map any module M_A/comodule X_C to its monoid A/comonoid C and the morphisms to their monoid/comonoid part respectively. In fact, G is a split fibration and V is a split opfibration: the descriptions of the global categories agree with the Grothendieck categories for specific (strict) functors

$$\begin{tikzcd}
\mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \arrow[r, "\mathbf{Mod}_{\mathcal{V}}"] & \mathbf{Cat} & \mathbf{Comon}(\mathcal{V}) \arrow[r, "\mathbf{Comod}_{\mathcal{V}}"] & \mathbf{Cat} \\
A \arrow[r, mapsto, dotted] \arrow[d, "f"'] & \mathbf{Mod}_{\mathcal{V}}(A) & C \arrow[r, mapsto, dotted] \arrow[d, "g"'] & \mathbf{Comod}_{\mathcal{V}}(C) \arrow[d, "g_!"] \\
B \arrow[r, mapsto, dotted] & \mathbf{Mod}_{\mathcal{V}}(B) \arrow[u, "f^*"'] & D \arrow[r, mapsto, dotted] & \mathbf{Comod}_{\mathcal{V}}(D)
\end{tikzcd}$$

where f^* and $g_!$ are the restriction and corestriction of scalars as in (3.25) and (3.27).

Remark. Under the assumptions of Proposition 3.4.4, the functor f^* has a left adjoint and the functor $g_!$ has a right adjoint. Thus by Remark 5.1.1, when $\mathcal{V}$ has and $A \otimes -$ preserves coequalizers for any monoid A, the fibration G is a bifibration. Dually, when $\mathcal{V}$ has and $- \otimes C$ preserves equalizers for any comonoid C, the opfibration V is a bifibration.

If we unravel the Grothendieck construction of Theorem 5.2.1, we have the following equivalent characterization of, for example, **Comod**:

· Objects are pairs (X, C) with $C \in \mathbf{Comon}(\mathcal{V})$ and $X \in \mathbf{Comod}_{\mathcal{V}}(C)$.

· Morphisms are pairs $(k, g) : (X, C) \to (Y, D)$ with

$$\begin{cases} g_!X \xrightarrow{k} Y & \text{in } \mathbf{Comod}_{\mathcal{V}}(D) \\ \quad C \xrightarrow{g} D & \text{in } \mathbf{Comon}(\mathcal{V}). \end{cases}$$

· Composition $X_C \xrightarrow{(k,g)} Y_D \xrightarrow{(l,h)} Z_E$ is given by

$$\begin{cases} (hg)_!X \xrightarrow{\theta} Z & \text{in } \mathbf{Comod}_{\mathcal{V}}(E) \\ \qquad C \xrightarrow{hg} E & \text{in } \mathbf{Comon}(\mathcal{V}) \end{cases}$$

where θ is the composite $(hg)_!X = h_!g_!X \xrightarrow{h_!l} h_!Y \xrightarrow{k} Z$.

· The identity morphism is

$$\begin{cases} X \xrightarrow{1_X} X & \text{in } \mathbf{Comod}_{\mathcal{V}}(C) \\ C \xrightarrow{1_C} C & \text{in } \mathbf{Comon}(\mathcal{V}) \end{cases}$$

since $(1_C)_!X = X$.

By comparing this with Definition 6.2.1, we deduce that $\mathbf{Comod} = \mathfrak{G}(\mathbf{Comod}_{\mathcal{V}})$ in a straightforward way. Dually $\mathbf{Mod} = \mathfrak{G}(\mathbf{Mod}_{\mathcal{V}})$, so objects M_A can be seen as pairs (M, A) with $A \in \mathbf{Mon}(\mathcal{V})$ and $M \in \mathbf{Mod}_{\mathcal{V}}(A)$, and morphisms p_f as

$$\begin{cases} M \xrightarrow{p} f^*N & \text{in } \mathbf{Mod}_{\mathcal{V}}(A) \\ A \xrightarrow{f} B & \text{in } \mathbf{Mon}(\mathcal{V}). \end{cases}$$

Since these presentations of the global categories are essentially the same, we can freely use the notation which is more convenient depending on the case. The fibre categories for $V = U_{\mathbf{Comod}_{\mathcal{V}}}$ and $G = P_{\mathbf{Mod}_{\mathcal{V}}}$ are respectively $\mathbf{Comod}_{\mathcal{V}}(C)$ and $\mathbf{Mod}_{\mathcal{V}}(A)$ and the canonical chosen cartesian and cocartesian liftings are

$$\mathrm{Cart}(f, N) : f^*N \xrightarrow{(1_{f^*N}, f)} N \text{ in } \mathbf{Mod}, \tag{6.9}$$

$$\mathrm{Cocart}(g, X) : X \xrightarrow{(1_{g_!X}, g)} g_!X \text{ in } \mathbf{Comod}.$$

Remark 6.2.2. There is another way of viewing the global category of modules $\mathbf{Mod}$ for a monoidal category $\mathcal{V}$. It is based on the observation that to give a lax functor of bicategories $\mathcal{MI} \to \mathcal{MV}$ which is identity on objects is to give an object in $\mathbf{Mod}$. I thank Steve Lack for explaining this point of view to me.

The bicategories are constructed as in the Remark 4.3.1(i), arising from the canonical actions of the monoidal categories $\mathcal{I}$, $\mathcal{V}$ on themselves via tensor product. For the unit monoidal category, we of course have that $\mathcal{MI}(0, 0) = \mathcal{MI}(0, 1) = \mathcal{MI}(1, 1) = \mathbf{1}$ and $\mathcal{MI}(1, 0) = \emptyset$. Such an identity-on-objects lax functor $\mathscr{F}$ would in particular consist of functors

$$\mathscr{F}_{0,1}, \mathscr{F}_{1,1} : 1 \rightrightarrows \mathcal{V}$$

which pick up two objects M and A in $\mathcal{V}$. The components of the natural transformations δ as in (2.3) give arrows $\mu : A \otimes M \to A$ and $m : A \otimes A \to A$ in $\mathcal{V}$, the components of γ as in (2.4) give $\eta : I \to A$ and the axioms ensure that (A, m, η) is a monoid in $\mathcal{V}$ and (M, μ) is an A-module.

Then, morphisms in **Mod** are icons, as described in Remark 2.3.1: if M_A, N_B are two identity-on-objects lax functors between $\mathcal{MI}$ and $\mathcal{MV}$, an icon between them consists in particular of natural transformations

$$\begin{tikzcd} \mathbf{1} \arrow[r, bend left=50, "A"{name=U}] \arrow[r, bend right=50, "B"'{name=D}] \arrow[phantom, from=U, to=D, "\Downarrow f"] & \mathcal{V} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbf{1} \arrow[r, bend left=50, "M"{name=U}] \arrow[r, bend right=50, "N"'{name=D}] \arrow[phantom, from=U, to=D, "\Downarrow p"] & \mathcal{V} \end{tikzcd}$$

which are two arrows $f : A \to B$ and $p : M \to N$ in $\mathcal{V}$, subject to conditions which coincide with those of Definition 6.2.1.

Dually, colax natural transformations $\mathcal{MI} \to \mathcal{MV}$ correspond to comodules over comonoids, and icons then turn out to be comodule morphisms. Therefore we have

$$\mathbf{Mod} = \mathbf{Bicat}_2(\mathcal{MI}, \mathcal{MV})_l$$
$$\mathbf{Comod} = \mathbf{Bicat}_2(\mathcal{MI}, \mathcal{MV})_c$$

where $\mathbf{Bicat}_2$ is the 2-category of bicategories, lax/colax functors and icons (see [**Lac10b**]).

We now explore some of the main properties of the global categories. First of all, if $\mathcal{V}$ is a symmetric monoidal category, **Comod** and **Mod** are symmetric monoidal categories as well. It is easy to verify that if s is the symmetry in $\mathcal{V}$, the object $X_C \otimes Y_D$ in $\mathcal{V}$ for $X_C, Y_D \in \mathbf{Comod}$ is a comodule over the comonoid $C \otimes D$ via the coaction

$$X \otimes Y \xrightarrow{\delta_X \otimes \delta_Y} X \otimes C \otimes Y \otimes D \xrightarrow{1 \otimes s \otimes 1} X \otimes Y \otimes C \otimes D. \tag{6.10}$$

The fact that $\mathbf{Comon}(\mathcal{V})$ is monoidal itself is evidently required, which holds again due to symmetry of $\mathcal{V}$. Notice that there is no appropriate way of endowing the fibre categories $\mathbf{Comod}_{\mathcal{V}}(C)$ with a monoidal structure in general, since for example, the tensor product of two C-comodules would end up as a $C \otimes C$-module by the above. Similarly, for $M_A, N_B \in \mathbf{Mod}$, the object $M_A \otimes N_B$ is a $A \otimes B$-module via the action

$$A \otimes B \otimes M \otimes N \xrightarrow{1 \otimes s \otimes 1} A \otimes M \otimes B \otimes N \xrightarrow{\mu_M \otimes \mu_N} M \otimes N.$$

The symmetry of **Mod** and **Comod** is inherited from $\mathcal{V}$. Moreover, in this case the functors V and G of (6.8) have the structure of a strict symmetric monoidal functor:

$$V(X_C \otimes Y_D) = C \otimes D = VX_C \otimes VY_D \tag{6.11}$$
$$G(M_A \otimes N_B) = A \otimes B = GM_A \otimes GN_B.$$

The monoidal unit in both cases is I, with a trivial I-action and coaction via r_I.

The following result, also mentioned at the end of [**Wis75**] for $\mathcal{V} = \mathbf{Mod}_R$, illustrates the structure of the global categories.

Proposition 6.2.3. *The functor* $F : \mathbf{Comod} \to \mathcal{V} \times \mathbf{Comon}(\mathcal{V})$ *which maps an object* X_C *to the pair* (X, C) *is comonadic.*

Proof. First notice that F 'consists of' the forgetful functor which discards the comodule structure from the object of $\mathcal{V}$, and the forgetful V which keeps the comonoid. Therefore this result is, in a sense, a generalization of Proposition 3.4.1.

Define a functor

$$\begin{tikzcd}
R:\ \mathcal{V}\times\mathbf{Comon}(\mathcal{V}) \arrow[r] & \mathbf{Comod} \\
(A,D) \arrow[r, mapsto, dotted] \arrow[d, "{(l,g)}"'] & (A\otimes D)_D \arrow[d, "(l\otimes g)_g"] \\
(B,E) \arrow[r, mapsto, dotted] & (B\otimes E)_E
\end{tikzcd}$$

where the D-action on the object $A\otimes D$ is given by $A\otimes D\xrightarrow{1\otimes\Delta}A\otimes D\otimes D$ with Δ the comultiplication of the comonoid D. It is not hard to establish a natural bijection

$$\begin{aligned}
(\mathcal{V}\times\mathbf{Comon}(\mathcal{V}))((X,C),(A,D)) &\cong \mathcal{V}(X,A)\times\mathbf{Comon}(\mathcal{V})(C,D)\\
&\cong \mathbf{Comod}(X_C,(A\otimes D)_D)
\end{aligned}$$

where $(X,C)=F(X_C)$ and $(A\otimes D)_D=R(A,D)$, so we obtain an adjunction

$$\begin{tikzcd}
\mathbf{Comod} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{V}\times\mathbf{Comon}(\mathcal{V}). \arrow[l, shift left, "R"]
\end{tikzcd}$$

This induces a comonad on $\mathcal{V}\times\mathbf{Comon}(\mathcal{V})$, namely $(FR,F\eta_R,\varepsilon)$, where the comultiplication and counit have components

$$\begin{aligned}
F\eta_{K(A,D)} &: (A\otimes D,D)\xrightarrow{(1\otimes\Delta,1)}(A\otimes D\otimes D,D)\\
\varepsilon_{(A,D)} &: (A\otimes D,D)\xrightarrow{(1\otimes\epsilon,1)}(A,D)
\end{aligned}$$

for the comonoid (D,Δ,ϵ). The category of coalgebras for this comonad is precisely **Comod**. □

This in particular implies that if $\mathcal{V}$ and $\mathbf{Comon}(\mathcal{V})$ are cocomplete categories, then **Comod** is also cocomplete. In fact, using results from Section 5.3 concerning fibrewise colimits, we can recover this as follows.

Corollary 6.2.4. *If $\mathcal{V}$ and $\mathbf{Comon}(\mathcal{V})$ have all colimits, then* **Comod** *has all colimits and $V:\mathbf{Comod}\to\mathbf{Comon}(\mathcal{V})$ strictly preserves them.*

Proof. Since every fibre $\mathbf{Comod}_{\mathcal{V}}(C)$ of the opfibration V is comonadic over $\mathcal{V}$, it has all colimits for any comonoid C. Moreover, the reindexing functors $\mathbf{Comod}_{\mathcal{V}}(g)=g_!$ preserve all colimits by the commutative diagram (3.28) for any comonoid arrow g. By Proposition 5.3.9, the opfibration $V:\mathbf{Comod}\to\mathbf{Comon}(\mathcal{V})$ has all opfibred colimits. Then, by the dual of Corollary 5.3.11, this is equivalent to the total category **Comod** being cocomplete and V being strictly cocontinuous. □

Colimits in **Comod** are therefore constructed as follows. If we consider a diagram $D:\mathcal{J}\to\mathbf{Comod}$, the composite functor

$$\mathcal{J}\xrightarrow{D}\mathbf{Comod}\xrightarrow{V}\mathbf{Comon}(\mathcal{V})$$

has a colimiting cocone $(\tau_j : VD_j \to \mathrm{colim}(VD)\,|\, j \in \mathcal{J})$ since $\mathbf{Comon}(\mathcal{V})$ is cocomplete. Define a new diagram

$$\begin{tikzcd}
\mathcal{J} \arrow[r, "H"] & \mathbf{Comod}_{\mathcal{V}}(\mathrm{colim}\, VD) \\
j \arrow[d, "\kappa"'] \arrow[r, mapsto, dotted] & (\tau_j)_! Dj = (\tau_{j'})_!(VD\kappa)_! Dj \arrow[d, "(\tau_{j'})_! D\kappa"] \\
j' \arrow[r, mapsto, dotted] & (\tau_{j'})_! Dj'
\end{tikzcd}$$

which, since the category $\mathbf{Comod}_{\mathcal{V}}(\mathrm{colim}\, VD)$ is cocomplete, also has a colimiting cocone $(\sigma_j : (\tau_j)_! Dj \to \mathrm{colim}\, H \,|\, j \in \mathcal{J})$. It turns out that

$$\big(D_j \xrightarrow{(\sigma_j, \tau_j)} \mathrm{colim}\, H \,|\, j \in \mathcal{J}\big)$$

is the colimiting cocone of D in **Comod**, and of course $V \,\mathrm{colim}\, D = \mathrm{colim}(VD)$.

Dually to the above results, we obtain the following.

PROPOSITION 6.2.5. *The global category of modules* **Mod** *is monadic over the category* $\mathcal{V} \times \mathbf{Mon}(\mathcal{V})$*, and so if* $\mathcal{V}$ *and* $\mathbf{Mon}(\mathcal{V})$ *are complete,* **Mod** *has all limits and* $G : \mathbf{Mod} \to \mathbf{Mon}(\mathcal{V})$ *strictly preserves them.*

Now suppose that $\mathcal{V}$ is a symmetric monoidal closed category. In Section 3.4 it was explained how the internal hom bifunctor induces a functor

$$\mathbf{Mod}_{CA}[-,-] : \mathbf{Comod}_{\mathcal{V}}(C)^{\mathrm{op}} \times \mathbf{Mod}_{\mathcal{V}}(A) \longrightarrow \mathbf{Mod}_{\mathcal{V}}([C,A])$$

as in (3.23), which is again the restriction of the internal hom on the cartesian product of the categories of C-comodules and A-modules. There is a way to lift this functor on the level of the global categories of comodules and modules, in the sense that there is a functor between the total categories

$$\begin{tikzcd}
\bar{H} :\ \mathbf{Comod}^{\mathrm{op}} \times \mathbf{Mod} \arrow[r] & \mathbf{Mod} \\
(\ X_C\ ,\ M_A\ ) \arrow[r, mapsto] & {[X,M]_{[C,A]}}
\end{tikzcd} \tag{6.12}$$

such that $\mathbf{Mod}_{CA}[-,-]$ are the functors induced between the fibres (see Remark 5.1.2). If $(k_g, l_f) : (X_C, M_A) \to (Y_D, N_B)$ is a morphism in the cartesian product, the fact that k and l commute with the corestricted and restricted actions accordingly forces the arrow $[k,l] : [X,M] \to [Y,N]$ in $\mathcal{V}$ to satisfy the appropriate property. Hence

$$\begin{cases} [X,M] \xrightarrow{[k,l]} [g,f]^*[Y,N] & \text{in } \mathbf{Mod}_{\mathcal{V}}[C,A] \\ [C,A] \xrightarrow{[g,f]} [D,B] & \text{in } \mathbf{Mon}(\mathcal{V}) \end{cases}$$

defines an arrow $\bar{H}(k,l)_{[g,f]} : [X,M]_{[C,A]} \to [Y,N]_{[D,B]}$ in **Mod**. In fact, the pair $(\bar{H}, H)$ is a fibred 1-cell depicted by the square

$$\begin{tikzcd}
\mathbf{Comod}^{\mathrm{op}} \times \mathbf{Mod} \arrow[r, "\bar{H}(-,-)"] \arrow[d, "V^{\mathrm{op}} \times G"'] & \mathbf{Mod} \arrow[d, "G"] \\
\mathbf{Comon}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Mon}(\mathcal{V}) \arrow[r, "H(-,-)"'] & \mathbf{Mon}(\mathcal{V}),
\end{tikzcd} \tag{6.13}$$

where of course the cartesian product $V^{\mathrm{op}} \times G$ is treated as a fibration. Commutativity is clear from the above construction, which ensures that

$$G([X,N]_{[C,B]}) = [VX_C, GN_B] = [C,B].$$

Moreover $\bar{H}$ is a cartesian functor: it maps a cartesian arrow of the domain, which is a pair of a cocartesian lifting in **Comod** and a cartesian lifting in **Mod**, to the arrow

$$\begin{tikzcd}
{[g_!Y, f^*N]} \arrow[r, "{\bar{H}(\mathrm{Cocart}(g,Y),\mathrm{Cart}(f,N))}"] \arrow[d, dotted] & {[Y,N]} \arrow[d, dotted] & \text{in } \mathbf{Mod} \\
{[C,A]} \arrow[r, "{[g,f]}"'] & {[D,B]} & \text{in } \mathbf{Mon}(\mathcal{V}).
\end{tikzcd}$$

By the canonical liftings (6.9) from the Grothendieck construction, that module arrow is specifically

$$\bar{H}((1_{g_!Y}, g), (1_{f^*N}, f)) = ([1_{g_!Y}, 1_{f^*N}], [g,f]) = (1_{[g_!Y, f^*N]}, [g,f])$$

by the definition of $\bar{H}$ and functoriality of $[-,-]$. On the other hand, the canonical cartesian lifting of $[Y,N]$ along $[g,f]$ is

$$\begin{tikzcd}
{[g,f]^*[Y,N]} \arrow[r, "{(1_{[g,f]^*[Y,N]},[g,f])}"] \arrow[d, dotted] & {[Y,N]} \arrow[d, dotted] & \text{in } \mathbf{Mod} \\
{[C,A]} \arrow[r, "{[g,f]}"'] & {[D,B]} & \text{in } \mathbf{Mon}(\mathcal{V}).
\end{tikzcd}$$

The above two arrows in **Mod** are essentially identical, both being $1_{[Y,N]} : [Y,N] \to [Y,N]$ as morphisms in $\mathcal{V}$ between the modules, and the $[C,A]$-actions on $[g_!Y, f^*N]$ and $[g,f]^*[Y,N]$ can be computed to be the same. Therefore $(\bar{H}, H)$ is actually a split fibred 1-cell.

Finally, suppose $\mathcal{V}$ is monoidal such that $\otimes$ preserves (filtered) colimits on both sides, and moreover locally presentable. It is not hard to see that the comonad on $\mathcal{V} \times \mathbf{Comon}(\mathcal{V})$ whose category of coalgebras is **Comod** (see Proposition 6.2.3) is finitary: if $(\lambda_j, \tau_j) : (X_j, C_j) \to (X, C)$ is a filtered colimiting cocone, then

$$(\lambda_j \otimes \tau_j, \tau_j) : (X_j \otimes C_j, C_j) \longrightarrow (X \otimes C, C)$$

is too, since $\otimes$ preserves colimits on both variables and $\mathbf{Comon}(\mathcal{V})$ is comonadic over $\mathcal{V}$. Dually, **Mod** is finitary monadic over $\mathcal{V} \times \mathbf{Mon}(\mathcal{V})$, since $(\lambda_j \otimes \tau_j, \tau_j) : (A_j \otimes M_j, A_j) \to (A \otimes M, A)$ is a filtered colimit when λ_j is a colimiting cocone in $\mathcal{V}$ and τ_j in $\mathbf{Mon}(\mathcal{V})$. This happens because the monadic $\mathbf{Mon}(\mathcal{V} \to \mathcal{V}$ creates all colimits that the finitary monad preserves (see Proposition 3.3.5(1)). Since $\mathcal{V}$, $\mathbf{Mon}(\mathcal{V})$ and $\mathbf{Comon}(\mathcal{V})$ are all locally presentable categories under the above assumptions, we can apply Theorem 3.4.3 for the global categories.

Theorem 6.2.6. *If $\mathcal{V}$ is a locally presentable monoidal category such that $(-\otimes-)$ is finitary on both entries,* **Mod** *and* **Comod** *are locally presentable.*

6.3. Universal measuring comodule and enrichment

The notion of a universal measuring comodule in the category of vector spaces $\mathbf{Vect}_k$ was first introduced by Batchelor in [**Bat00**], where emphasis was given to its applications. Very similarly to the context of measuring coalgebras, a k-linear map $\psi : X \to \mathrm{Hom}_k(M, N)$ is said to *measure* if it satisfies

$$\psi(x)(am) = \sum_{(x)} \phi x_{(1)}(a)\psi x_{(0)}(m)$$

again using sigma notation. Here X is a C-comodule, M an A-module and N a B-module, for (C, ϕ) a measuring coalgebra and A,B algebras. The pair (X, ψ) is called *measuring comodule.* The question that gave rise to this definition is whether the transpose arrow $\bar{\psi} : M \to \mathrm{Hom}_k(X, N)$ is a map of A-modules, using the symmetry in $\mathbf{Vect}_k$ and the module structure on $\mathrm{Hom}_k(X, N)$.

There is a category of measuring comodules for a fixed measuring coalgebra C, and it has a terminal object $Q(M, N)$ satisfying the property that there is a correspondence

$$\big\{C\text{-comodule maps } X \to Q(M,N)\big\} \leftrightarrow \big\{A\text{-module maps } M \to \mathrm{Hom}_k(X,N)\big\}. \tag{6.14}$$

The object $Q(M, N)$ is called *universal measuring comodule.* Initially, the goal is to extend the existence of the universal measuring comodule in a more general context than $\mathbf{Vect}_k$.

Consider a symmetric monoidal closed category $\mathcal{V}$. In the end of the previous section, we defined a functor of two variables $\bar{H} : \mathbf{Comod}^{\mathrm{op}} \times \mathbf{Mod} \to \mathbf{Mod}$ which maps a comodule and a module to their internal hom in $\mathcal{V}$. Since the aim is a generalization of the correspondence (6.14) in order to define the universal measuring comodule, in fact we need a natural isomorphism

$$\mathbf{Comod}(X, Q(M,N)) \cong \mathbf{Mod}(M, \bar{H}(X,N))$$

where $X = X_C$, $M = M_A$, $N = N_B$ and $\bar{H}(X, N) = [X, N]_{[C,B]}$. Thus it is enough to show that the functor $\bar{H}(-, N_B)^{\mathrm{op}} : \mathbf{Comod} \longrightarrow \mathbf{Mod}^{\mathrm{op}}$ for a fixed B-module N has a right adjoint.

Moreover, we intend to show that $Q(M, N)$ is a comodule over the universal measuring coalgebra, hence the assumptions on $\mathcal{V}$ have to also cover the existence of $P(A, B)$. The following result is an application of Theorem 5.3.7 in the abstract setting of (op)fibrations. A direct proof can be found at the end of this chapter.

Proposition 6.3.1. *Let $\mathcal{V}$ be a locally presentable symmetric monoidal closed category. There is an adjunction*

$$\begin{tikzcd}\mathbf{Comod} \arrow[r, shift left, "\bar{H}(-,N_B)^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] & \mathbf{Mod}^{\mathrm{op}} \arrow[l, shift left, "Q(-,N_B)"]\end{tikzcd}$$

between the global categories of modules and comodules, with a natural isomorphism

$$\mathbf{Comod}(X_C, Q(M,N)_{P(A,B)}) \cong \mathbf{Mod}(M_A, [X,N]_{[C,B]}). \tag{6.15}$$

Proof. The pair of bifunctors $(\bar{H}, H)$ depicted as (6.13) constitutes a fibred 1-cell between the fibrations $V^{\mathrm{op}} \times G$ and G, as shown earlier. This implies that the pair of functors $(\bar{H}(-, N_B), H(-, B))$ for a fixed monoid B and a B-module N is again a fibred 1-cell between V^{op} and G, and hence the opposite square

$$\begin{tikzcd}
\mathbf{Comod} \arrow[r, "\bar{H}(-,N_B)^{\mathrm{op}}"] \arrow[d, "V"'] & \mathbf{Mod}^{\mathrm{op}} \arrow[d, "G^{\mathrm{op}}"] \\
\mathbf{Comon}(\mathcal{V}) \arrow[r, "H(-,B)^{\mathrm{op}}"'] & \mathbf{Mon}(\mathcal{V})^{\mathrm{op}}
\end{tikzcd}$$

is an opfibred 1-cell between the opfibrations V and G^{op}. Also, by Proposition 6.1.1 there is an adjunction between the base categories

$$\begin{tikzcd}
\mathbf{Comon}(\mathcal{V}) \arrow[r, shift left, "H(-,B)^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] & \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \arrow[l, shift left, "P(-,B)"]
\end{tikzcd}$$

where P is the Sweedler hom functor.

In order for Lemma 5.3.6 to apply, we need the existence of a right adjoint of the composite functor

$$\mathbf{Comod}_{\mathcal{V}}(P(A,B)) \xrightarrow{\bar{H}(-,N_B)^{\mathrm{op}}_{P(A,B)}} \mathbf{Mod}^{\mathrm{op}}_{\mathcal{V}}([P(A,B),B]) \xrightarrow{(\varepsilon_A)_!} \mathbf{Mod}^{\mathrm{op}}_{\mathcal{V}}(A) \quad (6.16)$$

where

$$\varepsilon^B_A : H(P(A,B),B) \to A \quad \text{in} \quad \mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$$

are the components of the counit of the parametrized adjunction $H^{\mathrm{op}} \dashv P$. We already know that $\mathbf{Comod}_{\mathcal{V}}(C)$ is a locally presentable category by Proposition 3.4.2, so cocomplete with a small dense subcategory, namely the presentable objects. Moreover, the reindexing functors are always cocontinuous as seen in Section 3.4, hence so is $(\varepsilon_A)_!$ of the opfibration V^{op}. Finally, the following commutative diagram

$$\begin{tikzcd}
\mathbf{Comod} \arrow[r, "\bar{H}(-,N_B)^{\mathrm{op}}"] \arrow[d] & \mathbf{Mod}^{\mathrm{op}} \arrow[d] \\
\mathcal{V} \times \mathbf{Comon}(\mathcal{V}) \arrow[r, "{[-,N]^{\mathrm{op}} \times H(-,B)^{\mathrm{op}}}"'] & \mathcal{V}^{\mathrm{op}} \times \mathbf{Mon}(\mathcal{V})^{\mathrm{op}}
\end{tikzcd} \quad (6.17)$$

implies that $\bar{H}(-, N_B)^{\mathrm{op}}$ preserves all colimits: both functors at the bottom have right adjoints, and the vertical functors create all colimits by Propositions 6.2.3 and 6.2.5. Since the fibres of the total categories **Comod** and $\mathbf{Mod}^{\mathrm{op}}$ are closed under colimits, the restricted fibrewise functor $\bar{H}(-, N_B)^{\mathrm{op}}_{P(A,B)}$ is cocontinuous too.

Consequently, by Theorem 3.0.1 the composite (6.16) has a ‘fibrewise’ right adjoint

$$Q_A(-, N_B) : \mathbf{Mod}_{\mathcal{V}}(A)^{\mathrm{op}} \longrightarrow \mathbf{Comod}_{\mathcal{V}}(P(A,B))$$

and Theorem 5.3.7 implies that this lifts to a functor between the total categories $Q(-,N_B) : \mathbf{Mod}^{\mathrm{op}} \longrightarrow \mathbf{Comod}$ such that

$$\begin{tikzcd}
\mathbf{Comod} \arrow[r, shift left, "\bar{H}(-{,}N_B)^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] \arrow[d, "V"'] & \mathbf{Mod}^{\mathrm{op}} \arrow[l, shift left, "Q(-{,}N_B)"] \arrow[d, "G^{\mathrm{op}}"] \\
\mathbf{Comon}(\mathcal{V}) \arrow[r, shift left, "H(-{,}B)^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] & \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \arrow[l, shift left, "P(-{,}B)"]
\end{tikzcd}$$

is an adjunction in $\mathbf{Cat^2}$. The isomorphism (6.15) for the adjunction between the total categories, natural in X_C and M_A, makes this adjoint uniquely into a functor of two variables

$$Q(-,-) : \mathbf{Mod}^{\mathrm{op}} \times \mathbf{Mod} \longrightarrow \mathbf{Comod}$$

such that the isomorphism is natural in all three variables. In other words, Q is the parametrized adjoint of the bifunctor $\bar{H}^{\mathrm{op}}$. ☐

The bifunctor Q is called the *universal measuring comodule functor*. By construction of Q, the object $Q(M_A, N_B)$ has the structure of a $P(A,B)$-comodule.

Similarly, we can show that the symmetric monoidal category **Comod** has a monoidal closed structure.

Proposition 6.3.2. *The global category of comodules* **Comod** *for a locally presentable symmetric monoidal closed category* $\mathcal{V}$ *is a symmetric monoidal closed category.*

Proof. By the definition of the symmetric monoidal tensor product

$$\begin{aligned} \otimes :\ \mathbf{Comod} \times \mathbf{Comod} &\longrightarrow \mathbf{Comod} \\ (\ Y_D\ ,\ X_C\ ) &\longmapsto (Y \otimes X)_{D\otimes C} \end{aligned}$$

in **Comod** as in (6.10), we have a commutative square

$$\begin{tikzcd}
\mathbf{Comod} \arrow[r, "(-\otimes X_C)"] \arrow[d, "V"'] & \mathbf{Comod} \arrow[d, "V"] \\
\mathbf{Comon}(\mathcal{V}) \arrow[r, "(-\otimes C)"'] & \mathbf{Comon}(\mathcal{V}).
\end{tikzcd} \tag{6.18}$$

Actually, this is an opfibred 1-cell: the functor $(-\otimes X_C)$ for a fixed C-comodule X maps a cocartesian lifting to the right top arrow

$$\begin{tikzcd}
Y \arrow[r, "\mathrm{Cocart}(f{,}Y)"] \arrow[dd, dash, dotted] & f_!Y \arrow[dd, dash, dotted] & & Y\otimes X \arrow[r, "\mathrm{Cocart}(f{,}Y)\otimes 1"] \arrow[dr, "\mathrm{Cocart}"'] \arrow[dd, dash, dotted] & f_!Y\otimes X & \text{in } \mathbf{Comod} \\
& & \mapsto & & (f\otimes 1)_!(Y\otimes X) \arrow[u, dashed, "\exists!"'] \arrow[d, dash, dotted] & \\
D \arrow[r, "f"'] & E & & D\otimes C \arrow[r, "f\otimes 1"'] & E\otimes C & \text{in } \mathbf{Comon}(\mathcal{V}).
\end{tikzcd}$$

The two $(E\otimes C)$-comodules $f_!Y\otimes X$ and $(f\otimes 1)_!(Y\otimes X)$ are both $Y\otimes X$ as objects in $\mathcal{V}$, and the coactions induced in both cases are equal. Hence, by the canonical choice of cocartesian liftings for the opfibration $V:\mathbf{Comod}\to\mathbf{Comon}(\mathcal{V})$ and functoriality of the tensor product, $1_{(f\otimes 1)_!(Y\otimes X)}=1_{f_!Y}\otimes 1_X$ and so $(-\otimes X_C)$ is a cocartesian functor.

By Proposition 3.3.6, the category of comonoids for such a monoidal category $\mathcal{V}$ is monoidal closed with internal hom functor Hom via an adjunction

$$\begin{tikzcd}\mathbf{Comon}(\mathcal{V}) \arrow[r, shift left, "(-\otimes C)"] \arrow[r, phantom, "\perp"] & \mathbf{Comon}(\mathcal{V}) \arrow[l, shift left, "\textsc{Hom}(C,-)"]\end{tikzcd}$$

between the bases of (6.18). Finally, if ε is the counit for this adjunction, the composite functor

$$\mathbf{Comod}_{\mathcal{V}}(\textsc{Hom}(C,D))\xrightarrow{(-\otimes X_C)}\mathbf{Comod}_{\mathcal{V}}(\textsc{Hom}(C,D)\otimes C)\xrightarrow{(\varepsilon_D)_!}\mathbf{Comod}_{\mathcal{V}}(D)$$

has a right adjoint $\overline{\textsc{Hom}}_D(X_C,-)$. This follows from the adjoint functor Theorem 3.0.1, since $\mathbf{Comod}_{\mathcal{V}}(\textsc{Hom}(C,D))$ is locally presentable and the composite functor preserves all colimits. This is the case because reindexing functors are always cocontinuous, and the commutative diagram

$$\begin{tikzcd}\mathbf{Comod} \arrow[r, "(-\otimes X_C)"] \arrow[d, "F"'] & \mathbf{Comod} \arrow[d, "F"] \\ \mathcal{V}\times\mathbf{Comon}(\mathcal{V}) \arrow[r, "(-\otimes X)\times(-\otimes C)"] & \mathcal{V}\times\mathbf{Comon}(\mathcal{V})\end{tikzcd} \tag{6.19}$$

implies that $(-\otimes X_C)$ preserves all colimits, since the bottom arrow does by monoidal closedness of $\mathcal{V}$ and $\mathbf{Comon}(\mathcal{V})$, and F is comonadic.

By Theorem 5.3.7, the functors $\overline{\textsc{Hom}}_D(X_C,-)$ between the fibres assemble into a total adjoint $\overline{\textsc{Hom}}(X_C,-):\mathbf{Comod}\to\mathbf{Comod}$ such that

$$\begin{tikzcd}\mathbf{Comod} \arrow[r, shift left, "-\otimes X_C"] \arrow[r, phantom, "\perp"] \arrow[d, "V"'] & \mathbf{Comod} \arrow[l, shift left, "\overline{\textsc{Hom}}(X_C,-)"] \arrow[d, "V"] \\ \mathbf{Comon}(\mathcal{V}) \arrow[r, shift left, "-\otimes C"] \arrow[r, phantom, "\perp"] & \mathbf{Comon}(\mathcal{V}) \arrow[l, shift left, "\textsc{Hom}(C,-)"]\end{tikzcd}$$

is an adjunction in $\mathbf{Cat^2}$. Thus the uniquely defined parametrized adjoint

$$\overline{\textsc{Hom}}:\mathbf{Comod}^{\mathrm{op}}\times\mathbf{Comod}\longrightarrow\mathbf{Comod} \tag{6.20}$$

of $(-\otimes-)$ is the internal hom of the global category of comodules $\mathbf{Comod}$. □

Remark 6.3.3. An alternative approach for the existence of the functors Q and $\overline{\textsc{Hom}}$ would be to show that

$$\bar{H}^{\mathrm{op}}(-,N_B):\mathbf{Comod}\longrightarrow\mathbf{Mod}^{\mathrm{op}}$$

$$-\otimes X_C:\mathbf{Comod}\longrightarrow\mathbf{Comod}$$

have right adjoints via an adjoint functor theorem. Both functors are cocontinuous by diagrams (6.17) and (6.19) respectively, and the domain **Comod** is locally presentable by Theorem 6.2.6. Hence Theorem 3.0.1 directly establishes the existence of right adjoints. However, we prefer the method which employs the fibrational structure of the global categories, because it provides with a better understanding of the situation. For example, the above proposition ensures that $\overline{\mathrm{Hom}}(X_C, Y_D)$ is specifically a $\textsc{Hom}(C, D)$-comodule.

We can now once more combine the existence of the universal measuring comodule with the theory of actions of monoidal categories, in order to show how the functor Q induces an enrichment of the global category of modules in the global category of comodules.

For any symmetric monoidal closed category $\mathcal{V}$, the functor of two variables $\bar{H}(-,-) : \mathbf{Comod}^{\mathrm{op}} \times \mathbf{Mod} \longrightarrow \mathbf{Mod}$ defined as in (6.12) is in fact an action of the symmetric monoidal category $\mathbf{Comod}^{\mathrm{op}}$ on the ordinary category **Mod**. It is easy to see that there exist natural isomorphisms

$$\begin{aligned} [X \otimes Y, M]_{[C\otimes D,A]} &\xrightarrow{\sim} [X, [Y, M]]_{[C,[D,A]]} \\ [I, M]_{[I,A]} &\xrightarrow{\sim} M_A \end{aligned}$$

for any coalgebras C, D, algebras A, comodules X_C, Y_D and modules M_A that satisfy the axioms of an action. This follows from the facts that $[-,-]$ and $H(-,-)$ are actions and the monadic functor $\mathbf{Mod} \to \mathcal{V} \times \mathbf{Mon}(\mathcal{V})$ reflects isomorphisms. Therefore the opposite functor

$$\bar{H}^{\mathrm{op}} : \mathbf{Comod} \times \mathbf{Mod}^{\mathrm{op}} \longrightarrow \mathbf{Mod}^{\mathrm{op}} \tag{6.21}$$

is an action of the symmetric monoidal **Comod** on $\mathbf{Mod}^{\mathrm{op}}$.

Since we have an adjunction $\bar{H}(-, N_B)^{\mathrm{op}} \dashv Q(-, N_B)$ for any module N_B by Proposition 6.3.1, Corollaries 4.3.4 and 4.3.5 apply and give the following result.

Theorem 6.3.4. *Let $\mathcal{V}$ be a locally presentable symmetric monoidal closed category and Q the universal measuring comodule functor.*

1. *The opposite of the global category of comodules* $\mathbf{Mod}^{\mathrm{op}}$ *is enriched in the global category of comodules* **Comod***, with hom-objects*

$$\mathbf{Mod}^{\mathrm{op}}(M_A, N_B) = Q(N, M)_{P(B,A)}$$

where the **Comod***-enriched category is denoted with the same name.*

2. *The global category of modules* **Mod** *is a tensored and cotensored* **Comod***-enriched category, with hom-objects*

$$\mathbf{Mod}(M_A, N_B) = Q(M, N)_{P(A,B)}$$

and cotensor products $[X, N]_{[C,B]}$ for any C-comodule X and B-module N.

Proof. The only part left to show is that the functor

$$\bar{H}(X_C, -)^{\mathrm{op}} : \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \longrightarrow \mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$$

has a right adjoint for every comodule X_C. Consider the commutative square

$$\begin{tikzcd}
\mathbf{Mod} \arrow[r, "\bar{H}(X_C,-)"] \arrow[d] & \mathbf{Mod} \arrow[d] \\
\mathcal{V} \times \mathbf{Mon}(\mathcal{V}) \arrow[r, "{[X,-]\times H(C,-)}"'] & \mathcal{V} \times \mathbf{Mon}(\mathcal{V})
\end{tikzcd}$$

where the vertical functors are monadic, **Mod** is locally presentable by Theorem 6.2.6, $[X,-] \vdash (- \otimes X)$ in $\mathcal{V}$ and $H(C,-)$ has a left adjoint as in (6.7). By Dubuc's Adjoint Triangle Theorem, the top functor has a left adjoint $X_C \bar{\triangleright} -$ for all X_C's, inducing a bifunctor

$$\bar{\triangleright} : \mathbf{Comod} \times \mathbf{Mod} \longrightarrow \mathbf{Mod}$$

which gives the tensor products of the **Comod**-enriched category **Mod**. $\square$

We finish this chapter by giving a direct proof of Proposition 6.3.1, which can also be found in [**Vas12**]. We should note here that the proof of the more general Theorem 5.3.7 in the context of opfibrations actually relied heavily on this special case of modules and comodules. These objects' nature and the effect of the well-behaved reindexing functors on them illustrate the correspondences between the hom-sets clearly and give insight for the generalized result.

PROOF 2. Suppose that $\mathcal{V}$ is a locally presentable monoidal closed category, P is the Sweedler hom as in (6.3) and $\bar{H}$ is the restricted internal hom between the global categories as in (6.12). We are going to explicitly establish a bijective correspondence

$$\mathbf{Comod}(X, Q_A(M,N)) \cong \mathbf{Mod}(M, \bar{H}(X,N)) \tag{6.22}$$

for any C-comodule X, A-module M and B-module N. The object $Q_{(-)}(M,N)$ arises once more from the existence of a 'special case adjunction'

$$\begin{tikzcd}
& \mathbf{Mod}^{\mathrm{op}}_{\mathcal{V}}([P(A,B),B]) \arrow[dr, "\varepsilon_{A!}"] & \\
\mathbf{Comod}_{\mathcal{V}}(P(A,B)) \arrow[ur, "\bar{H}(-,N_B)^{\mathrm{op}}"] \arrow[rr, phantom, "\perp"] & & \mathbf{Mod}^{\mathrm{op}}_{\mathcal{V}}(A) \arrow[ll, bend left, "Q_A(-,N_B)"]
\end{tikzcd}$$

with a natural isomorphism for Z a $P(A,B)$-comodule

$$\big(\mathbf{Comod}_{\mathcal{V}}(P(A,B))\big)(Z, Q_A(M,N)) \cong \big(\mathbf{Mod}_{\mathcal{V}}(A)\big)(M, (\varepsilon_A)^*[Z,N]). \tag{6.23}$$

An arbitrary element of $\mathbf{Comod}(X_C, Q_A(M,N)_{P(A,B)})$

$$\begin{cases} h_!X \xrightarrow{k} Q_A(M,N) & \text{in } \mathbf{Comod}_{\mathcal{V}}(P(A,B)) \\ C \xrightarrow{h} P(A,B) & \text{in } \mathbf{Comon}(\mathcal{V}) \end{cases}$$

corresponds uniquely to a pair of arrows

$$\begin{cases} M \xrightarrow{t} (\varepsilon_A)^*[h_!X, N] & \text{in } \mathbf{Mod}_{\mathcal{V}}(A) \\ A \xrightarrow{\tilde{h}} [C,B] & \text{in } \mathbf{Mon}(\mathcal{V}) \end{cases} \tag{6.24}$$

as follows: the top one is obtained via the special case adjunction (6.23) since $h_!X$ is a $P(A,B)$-comodule, and the bottom one via the adjunction (6.2). Here, $(\varepsilon_A)^*[h_!X,N]$ is an A-module via the induced A-action on $[X,N]$

$$\begin{tikzcd}
A\otimes[X,N] \arrow[r, "\varepsilon_A\otimes 1"] \arrow[rrd, dashed, bend right] & {[P(A,B),B]\otimes[X,N]} \arrow[r, "{[h,1]\otimes 1}"] & {[C,B]\otimes[X,N]} \arrow[d, "\mu"] \\
 & & {[X,N]}
\end{tikzcd}$$

where μ is the canonical $[C,B]$-action on $[X_C,N_B]$ given by (3.24). By definition of the global category **Mod**, t is a morphism $M\to[X,N]$ in $\mathcal{V}$ which is compatible with the respective A-actions. Thus the diagram (3.21) which it has to satisfy corresponds under the adjunction $(-\otimes X)\dashv[X,-]$ to

$$\begin{tikzcd}
A\otimes M\otimes X \arrow[r, "1\otimes t\otimes\delta"] \arrow[dd, "\mu\otimes 1"'] & A\otimes[X,N]\otimes X\otimes C \arrow[r, "\varepsilon\otimes 1\otimes 1"] \arrow[rddd, dashed, bend right, "(*)"] & {[P(A,B),B]\otimes[X,N]\otimes X\otimes C} \arrow[d, "1\otimes h"] \\
 & & {[P(A,B),B]\otimes[X,N]\otimes X\otimes P(A,B)} \arrow[d, "1\otimes s"] \\
M\otimes X \arrow[rddd, "\bar{t}"'] & & {[P(A,B),B]\otimes P(A,B)\otimes[X,N]\otimes X} \arrow[d, "\mathrm{ev}\otimes 1"] \\
 & & B\otimes[X,N]\otimes X \arrow[d, "1\otimes\mathrm{ev}"] \\
 & & B\otimes N \arrow[ld, "\mu"] \\
 & N &
\end{tikzcd} \tag{6.25}$$

where $\bar{t}:M\otimes X\to N$ is the adjunct of t in $\mathcal{V}$.

The goal is to show that the pair (6.24) is actually an element of the set $\mathbf{Mod}(M,\bar{H}(X,N))$, which is of the general form

$$\begin{cases} M\to f^*[X,N] & \text{in } \mathbf{Mod}_{\mathcal{V}}(A) \\ A\xrightarrow{f}[C,B] & \text{in } \mathbf{Mon}(\mathcal{V}) \end{cases}$$

for some $f:A\to[C,B]$, so that a bijective correspondence (6.22) will be established. For that, it is enough to prove that t coincides with an A-module map $M\to\tilde{h}^*[X,N]$, since there is already a monoid arrow $\tilde{h}:A\to[C,B]$. So the question would be whether t satisfies the commutativity of a diagram

$$\begin{tikzcd}
A\otimes M \arrow[r, "1\otimes t"] \arrow[dd, "\mu"'] & A\otimes[X,N] \arrow[rd, "\tilde{h}\otimes 1"] \arrow[dd, dotted] & \\
 & & {[C,B]\otimes[X,N]} \arrow[ld, "\mu"] \\
M \arrow[r, "t"'] & {[X,N]} &
\end{tikzcd}$$

which again under the adjunction $(- \otimes X) \dashv [X, -]$ translates, by rearranging the terms appropriately, to the diagram

$$\begin{tikzcd}
A \otimes M \otimes X \arrow[r, "1\otimes t\otimes\delta"] \arrow[ddd, "\mu\otimes 1"'] & A \otimes [X,N] \otimes X \otimes C \arrow[r, "\tilde{h}\otimes 1"] \arrow[ddr, dashed] & {[C,B] \otimes [X,N] \otimes X \otimes C} \arrow[d, "1\otimes s\otimes 1"] \\
 & (**) & {[C,B] \otimes C \otimes [X,N] \otimes X} \arrow[d, "\mathrm{ev}\otimes 1\otimes 1"] \\
 & & B \otimes [X,N] \otimes X \arrow[d, "1\otimes \mathrm{ev}"] \\
M \otimes X \arrow[dr, "\bar{t}"'] & & B \otimes N \arrow[dl, "\mu"] \\
 & N. &
\end{tikzcd} \qquad (6.26)$$

By inspection of the commutative diagram (6.25) and this one (6.26), it suffices to show that the parts $(*)$ and $(**)$ are the same for the latter to commute as well. Since the term $[X, N]$ remains unchanged, this comes down to the commutativity of

$$\begin{tikzcd}
 & {[P(A,B),B] \otimes C} \arrow[r, "1\otimes h"] & {[P(A,B),B] \otimes P(A,B)} \arrow[dr, "\mathrm{ev}"] & \\
A \otimes C \arrow[ur, "\varepsilon\otimes 1"] \arrow[drr, "\tilde{h}\otimes 1"'] & & & B. \\
 & & {[C,B] \otimes C} \arrow[ur, "\mathrm{ev}"'] &
\end{tikzcd}$$

This is satisfied by Lemma 6.1.3, since $h = \hat{\tilde{h}}$.

Thus a bijection (6.22) is established, and by standard arguments of adjunctions via representing objects and Theorem 3.0.2, this results once again to the existence of a parametrized adjoint $Q(-,-)$ of $H^{\mathrm{op}}(-,-)$. ☐

In essence, the above proof establishes that the A-modules $(\varepsilon_A)^*[h_! X, N]$ and $(\tilde{h})^*[X, N]$ are essentially the same. As objects they are both $[X, N]$, and their A-actions can be verified to coincide, when we conveniently translate them under the usual tensor-hom adjunction. If we compare this with the proof of Lemma 5.3.6, the above fact follows from the final diagram (5.27), where the part on the right is actually equality since we are now dealing with split fibrations and opfibrations, and the part on the left follows from cocartesianess (on the nose) of the functor $\bar{H}$ as shown at the end of Section 6.2. However, since in the direct proof neither cocartesianess nor splitness is explicitly used or mentioned, Lemma 6.1.3 incorporates the necessary information for the proof to be completed.

CHAPTER 7

Enrichment of $\mathcal{V}$-Categories and $\mathcal{V}$-Modules

7.1. The bicategory of $\mathcal{V}$-matrices

The bicategory of $\mathcal{V}$-matrices was mentioned in Examples 2.1.2 for $\mathcal{V} = \mathbf{Set}$. We now give a detailed description of enriched matrices and the structure of the bicategory they form, unravelling Definition 2.1.1 in this specific case. The main references here are [**BCSW83**] and [**KL01**]. In the former, the more general bicategory $\mathcal{W}$-**Mat** of matrices enriched in a bicategory $\mathcal{W}$ was studied, leading to the theory of bicategory enriched categories. For the one-object case, *i.e.* monoidal categories, the main results are in works of Bénabou [**Bén73**] and Wolff [**Wol74**].

Suppose that $\mathcal{V}$ is a cocomplete monoidal category, such that the functors $A \otimes -$ and $- \otimes A$ preserve colimits, as is certainly the case if $\mathcal{V}$ is monoidal closed. For sets X and Y, a *$\mathcal{V}$-matrix* $S : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ from X to Y is a functor $S : Y \times X \to \mathcal{V}$ given by a family

$$\{S(y,x)\}_{(x,y) \in X \times Y}$$

of objects in $\mathcal{V}$, where the set $Y \times X$ is viewed as a discrete category.

The bicategory $\mathcal{V}$-**Mat** consists of (small) sets X, Y as objects, $\mathcal{V}$-matrices $S : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ as 1-cells and natural transformations

$$\begin{tikzcd} Y \times X \arrow[r, bend left, "S"{name=U}] \arrow[r, bend right, "S'"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\sigma"] & \mathcal{V} \ =:\ X \arrow[r, bend left, "S"{name=A}, "|" marking] \arrow[r, bend right, "S'"'{name=B}, "|" marking] \arrow[Rightarrow, from=A, to=B, "\sigma"] & Y \end{tikzcd}$$

as 2-cells between $\mathcal{V}$-matrices S and S'. These are given by families of arrows

$$\sigma_{y,x} : S(y,x) \to S'(y,x)$$

in $\mathcal{V}$, for every $(x,y) \in X \times Y$. Hence the hom-category for two objects X and Y is the category

$$\mathcal{V}\text{-}\mathbf{Mat}(X,Y) = \mathcal{V}^{Y \times X}$$

with (vertical) composition of 2-cells being 'componentwise' in $\mathcal{V}$ and the identity 2-cell $1_S : S \Rightarrow S$ consisting of identity morphisms $(1_S)_{x',x} = 1_{S(x',x)}$ in $\mathcal{V}$. The horizontal composition

$$\circ : \mathcal{V}\text{-}\mathbf{Mat}(Y,Z) \times \mathcal{V}\text{-}\mathbf{Mat}(X,Y) \to \mathcal{V}\text{-}\mathbf{Mat}(X,Z)$$

maps two composable $\mathcal{V}$-matrices $T : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ and $S : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ to their composite 1-cell $T \circ S : X \relbar\joinrel\mapstochar\joinrel\rightarrow Z$, given by the family of objects in $\mathcal{V}$

$$(T \circ S)(z,x) = \sum_{y \in Y} T(z,y) \otimes S(y,x) \tag{7.1}$$

for all $z \in Z$ and $x \in X$. A pair of 2-cells $(\tau : T \Rightarrow T', \sigma : S \Rightarrow S')$ is mapped to the 2-cell $\tau * \sigma : T \circ S \Rightarrow T' \circ S'$ with components arrows

$$(\tau * \sigma)_{z,x} : \sum_{y \in Y} T(z,y) \otimes S(y,x) \xrightarrow{\sum \tau_{z,y} \otimes \sigma_{y,x}} \sum_{y \in Y} T'(z,y) \otimes S'(y,x) \tag{7.2}$$

in $\mathcal{V}$. For each set X, the identity 1-cell is $1_X : X \nrightarrow X$, which is given by

$$1_X(x',x) = \begin{cases} I, & \text{if } x = x' \\ 0, & \text{otherwise} \end{cases}$$

where I is the unit object in $\mathcal{V}$ and 0 is the initial object.

For composable $\mathcal{V}$-matrices $X \overset{S}{\nrightarrow} Y \overset{T}{\nrightarrow} Z \overset{R}{\nrightarrow} W$, the associator α has components invertible 2-cells

$$\alpha^{R,T,S} : (R \circ T) \circ S \xrightarrow{\sim} R \circ (T \circ S)$$

in $\mathcal{V}$-**Mat**, given by the family $\{\alpha_{w,x}\}_{w,x}$ of composite isomorphisms

$$\begin{tikzcd}
\sum\limits_{y \in Y} \big(\sum\limits_{z \in Z} R(w,z) \otimes T(z,y) \big) \otimes S(y,x) \arrow[r, dashed] \arrow[d, "\cong"'] & \sum\limits_{z \in Z} R(w,z) \otimes \big(\sum\limits_{y \in Y} T(z,y) \otimes S(y,x) \big) \\
\sum\limits_{\substack{y \in Y \\ z \in Z}} \big((R(w,z) \otimes T(z,y)) \otimes S(y,x) \big) \arrow[r, "\sum a"'] & \sum\limits_{\substack{y \in Y \\ z \in Z}} \big(R(w,z) \otimes (T(z,y) \otimes S(y,x)) \big) \arrow[u, "\cong"']
\end{tikzcd}$$

in $\mathcal{V}$. The isomorphism a is the associativity constraint of $\mathcal{V}$ and the vertical invertible arrows express the fact that $\otimes$ commutes with colimits. This definition clearly makes the horizontal composition associative up to isomorphism. Finally, for each $\mathcal{V}$-matrix $S : X \nrightarrow Y$, the unitors λ, ρ have components invertible 2-cells

$$\lambda^S : 1_Y \circ S \xrightarrow{\sim} S, \quad \rho^S : S \circ 1_X \xrightarrow{\sim} S$$

given by families of isomorphisms

$$\lambda^S_{y,x} : \sum_{y' \in Y} 1_Y(y,y') \otimes S(y',x) \equiv I \otimes S(y,x) \xrightarrow{l_{S(y,x)}} S(y,x)$$

$$\rho^S_{y,x} : \sum_{x' \in X} S(y,x') \otimes 1_X(x',x) \equiv S(y,x) \otimes I \xrightarrow{r_{S(y,x)}} S(y,x)$$

where l and r are the right and left unit constraints of $\mathcal{V}$. The respective coherence condition is satisfied, thus these data indeed define a bicategory. Notice that only the existence of coproducts in $\mathcal{V}$ is enough for the formation of $\mathcal{V}$-**Mat**.

The hom-categories $\mathcal{V}$-**Mat**(X,X) of this bicategory for a fixed set X will play an important role in this chapter. The following proposition underlines some of the properties that these categories possess, and more specifically the ones that imply certain results with regard to categories of monoids and comonoids as seen in Chapter 3.

PROPOSITION 7.1.1. *Let $\mathcal{V}$ be a cocomplete monoidal category such that $\otimes$ preserves colimits on both entries. The category $\mathcal{V}$-$\mathbf{Mat}(X,X)$ for any set X*

(i) is cocomplete, and has all limits that exist in $\mathcal{V}$;
(ii) is a monoidal category, and $\otimes = \circ$ preserves colimits on both entries;
(iii) is locally presentable when $\mathcal{V}$ is;
(iv) is monoidal closed when $\mathcal{V}$ is monoidal closed with products.

PROOF. (*i*) Since $\mathcal{V}\text{-}\mathbf{Mat}(X,X) = [X \times X, \mathcal{V}]$, all limits and colimits can be formed pointwise from those in $\mathcal{V}$.

(*ii*) The hom-categories $\mathcal{K}(X,X)$ for any bicategory $\mathcal{K}$ obtain a monoidal structure via the horizontal composition, as in (3.3). The unit object is the identity $\mathcal{V}$-matrix 1_X, so $(\mathcal{V}\text{-}\mathbf{Mat}(X,X), \circ, 1_X)$ is a monoidal category.

Horizontal composition of $\mathcal{V}$-matrices preserves colimits on both entries: if $(G_j \to G \,|\, j \in \mathcal{J})$ is a colimiting cocone for a diagram of shape $\mathcal{J}$ in $\mathcal{V}\text{-}\mathbf{Mat}(X,X)$, this means that for any $x, y \in X$, the arrows $G_j(x,y) \to G(x,y)$ form colimiting cocones in $\mathcal{V}$. If we apply the functor

$$- \circ S : \mathcal{V}\text{-}\mathbf{Mat}(X,X) \to \mathcal{V}\text{-}\mathbf{Mat}(X,X)$$

for any $\mathcal{V}$-matrix $S : X \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} X$, we obtain a collection of 2-cells $(G_j \circ S \to G \circ S \,|\, j \in \mathcal{J})$ in $\mathcal{V}$-$\mathbf{Mat}$. For this to be a colimit, for any $x, z \in X$ the arrows

$$\sum_{y \in X} G_j(x,y) \otimes S(y,z) \longrightarrow \sum_{y \in X} \mathrm{colim}_j G_j(x,y) \otimes S(y,z)$$

must also form colimiting cocones in $\mathcal{V}$. Since by assumptions $(- \otimes A)$ preserves colimits for any $A \in \mathcal{V}$, we have isomorphisms

$$\begin{aligned}\sum_{y \in X} (\mathrm{colim}_j G_j(x,y)) \otimes S(y,z) &\cong \sum_{y \in X} \mathrm{colim}_j (G_j(x,y) \otimes S(y,z)) \\ &\cong \mathrm{colim}_j (\sum_{y \in X} G_j(x,y) \otimes S(y,z)),\end{aligned}$$

thus $- \circ S$ is cocontinuous. Similarly, $S \circ -$ preserves colimits for any $\mathcal{V}$-matrix, since $(A \otimes -)$ does in $\mathcal{V}$.

(*iii*) For each locally λ-presentable category $\mathcal{C}$, it is known that the functor category $\mathcal{C}^{\mathcal{A}} = [\mathcal{A}, \mathcal{C}]$ for any small category $\mathcal{A}$ is locally λ-presentable itself, see [**AR94**, 1.54]. Hence, for the discrete small category $X \times X$, the functor category $\mathcal{V}^{X \times X}$ is a locally presentable category.

(*iv*) We need to demonstrate a bijective correspondence between morphisms

$$\begin{array}{lr} \dfrac{S \circ T \xrightarrow{\qquad\quad} R}{S \xrightarrow{\qquad\quad} G(T,R)} & \begin{array}{l}\text{in } \mathcal{V}\text{-}\mathbf{Mat}(X,X) \\[1ex] \text{in } \mathcal{V}\text{-}\mathbf{Mat}(X,X).\end{array} \end{array} \tag{7.3}$$

We define the $\mathcal{V}$-matrix $G(T,R)$ from X to X to be given by the family of objects in $\mathcal{V}$

$$G(T,R)(x,y) := \prod_{z \in X} [T(y,z), R(x,z)]$$

where $[-,-]$ is the internal hom in $\mathcal{V}$. Then, an arrow $\sigma : S \to G(T,R)$ in $\mathcal{V}$-$\mathbf{Mat}(X,X)$ is given by a family of arrows

$$\sigma_{x,y} : S(x,y) \to \prod_{z\in X} [T(y,z), R(x,z)]$$

in $\mathcal{V}$, for each $x,y \in X$. Since $\mathcal{V}$ is monoidal closed, for any fixed z the arrow $S(x,y) \to [T(y,z), R(x,z)]$ corresponds uniquely to $S(x,y) \otimes T(y,z) \to R(x,z)$, which in turn gives a unique arrow in $\mathcal{V}$ from the sum over all y's in X

$$\rho_{x,z} : \sum_{y\in X} S(x,y) \otimes T(y,z) \to R(x,z).$$

These arrows form a family which defines a 2-cell $\rho : S \circ T \to R$ in $\mathcal{V}$-$\mathbf{Mat}(X,X)$, thus the correspondence (7.3) is now established.

Notice that this actually shows that $\mathcal{V}$-$\mathbf{Mat}(X,X)$ is left closed, but we can repeat the above argument using the (right) internal hom of the monoidal closed $\mathcal{V}$ appropriately, and show that $\mathcal{V}$-$\mathbf{Mat}(X,X)$ is (bi)closed. ☐

Recall that Proposition 3.3.5 presented some very useful properties for the categories of monoids and comonoids of admissible categories, *i.e.* locally presentable symmetric monoidal categories, such that tensoring on one side preserves all filtered colimits. However, as was also noted then, the results are still valid if we drop the symmetry condition and ask instead that both $A \otimes -$ and $- \otimes A$ preserve (filtered) colimits.

COROLLARY 7.1.2. *If $\mathcal{V}$ is a locally presentable monoidal category, where $\otimes$ preserves colimits in both entries, the forgetful functors*

$$S : \mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X)) \to \mathcal{V}\text{-}\mathbf{Mat}(X,X)$$
$$U : \mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X)) \to \mathcal{V}\text{-}\mathbf{Mat}(X,X)$$

are monadic and comonadic respectively, and all categories are locally presentable.

The existence of the free monoid and cofree comonoid functors will be of use in Section 7.3. As mentioned again in Chapter 3, in reality the free monoid construction requires less assumptions than the ones above, *i.e.* existence of coproducts which are preserved by the tensor product. Notice that the current setting only differs from the general one of Section 3.3, in that the categories of monoids and comonoids of the non-symmetric $(\mathcal{V}\text{-}\mathbf{Mat}(X,X), \circ, 1_X)$ cannot inherit its monoidal structure.

The bicategory $\mathcal{V}$-$\mathbf{Mat}$ is in fact a *monoidal bicategory* (see [**Car95**]) via a pseudofunctor

$$\otimes : \mathcal{V}\text{-}\mathbf{Mat} \times \mathcal{V}\text{-}\mathbf{Mat} \longrightarrow \mathcal{V}\text{-}\mathbf{Mat}.$$

This maps any two sets X and Y to their cartesian product $X \times Y$, any two matrices $\{S(y,x)\}_{y,x}$ and $\{T(z,w)\}_{z,w}$ to the $\mathcal{V}$-matrix with components

$$(S \otimes T)\big((y,z),(x,w)\big) = S(y,x) \otimes T(z,w) \tag{7.4}$$

and any 2-cells to their pointwise tensor product in $\mathcal{V}$. The monoidal unit is the unit $\mathcal{V}$-matrix $\mathcal{I} : 1 \nrightarrow 1$ where $1 = \{*\}$ is the singleton set, with $\mathcal{I}(*,*) = I$. This monoidal structure will be discussed in detail in the next chapter (see Proposition 8.2.6).

We now proceed to the definition of a specific lax functor which will later give rise to certain very important mappings for particular enrichment relations we want to establish. Intuitively, there is an analogy with the internal hom functor of our monoidal closed $\mathcal{V}$ in the previous chapter, which induced the mappings H and $\bar{H}$ between the categories of monoids/comonoids and modules/comodules.

Suppose that $\mathcal{V}$ is a cocomplete symmetric monoidal closed category with products. If $\mathcal{V}$-$\mathbf{Mat}^{\mathrm{co}}$ is the bicategory of $\mathcal{V}$-matrices with reversed 2-cells, define a lax functor of bicategories

$$\mathrm{Hom} : (\mathcal{V}\text{-}\mathbf{Mat})^{\mathrm{co}} \times \mathcal{V}\text{-}\mathbf{Mat} \longrightarrow \mathcal{V}\text{-}\mathbf{Mat} \tag{7.5}$$

as follows:

· each pair of sets (X, Y) is mapped to the set $\mathrm{Hom}(X, Y) := Y^X$ of functions from X to Y;

· for all pairs $(X, Y), (Z, W)$ there is a functor

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Mat}(X,Z)^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{Mat}(Y,W) \arrow[r, "\mathrm{Hom}_{(X,Y),(Z,W)}"] & \mathcal{V}\text{-}\mathbf{Mat}(Y^X, W^Z) \\
(\ S\ ,\ T\ ) \arrow[r, mapsto, dotted] \arrow[d, "{(\sigma,\tau)}"'] & \mathrm{Hom}(S,T) \arrow[d, "{\mathrm{Hom}(\sigma,\tau)}"] \\
(\ S'\ ,\ T'\ ) \arrow[r, mapsto, dotted] & \mathrm{Hom}(S',T')
\end{tikzcd} \tag{7.6}$$

where the $\mathcal{V}$-matrix $\mathrm{Hom}(S, T) : Y^X \xrightarrow{+} W^Z$ is given by the family

$$\mathrm{Hom}(S,T)(q,k) := \prod_{\substack{z\in Z\\ x\in X}} [S(z,x), T(qz,kx)] \tag{7.7}$$

of objects in $\mathcal{V}$, for all $q \in W^Z$ and $k \in Y^X$, where $[-,-]$ is the internal hom in $\mathcal{V}$. For $\sigma : S' \Rightarrow S$ and $\tau : T \Rightarrow T'$, the 2-cell

$$\begin{tikzcd}
Y^X \arrow[r, bend left, "{\mathrm{Hom}(S,T)}", ""{name=U, below}] \arrow[r, bend right, "{\mathrm{Hom}(S',T')}"', ""{name=D, above}] & W^Z
\arrow[Rightarrow, from=U, to=D, "{\mathrm{Hom}(\sigma,\tau)}"]
\end{tikzcd} \tag{7.8}$$

has components, for every $(q, k) \in W^Z \times Y^X$, arrows in $\mathcal{V}$

$$\mathrm{Hom}(\sigma,\tau)_{q,k} : \prod_{(z,x)} [S(z,x), T(qz,kx)] \longrightarrow \prod_{(z,x)} [S'(z,x), T'(qz,kx)].$$

For fixed z, x, these correspond under the usual tensor-hom adjunction in $\mathcal{V}$ to

$$\begin{tikzcd}
{[S(z,x),T(qz,kx)] \otimes S'(z,x)} \arrow[r, dashed] \arrow[d, "1\otimes\sigma_{z,x}"'] & T'(qz,kx) \\
{[S(z,x),T(qz,kx)] \otimes S(z,x)} \arrow[r, "\mathrm{ev}_{T(qz,kx)}"'] & T(qz,kx) \arrow[u, "\tau_{qz,kx}"']
\end{tikzcd}$$

where ev is the evaluation;

· for all $(X,Y),(Z,W),(U,V)$, there is a natural transformation δ with components, for $(R:Z \nrightarrow U,\ O:W \nrightarrow V)$ and $(S:X \nrightarrow Z,\ T:Y \nrightarrow W)$, 2-cells in $\mathcal{V}$-**Mat**

$$\begin{tikzcd}
& W^Z \arrow[dr, bend left=15, "\mathrm{Hom}(R,O)", "|" marking] \arrow[d, phantom, "\Downarrow \delta_{(S,T),(R,O)}"] & \\
Y^X \arrow[ur, bend left=15, "\mathrm{Hom}(S,T)", "|" marking] \arrow[rr, bend right=15, "\mathrm{Hom}(R\circ S,O\circ T)"', "|" marking] & {} & V^U
\end{tikzcd} \tag{7.9}$$

which are given by families of arrows in $\mathcal{V}$

$$\sum_{q\in W^Z} \mathrm{Hom}(R,O)(t,q)\otimes \mathrm{Hom}(S,T)(q,k) \xrightarrow{\delta_{t,k}} \prod_{(u,x)} [(R\circ S)(u,x),(O\circ T)(tu,kx)]$$

for all $(t,k)\in V^U\times Y^X$. These again can be understood via their transposes under the tensor-hom adjunction, *i.e.* composites of projections, inclusions, symmetries and evaluations, using the fact that the tensor product preserves sums;

· for all pairs of sets (X,Y), there is a natural transformation γ with components

$$\begin{tikzcd}
Y^X \arrow[r, bend left=30, "1_{Y^X}", "|" marking] \arrow[r, bend right=30, "\mathrm{Hom}(1_X,1_Y)"', "|" marking] \arrow[r, phantom, "\Downarrow \gamma_{(X,Y)}"] & Y^X
\end{tikzcd} \tag{7.10}$$

which for $q=k\in Y^X$ and $x'=x\in X$ consist of the isomorphisms

$$(\gamma_{(X,Y)})_{q,q}: I \longrightarrow [1_X(x,x),1_Y(kx,kx)]=[I,I].$$

The coherence axioms of Definition 2.1.3 are satisfied, therefore Hom is a lax functor of bicategories.

We now turn to some more technical points of the bicategory $\mathcal{V}$-**Mat**. Any function $f:X\to Y$ between two sets X, Y determines two $\mathcal{V}$-matrices, $f_*:X\nrightarrow Y$ and $f^*:Y\nrightarrow X$, given by

$$f_*(y,x)=f^*(x,y)=\begin{cases} I, & \text{if } f(x)=y \\ 0, & \text{otherwise}\end{cases} \tag{7.11}$$

for any $x\in X$, $y\in Y$. It can be easily verified that there is a natural bijection between 2-cells $f_*\circ S\Rightarrow T$ and $S\Rightarrow f^*\circ T$ for any $\mathcal{V}$-matrices $S:Z\nrightarrow X$ and $T:Z\nrightarrow W$, thus they form an adjunction $f_*\dashv f^*$ in the bicategory $\mathcal{V}$-**Mat**. The unit and counit of this adjunction are the 2-cells

$$\begin{tikzcd}
X \arrow[r, bend left=30, "1_X", "|" marking] \arrow[r, bend right=30, "f^*\circ f_*"', "|" marking] \arrow[r, phantom, "\Downarrow \check{\eta}"] & X
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
Y \arrow[r, bend left=30, "f_*\circ f^*", "|" marking] \arrow[r, bend right=30, "1_Y"', "|" marking] \arrow[r, phantom, "\Downarrow \check{\varepsilon}"] & Y
\end{tikzcd}$$

with components arrows in $\mathcal{V}$

$$\check{\varepsilon}_{y',y}:(f_*\circ f^*)(y',y)\to 1_Y(y',y)\equiv\begin{cases} \sum\limits_{x\in f^{-1}(y)} I\otimes I \xrightarrow{r_I} I, & \text{if } y=y' \\ \qquad 0\xrightarrow{!}0, & \text{if } y\neq y' \end{cases}$$

and

$$\breve{\eta}_{x',x} : 1_X(x',x) \to (f^* \circ f_*)(x',x) \equiv \begin{cases} I \xrightarrow{(r_I)^{-1}} I \otimes I, & \text{if } x' = x \\ 0 \xrightarrow{!} \begin{cases} I \otimes I, & fx = fx' \\ 0, & \text{else} \end{cases} & \text{if } x' \neq x \end{cases}$$

where ! is the unique arrow from the initial to any object. Notice that $\breve{\eta}$ and $\breve{\varepsilon}$ are isomorphisms if and only if the function f is a bijection.

These $\mathcal{V}$-matrices induced by functions between sets are of central importance to constructions in later sections. Below we show some useful properties.

Lemma 7.1.3. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. There exist isomorphisms*

$$\zeta^{g,f} : g_* \circ f_* \cong (gf)_* : \; X \mathrel{\relbar\joinrel\mapstochar\joinrel\rightarrow} Z$$

$$\xi^{g,f} : f^* \circ g^* \cong (gf)^* : \; Z \mathrel{\relbar\joinrel\mapstochar\joinrel\rightarrow} X$$

which are families of invertible arrows

$$\zeta^{g,f}_{z,x} = \xi^{g,f}_{x,z} : \begin{cases} I \otimes I \xrightarrow{r_I = l_I} I, & \textit{if } g(f(x)) = z \\ \quad 0 \xrightarrow{!} 0, & \textit{otherwise} \end{cases} \tag{7.12}$$

for each pair of elements $(x,z) \in X \times Z$.

Proof. In general, for any $\mathcal{V}$-matrix $S : Y \mathrel{\relbar\joinrel\mapstochar\joinrel\rightarrow} Z$, the composite 1-cell $S \circ f_*$ is computed to be the family

$$(S \circ f_*)(z,x) = \sum_{y \in Y} S(z,y) \otimes f_*(y,x) = \sum_{y = fx} S(z,y) \otimes I = S(z,fx) \otimes I \overset{r}{\cong} S(z,fx)$$

of objects in $\mathcal{V}$, for any $(z,x) \in Z \times X$. Similarly, for a $\mathcal{V}$-matrix $T : Z \mathrel{\relbar\joinrel\mapstochar\joinrel\rightarrow} Y$, the composite $\mathcal{V}$-matrix $f^* \circ T$ is the family

$$(f^* \circ T)(x,z) = \sum_{y \in Y} f^*(x,y) \otimes T(y,z) = I \otimes T(fx,z) \overset{l}{\cong} T(fx,z)$$

of objects in $\mathcal{V}$, for all $(x,z) \in X \times Z$.

Using the above technique, we can explicitly write the families of objects in $\mathcal{V}$ which define the $\mathcal{V}$-matrices $g_* \circ f_*$ and $f^* \circ g^*$

$$(g_* \circ f_*)(z,x) = (f^* \circ g^*)(x,z) = \begin{cases} I \otimes I, & \text{if } g(f(x)) = z \\ 0, & \text{otherwise} \end{cases}$$

for any pairs of elements $(x,z) \in X \times Z$. We can now provide isomorphisms

$$\begin{tikzcd} & Y \arrow[dr, bend left, "g_*", "\shortmid"{marking}] & \\ X \arrow[ur, bend left, "f_*", "\shortmid"{marking}] \arrow[rr, bend right, "(gf)_*"', "\shortmid"{marking}] & \Downarrow \zeta^{g,f} & Z \end{tikzcd}$$

and

$$\begin{tikzcd} & Y \arrow[dr, bend left, "f^*", "\shortmid"{marking}] & \\ Z \arrow[ur, bend left, "g^*", "\shortmid"{marking}] \arrow[rr, bend right, "(gf)^*"', "\shortmid"{marking}] & \Downarrow \xi^{g,f} & X \end{tikzcd}$$

which consist of families of invertible arrows in $\mathcal{V}$ exactly the (7.12). ☐

Based on the above formulas, it is straightforward to show that ζ and ξ satisfy the following relations, which clarify how the composition of three such matrices works.

Lemma 7.1.4. *Consider three composable functions $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$. Then*

$$\begin{tikzcd}
& Y \arrow[r, "g_*", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\zeta^{g,f}"] & Z \arrow[dr, "h_*", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\zeta^{gf,h}"] & & & & Y \arrow[r, "g_*", "\shortmid" marking] \arrow[drr, "(hg)_*"', "\shortmid" marking] \arrow[d, phantom, "\Downarrow\zeta^{f,hg}"] & Z \arrow[dr, "h_*", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\zeta^{g,h}"] & \\
X \arrow[ur, "f_*", "\shortmid" marking] \arrow[urr, "(gf)_*"', "\shortmid" marking] \arrow[rrr, bend right, "(hgf)_*"', "\shortmid" marking] & & & W & = & X \arrow[ur, "f_*", "\shortmid" marking] \arrow[rrr, bend right, "(hgf)_*"', "\shortmid" marking] & & & W
\end{tikzcd}$$

and

$$\begin{tikzcd}
& Z \arrow[r, "g^*", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\xi^{g,h}"] & Y \arrow[dr, "f^*", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\xi^{f,hg}"] & & & & Z \arrow[r, "g^*", "\shortmid" marking] \arrow[drr, "(gf)^*"', "\shortmid" marking] \arrow[d, phantom, "\Downarrow\xi^{gf,h}"] & Y \arrow[dr, "f^*", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\xi^{g,f}"] & \\
W \arrow[ur, "h^*", "\shortmid" marking] \arrow[urr, "(hg)^*"', "\shortmid" marking] \arrow[rrr, bend right, "(hgf)^*"', "\shortmid" marking] & & & X & = & W \arrow[ur, "h^*", "\shortmid" marking] \arrow[rrr, bend right, "(hgf)^*"', "\shortmid" marking] & & & X.
\end{tikzcd}$$

7.2. The category of $\mathcal{V}$-graphs

Graphs, with variations on their exact meaning depending on the mathematical context they arise, have been of use for a very long time. For the needs of this thesis, we study the case of graphs enriched in a monoidal category, in order to better understand $\mathcal{V}$-categories. In this setting, enriched categories are enriched graphs with extra structure, and $\mathcal{V}$-cocategories will also naturally fit in later.

As a primary example, in [**ML98**, 11.7] the notion of a small (directed) graph consisting of a set of objects and a set of arrows was employed to describe the free category construction, in analogy with the free monoid construction on a set. Moreover, the idea of O-graphs with a fixed set of objects O inspires the fibrational view of these categories, which is going to be explicitly described in the following sections. For the main results regarding $\mathcal{V}$-**Grph** and $\mathcal{V}$-**Cat** from a more traditional point of view, Wolff's [**Wol74**] is a classic reference for $\mathcal{V}$ a symmetric monoidal closed category, whereas for the description of $\mathcal{V}$-graphs in terms of $\mathcal{V}$-matrices, we again closely follow [**BCSW83, KL01**].

A (small) $\mathcal{V}$-*graph* $\mathcal{G}$ consists of a set of objects ob$\mathcal{G}$, and for every pair of objects $A, B \in \text{ob}\mathcal{G}$ an object $\mathcal{G}(A,B) \in \mathcal{V}$. If $\mathcal{G}$ and $\mathcal{H}$ are $\mathcal{V}$-graphs, a $\mathcal{V}$-*graph morphism* $F : \mathcal{G} \to \mathcal{H}$ consists of a function $f : \text{ob}\mathcal{G} \to \text{ob}\mathcal{H}$ between their sets of objects, together with arrows in $\mathcal{V}$

$$F_{A,B} : \mathcal{G}(A,B) \to \mathcal{H}(fA, fB) \tag{7.13}$$

for each pair of objects A, B in $\mathcal{G}$. These data, with appropriate compositions and identities, form a category $\mathcal{V}$-**Grph**.

Notably, the above definition does not require any assumptions on the monoidal category $\mathcal{V}$. However, the context of the bicategory $\mathcal{V}$-**Mat** is very convenient for connecting relations between the above mentioned categories to be exhibited. For this reason, we proceed to the presentation of equivalent characterizations in the language of $\mathcal{V}$-matrices. Inevitably, we have to impose appropriate conditions on $\mathcal{V}$ as in the previous section, namely cocompleteness and $\otimes$ preserving colimits on both variables.

One can easily deduce that a $\mathcal{V}$-graph $\mathcal{G}$ as described above is an endoarrow in the bicategory $\mathcal{V}$-**Mat**, *i.e.* a set $X = \mathrm{ob}\mathcal{G}$ together with a $\mathcal{V}$-matrix $G : X \looparrowright X$ given by a family of objects $G(x', x)$ in $\mathcal{V}$, for all $x', x \in X$. Such a $\mathcal{V}$-graph will be denoted as (G, X) or $\mathcal{G}_X$. Furthermore, a morphism of $\mathcal{V}$-graphs between (G, X) and (H, Y) can be viewed as a function $f : X \to Y$ between their sets of objects, equipped with a 2-cell

$$\begin{tikzcd} X \arrow[r, bend left, "G", "\shortmid"{marking}, ""{name=U, below}] \arrow[r, bend right, "f^*\circ H\circ f_*"', "\shortmid"{marking}, ""{name=D, above}] & X \arrow[Rightarrow, from=U, to=D, "\phi"] \end{tikzcd}$$

in $\mathcal{V}$-**Mat**, where f_* and f^* are as in (7.11). This is the case, because the composite $\mathcal{V}$-matrix

$$X \xrightarrow{f_*} Y \xrightarrow{H} Y \xrightarrow{f^*} X$$

is given by the family of objects, for all $x', x \in X$,

$$\begin{aligned} (f^*Hf_*)(x', x) &= \sum_{y\in Y} f^*(x', y) \otimes (Hf_*)(y, x) = I \otimes (Hf_*)(f(x'), x) \\ &= I \otimes \sum_{y\in Y} H(f(x'), y) \otimes f_*(y, x) = I \otimes H(f(x'), f(x)) \otimes I \\ &\cong H(f(x'), f(x)). \end{aligned}$$

Hence the 2-cell ϕ has components arrows in $\mathcal{V}$

$$\phi_{x',x} : G(x', x) \longrightarrow I \otimes H(fx', fx) \otimes I \cong H(fx', fx)$$

for $x', x \in X$. This is essentially (7.13), in the sense that the arrows $F_{x',x}$ and $\phi_{x'x}$, are in bijective correspondence. We write $F = (\phi, f)$ for this way of viewing $\mathcal{V}$-graph morphisms.

In fact, because of the adjunction $f_* \dashv f^*$, the 'mates correspondence' of Proposition 2.3.7 gives a bijection between 2-cells

$$\begin{tikzcd} X \arrow[r, "G", "\shortmid"{marking}] \arrow[d, "f_*"', "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow\phi"] & X \\ Y \arrow[r, "H"', "\shortmid"{marking}] & Y, \arrow[u, "f^*"', "\shortmid"{marking}] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "G", "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow\psi"] & X \arrow[d, "f_*", "\shortmid"{marking}] \\ Y \arrow[u, "f^*", "\shortmid"{marking}] \arrow[r, "H"', "\shortmid"{marking}] & Y \end{tikzcd} \tag{7.14}$$

in the bicategory $\mathcal{V}$-**Mat**. By computing as before, the composite $\mathcal{V}$-matrix

$$Y \xrightarrow{f^*} X \xrightarrow{G} X \xrightarrow{f_*} Y$$

is the family of objects in $\mathcal{V}$, for each $y, y' \in Y$,

$$(f_*Gf^*)(y', y) = \sum_{\substack{fx'=y' \\ fx=y}} I \otimes G(x', x) \otimes I \cong \sum_{\substack{fx'=y' \\ fx=y}} G(x', x).$$

So the components of ψ are the arrows in $\mathcal{V}$

$$\psi_{y',y} : \sum_{\substack{fx'=y' \\ fx=y}} I \otimes G(x',x) \otimes I \longrightarrow H(y',y)$$

which, for fixed $x \in f^{-1}(y)$ and $x' \in f^{-1}(y')$, correspond uniquely to the components $\phi_{x',x}$. Hence, a $\mathcal{V}$-graph arrow can equivalently be given as a pair $(\psi, f) : (G,X) \to (H,Y)$ where $f : X \to Y$ is a function and $\psi : f_* G f^* \Rightarrow H$ a 2-cell in $\mathcal{V}$-**Mat**.

In the established terminology, the composition of two $\mathcal{V}$-graph morphisms

$$\mathcal{G}_X \xrightarrow{F=(\phi,f)} \mathcal{H}_Y \xrightarrow{K=(\chi,k)} \mathcal{J}_Z$$

is given by the function $kf : X \to Y \to Z$ and the composite 2-cell

$$\begin{tikzcd}
X \arrow[r, "G", "|" marking] \arrow[d, "f_*"', "|" marking] \arrow[dd, bend right=60, "(kf)_*"', "|" marking] \arrow[d, phantom, bend right=40, "\cong"'] \arrow[dr, phantom, "\swarrow\!\!\!\!\!/\, \phi"] & X \\
Y \arrow[r, "H"', "|" marking] \arrow[d, "k_*"', "|" marking] \arrow[dr, phantom, "\swarrow\!\!\!\!\!/\, \chi"] & Y \arrow[u, "f^*"', "|" marking] \arrow[u, phantom, bend right=40, "\cong"] \\
Z \arrow[r, "J"', "|" marking] & Z \arrow[u, "k^*"', "|" marking] \arrow[uu, bend right=60, "(kf)^*"', "|" marking]
\end{tikzcd}$$

where the isomorphisms are $\xi^{f,k}$ and $\zeta^{f,k}$ from Lemma 7.1.3. The identity arrow on (G,X) is given by the identity function $\mathrm{id}_X : X \to X$ on the set, and the 2-cell

$$\begin{tikzcd}
X \arrow[r, "G", "|" marking] \arrow[d, "(\mathrm{id}_X)_*"', "|" marking] \arrow[dr, phantom, "\Downarrow i_G"] & X \\
X \arrow[r, "G"', "|" marking] & X \arrow[u, "(\mathrm{id}_X)^*"', "|" marking]
\end{tikzcd}$$

with components arrows in $\mathcal{V}$

$$(i_G)_{x',x} : G(x',x) \xrightarrow{l^{-1}r^{-1}} I \otimes G(x',x) \otimes I \cong G(x',x),$$

evidently isomorphic to the identity arrows $1_{G(x',x)} : G(x',x) \to G(x',x)$. We write $1_{(G,X)} = (i_G, \mathrm{id}_X)$. Notice that in fact, the $\mathcal{V}$-matrices $(\mathrm{id}_X)_*, (\mathrm{id}_X)^*$ are the same as the identity 1-cell $1_X : X \;\mathrlap{\to}{\,+}\; X$ on X:

$$(\mathrm{id}_X)_*(x',x) = (\mathrm{id}_X)^*(x',x) = \begin{cases} I, & \text{if } x = x' \\ 0, & \text{otherwise.} \end{cases} \tag{7.15}$$

We can encode the above data in the following isomorphic characterization of the category of $\mathcal{V}$-graphs and $\mathcal{V}$-graph morphisms.

Definition 7.2.1. The category of small $\mathcal{V}$-graphs $\mathcal{V}$-**Grph** has objects pairs $(G,X) \in \mathcal{V}\text{-}\mathbf{Mat}(X,X) \times \mathbf{Set}$ and arrows (in bijection with) pairs $(\phi, f) : (G,X) \to (H,Y)$ where

$$\begin{cases} \phi : G \to f^* H f_* & \text{in } \mathcal{V}\text{-}\mathbf{Mat}(X,X) \\ f : X \to Y & \text{in } \mathbf{Set} \end{cases}$$

or equivalently pairs (ψ, f) where

$$\begin{cases} \psi : f_* G f^* \to H & \text{in } \mathcal{V}\text{-}\mathbf{Mat}(Y,Y) \\ f : X \to Y & \text{in } \mathbf{Set}. \end{cases}$$

From now on, we will use either description of $\mathcal{V}$-graph morphisms according to our needs, and the choice will be evident by the context and notation. In particular, we will usually denote a $\mathcal{V}$-graph morphism in the classic sense as $F_f : \mathcal{G}_X \to \mathcal{H}_Y$ and $(\phi, f) : (G, X) \to (H, Y)$ in the $\mathcal{V}$-matrices view. There is an evident forgetful functor $Q : \mathcal{V}\text{-}\mathbf{Grph} \to \mathbf{Set}$ which sends each graph (G, X) to its set of objects X, and each arrow (ϕ, f) to the function between the objects f.

We now continue with the basic properties of $\mathcal{V}$-**Grph**. First of all, when $\mathcal{V}$ is complete, it is straightforward to construct limits inside $\mathcal{V}$-**Grph**. Indeed, a diagram of shape $\mathcal{J}$ in $\mathcal{V}$-**Grph**

$$\begin{tikzcd}
D: \mathcal{J} \arrow[r] & \mathcal{V}\text{-}\mathbf{Grph} \\
j \arrow[r, mapsto, dotted] \arrow[d, "\theta"'] & (\mathcal{G}_j)_{X_j} \arrow[d, "(F_\theta)_{f_\theta}"] \\
k \arrow[r, mapsto, dotted] & (\mathcal{G}_k)_{X_k}
\end{tikzcd}$$

has as limit the graph $\mathcal{G}_X$ constructed as follows. The set of objects is the limit X of the composite diagram

$$\mathcal{J} \xrightarrow{D} \mathcal{V}\text{-}\mathbf{Grph} \xrightarrow{Q} \mathbf{Set},$$

thus if π_j are the projections from X, we have $\pi_k = f_\theta \pi_j$ in **Set** for every θ. Then, for any $x, x' \in X$ the hom-object $G(x', x)$ is the following limit in $\mathcal{V}$:

$$\begin{tikzcd}
\mathcal{G}_j(\pi_j x', \pi_j x) \arrow[d, "(F_\theta)_{\pi_j x', \pi_j x}"'] & G(x', x) \arrow[l, dotted, "(\Pi_j)_{x',x}"'] \arrow[dl, dotted, "(\Pi_k)_{x',x}"] \\
G_k(f_\theta \pi_j x', f_\theta \pi_j x). &
\end{tikzcd}$$

The cocone $\big(\mathcal{G}_X \xrightarrow{(\Pi_j)_{\pi_j}} (\mathcal{G}_j)_{X_j} \,|\, j \in \mathcal{J}\big)$ now satisfies the required universal property.

On the other hand, when $\mathcal{V}$ is cocomplete, the category $\mathcal{V}\text{-}\mathbf{Mat}(X, X)$ for any set X is cocomplete as well, which leads to the following construction of colimits in $\mathcal{V}$-**Grph**.

PROPOSITION 7.2.2 ([**KL01**]). *The category* $\mathcal{V}$-**Grph** *is cocomplete when* $\mathcal{V}$ *is.*

PROOF. Suppose $\mathcal{J}$ is a small category and F is a diagram of shape $\mathcal{J}$ in $\mathcal{V}$-**Grph** given by

$$\begin{tikzcd}
F: \mathcal{J} \arrow[r] & \mathcal{V}\text{-}\mathbf{Grph} \\
j \arrow[r, mapsto, dotted] \arrow[d, "\theta"'] & (G_j, X_j) \arrow[d, "(\psi_\theta, f_\theta)"] \\
k \arrow[r, mapsto, dotted] & (G_k, X_k).
\end{tikzcd} \tag{7.16}$$

By Definition 7.2.1, f_θ is a function between the sets of objects and $(f_\theta)_* G_j (f_\theta)^* \overset{\psi_\theta}{\Rightarrow} G_k$ is a 2-cell in $\mathcal{V}$-**Mat**. Again, the composite

$$\mathcal{J} \xrightarrow{F} \mathcal{V}\text{-}\mathbf{Grph} \xrightarrow{Q} \mathbf{Set}$$

has a colimiting cocone $(\tau_j : X_j \to X \mid j \in \mathcal{J})$ in the cocomplete **Set**. Notice that, since $\tau_j = f_\theta \tau_k$ for any $f_\theta : X_j \to X_k$, we have isomorphisms of $\mathcal{V}$-matrices

$$\begin{tikzcd}
X_j \arrow[rr, "(\tau_j)_*", "\shortmid" marking] \arrow[dr, "(f_\theta)_*"', "\shortmid" marking] & & X, & X \arrow[rr, "(\tau_j)^*", "\shortmid" marking] \arrow[dr, "(\tau_k)^*"', "\shortmid" marking] & & X_j \\
& X_k \arrow[ur, "(\tau_k)_*"', "\shortmid" marking] \arrow[u, phantom, "\overset{\zeta}{\cong}"] & & & X_k \arrow[ur, "(f_\theta)^*"', "\shortmid" marking] \arrow[u, phantom, "\overset{\xi}{\cong}"] &
\end{tikzcd}$$

where ζ and ξ are defined as in Lemma 7.12. Now consider the functor

$$\begin{tikzcd}
K: & \mathcal{J} \arrow[r] & \mathcal{V}\text{-}\mathbf{Mat}(X,X) \\
& j \arrow[r, mapsto, dotted] \arrow[d, "\theta"'] & (\tau_j)_* G_j (\tau_j)^* \cong (\tau_k)_* (f_\theta)_* G_j (f_\theta)^* (\tau_k)^* \arrow[d, "(\tau_k)_* \psi_\theta (\tau_k)^*"] \\
& k \arrow[r, mapsto, dotted] & (\tau_k)_* G_k (\tau_k)^*
\end{tikzcd} \tag{7.17}$$

which explicitly maps an arrow $\theta : j \to k$ in $\mathcal{J}$ to the composite 2-cell

$$\begin{tikzcd}
X \arrow[r, "(\tau_j)^*", "\shortmid" marking] \arrow[dr, bend right, "(\tau_k)^*"', "\shortmid" marking] & X_j \arrow[r, "G_j", "\shortmid" marking] & X_j \arrow[r, "(\tau_j)_*", "\shortmid" marking] \arrow[d, "(f_\theta)_*", "\shortmid" marking] & X. \\
& X_k \arrow[u, "(f_\theta)^*"', "\shortmid" marking] \arrow[r, "G_k"', "\shortmid" marking] \arrow[ur, phantom, "\Downarrow \psi_\theta"] \arrow[ul, phantom, "\overset{\xi}{\cong}"] & X_k \arrow[ur, bend right, "(\tau_k)_*"', "\shortmid" marking] \arrow[ur, phantom, "\overset{\zeta}{\cong}"] &
\end{tikzcd} \tag{7.18}$$

The colimit of K is formed pointwise in $\mathcal{V}$-**Mat**$(X, X) = [X \times X, \mathcal{V}]$, so there is a colimiting cocone $(\lambda_j : (\tau_j)_* G_j (\tau_j)^* \to G \mid j \in \mathcal{J})$. These data allow us to form a new cocone

$$\big((G_j, X_j) \xrightarrow{(\lambda_j, \tau_j)} (G, X) \mid j \in \mathcal{J}\big)$$

for the initial diagram F in $\mathcal{V}$-**Grph**, since (G, X) is an endoarrow in $\mathcal{V}$-**Mat** by construction, and also the pairs (λ_j, τ_j) commute accordingly with the (ψ_θ, f_θ)'s. This cocone which can be checked to be colimiting, since τ_j and λ_j are. Therefore (G, X) satisfies the universal property of a colimit of F in $\mathcal{V}$-**Grph**. □

The above construction is presented in [**BCSW83**], again in the more general case of enrichment in a bicategory. The existence of all colimits in $\mathcal{V}$-**Grph** was also shown in [**Wol74**] via the explicit construction of coproducts and coqualizers.

The category $\mathcal{V}$-**Grph** has a monoidal structure inherited from $\mathcal{V}$: given two $\mathcal{V}$-graphs $\mathcal{G}_X$ and $\mathcal{H}_Y$, their tensor product $\mathcal{G} \otimes \mathcal{H}$ is defined to be the $\mathcal{V}$-graph with set of objects $X \times Y$ and hom-objects

$$(\mathcal{G} \otimes \mathcal{H})((z, w), (x, y)) := G(z, x) \otimes H(w, y).$$

Of course, this comes from the monoidal structure of the bicategory $\mathcal{V}$-**Mat** as in (7.4). Similarly, we can define the tensor product of two $\mathcal{V}$-graph morphisms: given

$\mathcal{V}$-graph arrows $F_f : \mathcal{G}_X \to \mathcal{H}_Y$, $D_d : \mathcal{G}'_{X'} \to \mathcal{H}'_{Y'}$, their tensor product

$$F \otimes D : \mathcal{G} \otimes \mathcal{H} \to \mathcal{G}' \otimes \mathcal{H}'$$

is given by the function $f \times d : X \times Y \to X' \times Y'$ between their sets of objects, and for every $x, z \in X$, $y, w \in Y$, arrows

$$(F \otimes D)_{(z,w),(x,y)} : G(z,x) \otimes H(w,y) \xrightarrow{F_{z,x} \otimes D_{w,y}} G(fz, fx) \otimes H(dw, dy)$$

in $\mathcal{V}$. The monoidal unit is the unit $\mathcal{V}$-graph $\mathcal{I}$ with one object, and hom-object $\mathcal{I}(*,*) = I$. Also, symmetry is also evidently inherited from $\mathcal{V}$.

Furthermore, the category of $\mathcal{V}$-graphs is a monoidal closed category if we assume certain extra conditions on $\mathcal{V}$.

PROPOSITION 7.2.3. *Suppose $\mathcal{V}$ is a monoidal closed category with small products. The functor*

$$^g\mathrm{Hom} : \mathcal{V}\text{-}\mathbf{Grph}^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{Grph} \to \mathcal{V}\text{-}\mathbf{Grph}$$

which maps a pair $(\mathcal{G}_X,\ \mathcal{H}_Y)$ to the $\mathcal{V}$-graph $^g\mathrm{Hom}(\mathcal{G}, \mathcal{H})_{Y^X}$ with

$$^g\mathrm{Hom}(\mathcal{G}, \mathcal{H})(k, s) := \prod_{\substack{x' \in X \\ x \in X}} [\mathcal{G}(x', x), \mathcal{H}(kx', sx)]$$

*for $k, s \in Y^X$ is the internal hom of $\mathcal{V}$-**Grph**.*

PROOF. In order to establish an adjunction $(- \otimes \mathcal{H}_Y) \dashv {}^g\mathrm{Hom}(\mathcal{H}_Y, -)$ for any $\mathcal{V}$-graph $\mathcal{H}_Y$, take a $\mathcal{V}$-graph morphism $F_f : \mathcal{G}_X \to {}^g\mathrm{Hom}(\mathcal{H}_Y, \mathcal{J}_Z)$. This consists of a function $f : X \to Z^Y$ between the sets of objects, and arrows

$$F_{x',x} : \mathcal{G}(x', x) \to \prod_{y,y' \in Y} [(\mathcal{H}(y', y), \mathcal{J}(f_{x'}y', f_x y))$$

in $\mathcal{V}$ between the hom-objects, where $f_x = f(x) : Y \to Z$, for all $x, x' \in X$. These arrows correspond bijectively, under the tensor-hom adjunction in $\mathcal{V}$ for a fixed pair of elements $(y, y') \in Y$, to

$$\mathcal{G}(x', x) \otimes \mathcal{H}(y', y) \to \mathcal{J}(f_{x'}y', f_x y)$$

since $\mathcal{V}$ is monoidal closed. The category **Set** is cartesian closed, thus the function f corresponds uniquely to a function $\bar{f} : X \times Y \to Z$. This function together with the arrows above written as

$$\bar{F}_{(x',y'),(x,y)} : \mathcal{G}(x', x) \otimes \mathcal{H}(y', y) \to \mathcal{J}\big(\bar{f}(x', y'), \bar{f}(x, y)\big)$$

determines a $\mathcal{V}$-graph morphism $\bar{F}_{\bar{f}} : \mathcal{G}_X \otimes \mathcal{H}_Y \to \mathcal{J}_Z$ which establishes a bijective correspondence

$$\mathcal{V}\text{-}\mathbf{Grph}(\mathcal{G}_X \otimes \mathcal{H}_Y, \mathcal{J}_Z) \cong \mathcal{V}\text{-}\mathbf{Grph}(\mathcal{G}_X, {}^g\mathrm{Hom}(\mathcal{H}_Y, \mathcal{J}_Z)).$$

Moreover, this bijection is natural in $\mathcal{G}_X$, hence $^g\mathrm{Hom}(\mathcal{H}, \mathcal{J})$ is the object function of a right adjoint functor $^g\mathrm{Hom}(\mathcal{H}, -)$ of $(- \otimes \mathcal{H})$. Hence the induced functor of two variables $^g\mathrm{Hom}$ is the parametrized adjoint of $\otimes$.

Explicitly, ${}^g\mathrm{Hom}$ on a pair of $\mathcal{V}$-arrows $(F_f : \mathcal{J}_Z \to \mathcal{G}_X, D_d : \mathcal{H}_Y \to \mathcal{M}_W)$ gives a $\mathcal{V}$-graph morphism

$$ {}^g\mathrm{Hom}(F,D) : {}^g\mathrm{Hom}(\mathcal{G},\mathcal{H})_{Y^X} \longrightarrow {}^g\mathrm{Hom}(\mathcal{J},\mathcal{M})_{W^Z}. \tag{7.19}$$

This consists of the function 'pre-composing with f and post-composing with d' $d^f : Y^X \to W^Z$ between the sets of objects, and for each pair $(k,s) \in Y^X$ an arrow

$$ {}^g\mathrm{Hom}(F,D)_{k,s} : {}^g\mathrm{Hom}(\mathcal{G},\mathcal{H})(k,s) \longrightarrow {}^g\mathrm{Hom}(\mathcal{J},\mathcal{M})(d^f(k),d^f(s)) \equiv \\ \prod_{x,x' \in X} [G(x',x), H(kx',sx)] \to \prod_{z,z' \in Z} [J(z',z), M(dkfz',dsfz)].$$

For fixed $z, z' \in Z$, the latter corresponds uniquely under the usual tensor-hom adjunction to the composite

$$\begin{tikzcd}
\prod\limits_{x,x' \in X} [\mathcal{G}(x',x), \mathcal{H}(kx',sx)] \otimes \mathcal{J}(z',z) \arrow[r, dashed] \arrow[d, "1\otimes F_{z,z'}"'] & \mathcal{M}(dkfz',dsfz). \\
\prod\limits_{x,x' \in X} [\mathcal{G}(x',x), \mathcal{H}(kx',sx)] \otimes \mathcal{G}(fz',fz) \arrow[d, "\pi_{fz',fz}\otimes 1"'] & \\
{[\mathcal{G}(fz',fz), \mathcal{H}(kfz',sfz)] \otimes \mathcal{G}(fz',fz)} \arrow[r, "\mathrm{ev}"] & \mathcal{H}(kfz',sfz) \arrow[uu, "D_{kfz',sfz}"']
\end{tikzcd}$$

□

In the above proof, there was no need to move to the world of $\mathcal{V}$-matrices. If we did, however, it would be clear that the mapping of the functor ${}^g\mathrm{Hom}$ on two objects (G,X) and (H,Y) is in fact the mapping of the functor $\mathrm{Hom}_{(X,Y),(X,Y)}$ (7.6) provided by the lax functor of bicategories $\mathrm{Hom} : (\mathcal{V}\text{-}\mathbf{Mat})^{\mathrm{co}} \times \mathcal{V}\text{-}\mathbf{Mat} \to \mathcal{V}\text{-}\mathbf{Mat}$ defined explicitly in the previous section. For the mapping on morphisms though, the definition of $\mathrm{Hom}(\sigma,\tau)$ as in (7.8) is not sufficient, because the morphisms in $\mathcal{V}$-**Grph** are not just between endoarrows in $\mathcal{V}$-**Mat** with the same set of objects. Hence, in terms of $\mathcal{V}$-matrices, for $F = (\phi, f)$ and $D = (\chi, d)$ as in Definition 7.2.1, the $\mathcal{V}$-graph arrow ${}^g\mathrm{Hom}((\phi,f),(\chi,d))$ is the pair $([[\phi,\chi]], d^f)$ where

$$\begin{tikzcd}
Y^X \arrow[r, "\mathrm{Hom}(G,H)", "\shortmid"{marking}] \arrow[d, "(d^f)_*"', "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow {[[\phi,\chi]]}"] & Y^X \\
W^Z \arrow[r, "\mathrm{Hom}(J,M)"', "\shortmid"{marking}] & W^Z \arrow[u, "(d^f)^*"', "\shortmid"{marking}]
\end{tikzcd} \tag{7.20}$$

has components isomorphic to ${}^g\mathrm{Hom}(F_f, D_d)_{k,s}$ up to tensoring with I's on both sides of the codomain product.

Another important property of $\mathcal{V}$-**Grph** is the fact that it inherits local presentability from $\mathcal{V}$. The detailed arguments and constructions for this result can be found in [**KL01**].

Proposition 7.2.4. [**KL01**, 4.4] *The category* $\mathcal{V}$-**Grph** *is locally* λ*-presentable when* $\mathcal{V}$ *is so.*

Proof. (Sketch) Suppose $\mathcal{V}$ is a locally λ-presentable category. Then, if the set $\mathscr{G}$ of objects constitutes a strong generator of $\mathcal{V}$, it can be shown that the set

$$\{(\bar{G}, 2) \ / \ G \in \mathscr{G} \text{ or } G = 0\}$$

constitutes a strong generator of $\mathcal{V}$-**Grph**, where the graph $(\bar{G}, 2)$ has as set of objects $2 = \{0, 1\}$ and consists of the objects

$$\bar{G}(0,0) = G, \ \ \bar{G}(0,1) = \bar{G}(1,0) = \bar{G}(1,1) = 0$$

in $\mathcal{V}$. Also, this set is λ-presentable, in the sense that the hom-functors

$$\mathcal{V}\text{-}\mathbf{Grph}((\bar{G}, 2), -) : \mathcal{V}\text{-}\mathbf{Grph} \to \mathbf{Set}$$

preserve λ-filtered colimits. □

7.3. $\mathcal{V}$-categories and $\mathcal{V}$-cocategories

In Chapter 4, we recalled what it means for a category $\mathcal{A}$ to be $\mathcal{V}$-enriched for a monoidal category $\mathcal{V}$. In this section, we are going to re-define $\mathcal{V}$-categories from a slightly different perspective, in the context of $\mathcal{V}$-matrices. This is of importance because it allows us, just by dualizing certain arguments, to later construct the category of $\mathcal{V}$-cocategories in a natural way. Evidently, the motivation for this is that enriched categories and cocategories generalize monoids and comonoids in a monoidal category, since for example it is well-known that a one-object $\mathcal{V}$-category is precisely an object in $\mathbf{Mon}(\mathcal{V})$.

Notice that strictly speaking, composition in the bicategory $\mathcal{V}$-**Mat** (7.1) results in the opposite convention (7.21) to that preferred by Kelly (4.1) for the composition law in an enriched category. Similar issues arise regarding $\mathcal{V}$-modules later. There seems to be no consistent practice in these matters.

Following once again the approach of [**BCSW83**], a $\mathcal{V}$-category is defined to be a monad in the bicategory $\mathcal{V}$-**Mat**. Unravelling Definition 2.2.1, it consists of a set X together with an endoarrow $A : X \twoheadrightarrow X$, *i.e.* it is a $\mathcal{V}$-graph with set of objects ob$\mathcal{A} = X$, equipped with two 2-cells, the multiplication and the unit

$$\begin{tikzcd}
& X \arrow[dr, "A", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow M"] & \\
X \arrow[ur, "A", "\shortmid"{marking}] \arrow[rr, bend right=10, "A"', "\shortmid"{marking}] & {} & X
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \arrow[r, bend left, "1_X", "\shortmid"{marking}] \arrow[r, bend right, "A"', "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow\eta"] & X
\end{tikzcd}$$

satisfying the following axioms:

$$\begin{tikzcd}
& X \arrow[r, "A", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow M"] & X \arrow[dr, bend left, "A", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow M"] & \\
X \arrow[ur, bend left, "A", "\shortmid"{marking}] \arrow[urr, bend right, "A"', "\shortmid"{marking}] \arrow[rrr, bend right, "A"', "\shortmid"{marking}] & {} & {} & X
\end{tikzcd}
=
\begin{tikzcd}
& X \arrow[r, "A", "\shortmid"{marking}] \arrow[drr, bend right, "A"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow M"] & X \arrow[dr, bend left, "A", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow M"] & \\
X \arrow[ur, bend left, "A", "\shortmid"{marking}] \arrow[rrr, bend right, "A"', "\shortmid"{marking}] & {} & {} & X,
\end{tikzcd}$$

$$\begin{tikzcd}
& X \arrow[dr, bend left, "A"] \arrow[d, phantom, "\Downarrow M"] & \\
X \arrow[ur, bend left, "1_X"] \arrow[ur, "A"'] \arrow[ur, phantom, "\Downarrow\eta"] \arrow[rr, bend right, "A"'] & {} & X
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, bend left, "A"] \arrow[r, bend right, "A"'] \arrow[r, phantom, "\Downarrow 1_A"] & X
\end{tikzcd}
=
\begin{tikzcd}
& X \arrow[dr, bend left, "1_X"] \arrow[dr, "A"'] \arrow[dr, phantom, "\Downarrow\eta"] \arrow[d, phantom, "\Downarrow M"] & \\
X \arrow[ur, bend left, "A"] \arrow[rr, bend right, "A"'] & {} & X.
\end{tikzcd}$$

Notice that in the above diagrams, the associator and the unitors of the bicategory $\mathcal{V}$-**Mat** which are essential for the domains and codomains of the equal 2-cells to coincide, are suppressed. In terms of components, they are given by

$$M_{z,y,x} : \sum_{y\in X} A(z,y)\otimes A(y,x) \longrightarrow A(z,y) \tag{7.21}$$
$$\eta_x : I \longrightarrow A(x,x)$$

which are the usual composition law and identity elements. If we also express the above relations that M and η have to satisfy in terms of components of the 2-cells involved, we re-obtain the associativity and unit axioms of an enriched category. Also by Remark 3.3.1, a monad in a bicategory is the same as a monoid in the appropriate endoarrow hom-category, *i.e.* a $\mathcal{V}$-category $\mathcal{A}$ with set of objects X is a monoid in the monoidal category ($\mathcal{V}$-$\mathbf{Mat}(X,X)$,$\circ$,1_X). Denote a $\mathcal{V}$-category as a pair (A,X) or $\mathcal{A}_X$.

A $\mathcal{V}$-functor $F:\mathcal{A}\to\mathcal{B}$ between two $\mathcal{V}$-categories $\mathcal{A}_X$ and $\mathcal{B}_Y$ was again defined in Section 4.1, and in fact is a $\mathcal{V}$-graph morphism $F_f:\mathcal{A}_X\to\mathcal{B}_Y$ (in the classic sense) which respects the composition law and the identities. In the current context of $\mathcal{V}$-matrices, a $\mathcal{V}$-functor can be defined to be a morphism of $\mathcal{V}$-graphs $(\phi,f):(A,X)\to(B,Y)$ as in Definition 7.2.1, which satisfies

$$\begin{tikzcd}
X \arrow[r, "A"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\swArrow\hat{\phi}"] & X \arrow[r, "A"] \arrow[d, "f_*"] \arrow[dr, phantom, "\swArrow\hat{\phi}"] & X \arrow[d, "f_*"] \\
Y \arrow[r, "B"'] \arrow[rr, bend right=50, "B"'] & Y \arrow[r, "B"'] \arrow[d, phantom, "\Downarrow M"] & Y \\
& {} &
\end{tikzcd}
=
\begin{tikzcd}
& X \arrow[dr, "A"] \arrow[d, phantom, "\Downarrow M"] & \\
X \arrow[ur, "A"] \arrow[rr, "A"'] \arrow[d, "f_*"'] & {} \arrow[d, phantom, "\swArrow\hat{\phi}"] & X \arrow[d, "f_*"] \\
Y \arrow[rr, "B"'] & {} & Y,
\end{tikzcd} \tag{7.22}$$

$$\begin{tikzcd}
X \arrow[rr, bend left=40, "1_X"] \arrow[rr, "A"'] \arrow[d, "f_*"'] & \Downarrow\eta & X \arrow[d, "f_*"] \\
Y \arrow[rr, "B"'] & \Downarrow\hat{\phi} & Y
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[rr, "1_X"] \arrow[d, "f_*"'] \arrow[drr, dotted, "f_*" description] & \cong \quad\quad \cong & X \arrow[d, "f_*"] \\
Y \arrow[rr, "1_Y"] \arrow[rr, bend right=40, "B"'] & \Downarrow\eta & Y.
\end{tikzcd}$$

Here, the 2-cell $\hat{\phi}:f_*A\Rightarrow Bf_*$ corresponds bijectively to ϕ via mates correspondence 'on the one side', *i.e.* by pasting the counit $\check{\varepsilon}$ of $f_*\dashv f^*$ on the right. This description agrees with the standard $\mathcal{V}$-functor definition up to isomorphism again: the 2-cell $\bar{\phi}$

has components

$$\bar{\phi}_{y,x} : I \otimes A(x',x) \to B(fx', fx) \otimes I$$

for $x' \in f^{\text{-}1}y$, and the equality of the above pasted diagrams agrees with the commutative diagrams (4.3) up to tensoring the objects with I's and composing the arrows with the left and right unit constraints of $\mathcal{V}$.

Remark 7.3.1. The pair $(f_*, \hat{\phi})$ is a special case of 'colax monad functor' between the monads (A, X) and (B, Y) in the bicategory $\mathcal{V}$-**Mat**, as in Definition 2.2.7. However, it is not true that any colax monad functor given by the data

$$\begin{tikzcd}
X \arrow[r, "A", "\shortmid"{marking}] \arrow[d, "S"', "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow\chi"] & X \arrow[d, "S", "\shortmid"{marking}] \\
Y \arrow[r, "B"', "\shortmid"{marking}] & Y
\end{tikzcd}$$

for some $\mathcal{V}$-matrix S can be seen as a $\mathcal{V}$-functor, since it is obviously not true that any $S : X \nrightarrow Y$ is of the form f_* for some function $f : X \to Y$. This explains why the category $\mathcal{V}$-**Cat** cannot be characterized as **Mnd**($\mathcal{V}$-**Mat**), even if they have the same objects. Similar issues were discussed in a bigger depth in [**GS13**], employing the theory of *proarrow equipments.*

There is a 2-dimensional aspect for all the basic categories we study in this chapter, including $\mathcal{V}$-**Cat**. However, we choose to omit its description in this treatment, because it is not of central importance for our main results. More specifically, for the enrichment relations and the fibrational structures we explore, the 2-categorical structure of those categories is unnecessary.

Since a $\mathcal{V}$-category with set of objects X can be seen as a monoid in the monoidal category $\mathcal{V}$-**Mat**(X, X), a similar characterization for $\mathcal{V}$-functors could be attempted, in order to obtain a result analogous to Definition 7.2.1 for $\mathcal{V}$-**Grph**. The following is indicative of how to proceed.

Lemma 7.3.2. *Let (B, Y) be a $\mathcal{V}$-category. If $f : X \to Y$ is any function, the composite $\mathcal{V}$-matrix*

$$X \xrightarrow{f_*} Y \xrightarrow{B} Y \xrightarrow{f^*} X$$

*is a monoid in $\mathcal{V}$-**Mat**(X, X),* i.e. *the pair (f^*Bf_*, X) constitutes a $\mathcal{V}$-category.*

Proof. The multiplication $M' : f^*Bf_*f^*Bf_* \to f^*Bf_*$ is given by the composite 2-cell

$$\begin{tikzcd}
 & & & X \arrow[dr, "f_*", "\shortmid"{marking}] & & & \\
 & & Y \arrow[ur, "f^*", "\shortmid"{marking}] \arrow[rr, bend right, "1_Y"', "\shortmid"{marking}] & \Downarrow\check{\varepsilon} & Y \arrow[dr, "B", "\shortmid"{marking}] & & \\
X \arrow[r, "f_*", "\shortmid"{marking}] & Y \arrow[ur, "B", "\shortmid"{marking}] \arrow[rrrr, bend right, "B"', "\shortmid"{marking}] & & \Downarrow M & & Y \arrow[r, "f^*", "\shortmid"{marking}] & X
\end{tikzcd}$$

and the unit $\eta' : 1_X \to f^*Bf_*$ is given by the composite 2-cell

$$\begin{tikzcd}
X \arrow[rrrr, bend left, "1_X"] \arrow[dr, "f_*"'] & & \Downarrow\check{\eta} & & X \\
& Y \arrow[rr, bend left, "1_Y"] \arrow[rr, bend right, "B"'] & \Downarrow\eta & Y, \arrow[ur, "f^*"'] &
\end{tikzcd}$$

where $\check{\varepsilon}$ and $\check{\eta}$ are the counit and unit of the adjunction $f_* \dashv f^*$ in $\mathcal{V}$-**Mat**, and M and η the structure maps of the monoid B. Using pasting operations, the new multiplication and unit can be expressed as

$$M' = f^*\big(M \cdot (B\check{\varepsilon}B)\big)f_*,$$
$$\eta' = (f^*\eta f_*) \cdot \check{\eta}.$$

The associativity and unit axioms follow from the ones for the multiplication and unit of the monoid $B : Y \nrightarrow Y$ and the triangular identities for $\check{\eta}$ and $\check{\varepsilon}$. $\square$

It is not hard to see that the diagrams (7.22) which a $\mathcal{V}$-functor $F = (\phi, f) : (A, X) \to (B, Y)$ has to satisfy, coincide with the ones that an arrow in **Mon**($\mathcal{V}$-**Mat**(X, X)) between the monoids A and f^*Bf_* has to satisfy. For example, associativity can be written, using mates correspondence, as

$$\begin{tikzcd}
X \arrow[r, "A"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\Downarrow\phi"] & X \arrow[rr, "A"] \arrow[dr, "f_*"] & \Downarrow\check{\varepsilon} \arrow[r, phantom, "\Downarrow\phi"] & X \arrow[dr, "f_*"] & \Downarrow\check{\varepsilon} \\
Y \arrow[r, "B"'] \arrow[rrr, bend right, "B"'] & Y \arrow[u, "f^*"'] \arrow[r, "1_Y"'] & Y \arrow[r, "B"'] \arrow[d, phantom, "\Downarrow M"] & Y \arrow[u, "f^*"'] \arrow[r, "1_Y"'] & Y \\
& & {} & &
\end{tikzcd}
=
\begin{tikzcd}
& X \arrow[dr, "A"] \arrow[d, phantom, "\Downarrow M"] & & \\
X \arrow[ur, "A"] \arrow[rr, "A"] \arrow[d, "f_*"'] & \Downarrow\phi & X \arrow[dr, "f_*"] & \Downarrow\check{\varepsilon} \\
Y \arrow[rr, "B"'] & & Y \arrow[u, "f^*"'] \arrow[r, "1_Y"'] & Y
\end{tikzcd}$$

which implies the commutativity of the first diagram in (3.14) for a monoid morphism, taking into account the form of multiplication M' of f^*Bf_*. Therefore, the following characterization of the category of $\mathcal{V}$-categories is established.

Lemma 7.3.3. *The objects of* $\mathcal{V}$-**Cat** *are pairs*

$$(A, X) \in \mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(X, X)) \times \mathbf{Set}$$

and morphisms are pairs $(\phi, f) : (A, X) \to (B, Y)$ *where*

$$\begin{cases} \phi : A \to f^*Bf_* & \textit{in } \mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(X, X)) \\ f : X \to Y & \textit{in } \mathbf{Set}. \end{cases}$$

As in the case of $\mathcal{V}$-**Grph** in the previous section, the category $\mathcal{V}$-**Cat** as presented in Chapter 4 is in fact isomorphic with the category described above, in the sense that there is a bijection between objects (*i.e.* the identity) and a bijection between arrows of these categories.

We already saw how $\mathcal{V}$-**Cat** inherits a (symmetric) monoidal structure from $\mathcal{V}$. The tensor product of the $\mathcal{V}$-categories $\mathcal{A}_X$ and $\mathcal{B}_Y$ is defined to be the $\mathcal{V}$-graph

$(\mathcal{A}\otimes\mathcal{B})_{X\times Y}$, given by the family of objects $\{\mathcal{A}(z,x)\otimes\mathcal{B}(w,y)\}$ in $\mathcal{V}$ for all $x,z\in X$ and $y,w\in Y$, with composition law and identities as given in Section 4.1.

Similarly to the free monoid construction on an object in a monoidal category $\mathcal{V}$, briefly discussed in Section 3.3, we now proceed to the description of an endofunctor on $\mathcal{V}$-**Grph** inducing the 'free $\mathcal{V}$-category' monad. The following proof can also be found in [**BCSW83, KL01**].

Proposition 7.3.4. *Let $\mathcal{V}$ be a monoidal category with coproducts, such that $\otimes$ preserves them on both sides. The functor*

$$\tilde{S}:\mathcal{V}\text{-}\mathbf{Cat}\to\mathcal{V}\text{-}\mathbf{Grph}$$

which forgets composition and identities has a left adjoint $\tilde{L}$, which maps a $\mathcal{V}$-graph $G:X\looparrowright X$ to the geometric series

$$\sum_{n\in\mathbb{N}}G^{\otimes n}:X\looparrowright X.$$

Proof. Recall that by Proposition 7.1.1, $\mathcal{V}$-$\mathbf{Mat}(X,X)$ admits the same class of colimits as $\mathcal{V}$, and also $\otimes=\circ$ preserves colimits on both sides. Hence, the forgetful functor S from its category of monoids has a left adjoint, namely the 'free monoid' functor, as in Proposition 3.3.4:

$$\begin{tikzcd}
L: \mathcal{V}\text{-}\mathbf{Mat}(X,X) \arrow[r] & \mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X)) \\
G \arrow[r, mapsto] & \sum_{n\in\mathbb{N}}G^n.
\end{tikzcd}$$

By Lemma 7.3.3, we deduce that this geometric series is in fact a $\mathcal{V}$-category with set of objects X. We now claim that the mapping

$$\begin{tikzcd}
\tilde{L}: \mathcal{V}\text{-}\mathbf{Grph} \arrow[r] & \mathcal{V}\text{-}\mathbf{Cat} \\
(G,X) \arrow[r, mapsto] & (LG,X)
\end{tikzcd}\tag{7.23}$$

induces a left adjoint of the forgetful functor $\tilde{S}$. For that, it is enough to show that the $\mathcal{V}$-graph morphism $\tilde{\eta}:(G,X)\to\tilde{S}\tilde{L}(G,X)$ which is the identity function on objects and the injection 2-cell of the summand G into the series, has the following universal property: if (B,Y) is a $\mathcal{V}$-category and F is a $\mathcal{V}$-graph arrow from (G,X) to its underlying $\mathcal{V}$-graph $\tilde{S}(B,Y)$, then there exists a unique $\mathcal{V}$-functor $H:(LG,X)\to(B,Y)$ such that the diagram

$$\begin{tikzcd}
(G,X) \arrow[rr, "\tilde{\eta}"] \arrow[dr, "F"'] & & \tilde{S}(\sum\limits_{n\in\mathbb{N}}G^n,X) \arrow[dl, dotted, "\tilde{S}H"] \\
& \tilde{S}(B,Y) &
\end{tikzcd}\tag{7.24}$$

commutes.

By Definition 7.2.1, a $\mathcal{V}$-graph functor F can be seen as a pair (ϕ,f) where $\phi:G\to f^*Bf_*$ is an arrow in $\mathcal{V}$-$\mathbf{Mat}(X,X)$, and furthermore Lemma 7.3.2 ensures that f^*Bf_* obtains a monoid structure. Since LG is the free monoid on the object G of $\mathcal{V}$-$\mathbf{Mat}(X,X)$, ϕ extends uniquely to a monoid morphism $\chi:LG\to f^*Bf_*$

such that the diagram

$$\begin{tikzcd}
G \arrow[rr, "\eta"] \arrow[dr, "\phi"'] & & \sum\limits_{n\in\mathbb{N}} G^n \arrow[dl, dotted, "S\chi"] \\
& f^*Bf_* &
\end{tikzcd}$$

commutes in the category $\mathcal{V}$-$\mathbf{Mat}(X,X)$, where η and S are respectively the unit and forgetful functor of the 'free monoid' adjunction $L \dashv S$.

By Lemma 7.3.3, this 2-cell $\chi : \sum_{n\in\mathbb{N}} G^n \Rightarrow f^*Bf_*$ in $\mathcal{V}$-$\mathbf{Mat}$, together with the function f, determine a $\mathcal{V}$-functor $H = (\chi, f) : (LG, X) \to (B, Y)$ satisfying the universal property (7.24). These data are sufficient to define an adjoint functor $\tilde{L}$ with object function (7.23), thus the 'free $\mathcal{V}$-category' adjunction

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Grph} \arrow[r, shift left, "\tilde{L}"] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathbf{Cat} \arrow[l, shift left, "\tilde{S}"]
\end{tikzcd}$$

is established. ☐

The above result was also given earlier in [**Wol74**, Proposition 2.2] but constructively, in the sense that the explicit description of the free $\mathcal{V}$-category along with its composition and identities is provided, and the universal property is shown without the use of $\mathcal{V}$-matrices. As a result, in that plain context, just the existence of coproducts in $\mathcal{V}$ suffices to establish the free $\mathcal{V}$-category adjunction, without requiring $\otimes$ to preserve them. Also, as proved in detail in [**Wol74**] and later generalized in [**BCSW83**] for categories enriched in bicategories, $\mathcal{V}$-**Cat** has and the forgetful functor $\tilde{S}$ reflects split coequalizers when $\mathcal{V}$ is cocomplete. By Beck's monadicity theorem, since $\tilde{S}$ also reflects isomorphisms, we have the following well-known result.

Proposition 7.3.5. *If $\mathcal{V}$ is a cocomplete monoidal category (such that $\otimes$ preserves colimits on both variables), the forgetful $\tilde{S} : \mathcal{V}$-**Cat** $\to \mathcal{V}$-**Grph** is monadic.*

Consequently, the category $\mathcal{V}$-**Cat** is isomorphic to the category of $\tilde{S}\tilde{L}$-algebras on $\mathcal{V}$-**Grph**. As mentioned earlier, $\mathcal{V}$-**Grph** is complete when $\mathcal{V}$ is, thus

Corollary 7.3.6. *The category $\mathcal{V}$-**Cat** is complete when $\mathcal{V}$ is.*

The fact that $\mathcal{V}$-**Cat** also has all colimits follows from a result by Linton in [**Lin69**], which states that if the category of algebras for a monad has coequalizers of reflexive pairs and $\mathcal{A}$ has all small coproducts, then $\mathcal{A}^T$ has all small colimits. By Proposition 7.2.2 $\mathcal{V}$-**Grph** admits all colimits if $\mathcal{V}$ does, hence the following is true.

Corollary 7.3.7. *The category $\mathcal{V}$-**Cat** is cocomplete when $\mathcal{V}$ is.*

Finally, $\mathcal{V}$-**Cat** also inherits local presentability from $\mathcal{V}$-**Grph**. As shown in [**KL01**], the monad $\tilde{S}\tilde{L}$ is finitary. Thus by a result of Gabriel and Ulmer [**GU71**, Satz 10.3] which states that if $\mathcal{A}$ is locally presentable, then $\mathcal{A}^T$ for a finitary monad is locally presentable, we obtain the following result.

Theorem. [**KL01**, 4.5] *If $\mathcal{V}$ is a monoidal closed category whose underlying ordinary category is locally λ-presentable, then $\mathcal{V}$-**Cat** is also λ-presentable.*

We can now turn to the 'dualization' of the concept of a $\mathcal{V}$-category in the context of the bicategory $\mathcal{V}$-**Mat**. Henceforth $\mathcal{V}$ is a monoidal category with coproducts, such that the tensor product $\otimes$ preserves them on both entries. The definition below follows Definition 2.2.5.

Definition 7.3.8. A (small) $\mathcal{V}$-*cocategory* $\mathcal{C}$ is a comonad in the bicategory $\mathcal{V}$-**Mat**. Thus it consists of a set X with an endoarrow $C : X \to\!\!\!\!\!+ X$, *i.e.* a $\mathcal{V}$-graph with set of objects ob$\mathcal{C} = X$, equipped with two 2-cells, the comultiplication and the counit

$$\begin{tikzcd} X \arrow[rr, bend left, "C"] \arrow[dr, "C"'] & \Downarrow\Delta & X \\ & X \arrow[ur, "C"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, bend left, "C"] \arrow[r, bend right, "1_X"'] \arrow[r, phantom, "\Downarrow\epsilon"] & X \end{tikzcd}$$

satisfying the following axioms:

$$\begin{tikzcd} X \arrow[rrr, bend left, "C"] \arrow[dr, "C"'] \arrow[drr, bend left, "C"] & \Downarrow\Delta & \Downarrow\Delta & X & = & X \arrow[rrr, bend left, "C"] \arrow[dr, "C"'] & \Downarrow\Delta & \Downarrow\Delta & X \\ & X \arrow[r, "C"'] & X \arrow[ur, "C"'] & & & & X \arrow[r, "C"'] \arrow[urr, bend left, "C"] & X, \arrow[ur, "C"'] & \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[rr, bend left, "C"] \arrow[dr, "1_X"'] \arrow[dr, bend left, "C"] \arrow[dr, phantom, "\Downarrow\epsilon"'] & \Downarrow\Delta & X & = & X \arrow[r, bend left, "C"] \arrow[r, bend right, "C"'] \arrow[r, phantom, "\Downarrow 1_C"] & X & = & X \arrow[rr, bend left, "C"] \arrow[dr, "C"'] & \Downarrow\Delta & X \\ & X \arrow[ur, "C"'] & & & & & & & X. \arrow[ur, "1_X"'] \arrow[ur, bend left, "C"] \arrow[ur, phantom, "\Downarrow\epsilon"] & \end{tikzcd}$$

In terms of components, the *cocomposition* of a $\mathcal{V}$-cocategory $\mathcal{C}$ is given by

$$\Delta_{x,z} : \mathcal{C}(x,z) \to \sum_{y \in X} \mathcal{C}(x,y) \otimes \mathcal{C}(y,x)$$

for any two objects $x, z \in X$, and the *coidentity elements* are given by

$$\epsilon_{x,y} : \mathcal{C}(x,y) \to 1_X(x,y) \equiv \begin{cases} \mathcal{C}(x,x) \xrightarrow{\epsilon_{x,x}} I, & \text{if } x = y \\ \mathcal{C}(x,y) \xrightarrow{\epsilon_{x,y}} 0, & \text{if } x \neq y \end{cases}$$

for all objects $x \in X$. The commutative diagrams expressing the coassociativity and counit axioms are

$$\begin{tikzcd}[column sep=small] & \mathcal{C}(x,w) \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ \sum\limits_z \mathcal{C}(x,z) \otimes \mathcal{C}(z,w) \arrow[d, "\sum\limits_z \Delta\otimes 1"'] & & \sum\limits_y \mathcal{C}(x,y) \otimes \mathcal{C}(y,w) \arrow[d, "\sum\limits_y 1\otimes\Delta"] \\ \sum\limits_z (\sum\limits_y \mathcal{C}(x,y) \otimes \mathcal{C}(y,z)) \otimes \mathcal{C}(z,w) \arrow[rr, "\cong", "\alpha"'] & & \sum\limits_y \mathcal{C}(x,y) \otimes (\sum\limits_z \mathcal{C}(y,z) \otimes \mathcal{C}(y,w)) \end{tikzcd}$$

$$\begin{tikzcd}
\sum\limits_z \mathcal{C}(x,z)\otimes\mathcal{C}(z,y) \arrow[d, "\sum\limits_z \epsilon\otimes 1"'] & \mathcal{C}(x,y) \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[dl, "\lambda^{-1}"] \arrow[dr, "\rho^{-1}"'] & \sum\limits_z \mathcal{C}(x,z)\otimes\mathcal{C}(z,y) \arrow[d, "\sum\limits_z 1\otimes\epsilon"] \\
I\otimes\mathcal{C}(x,y) & & \mathcal{C}(x,y)\otimes I
\end{tikzcd}$$

where α is the associator and λ, ρ are the unitors of $\mathcal{V}$-**Mat**. The vertical arrows of the latter diagram are explicitely the unique morphisms making the left and right parts of the diagram commute:

$$\begin{tikzcd}
 & & \sum\limits_z C(x,z)\otimes\mathcal{C}(z,y) \arrow[dl, dotted, "\sum\limits_z \epsilon_{x,z}\otimes 1"'] \arrow[dr, dotted, "\sum\limits_z 1\otimes\epsilon_{z,y}"] & & \\
\mathcal{C}(x,x)\otimes\mathcal{C}(x,y) \arrow[urr, hook, "i"] \arrow[r, "\epsilon_{x,x}\otimes 1"'] & I\otimes\mathcal{C}(x,y) & & \mathcal{C}(x,y)\otimes I. & \mathcal{C}(x,y)\otimes\mathcal{C}(y,y) \arrow[ull, hook', "i"'] \arrow[l, "1\otimes\epsilon_{y,y}"]
\end{tikzcd}$$

As for comonads in any bicategory, a $\mathcal{V}$-cocategory $\mathcal{C}$ with ob$\mathcal{C}=X$ is the same as a comonoid in the monoidal category $(\mathcal{V}\text{-}\mathbf{Mat}(X,X),\circ,1_X)$. Thus a one-object $\mathcal{V}$-cocategory is the same as a comonoid in the monoidal category $\mathcal{V}$. We denote such a $\mathcal{V}$-cocategory as $\mathcal{C}_X$ or (C,X). Analogously to $\mathcal{V}$-graphs and $\mathcal{V}$-categories, the notation (C,X) is preferred for the $\mathcal{V}$-matrices context, whereas $\mathcal{C}_X$ for the dual to the 'classic presentation' which basically corresponds to the componentwise version. The latter can evidently be expressed without the explicit use of $\mathcal{V}$-matrices.

The next step is to define the appropriate morphisms between $\mathcal{V}$-cocategories. For $\mathcal{V}$-graph arrows and $\mathcal{V}$-functors, morphisms F were initially defined in the standard way, *i.e.* consisting of certain arrows in $\mathcal{V}$ as in (7.13) and (4.2). Then, using the formulation in terms of $\mathcal{V}$-matrices, F was expressed as a pair (ϕ,f), where ϕ is a 2-cell in $\mathcal{V}$-**Mat** with components *isomorphic* arrows to the previous ones. This led to the characterization of Definition 7.2.1 for $\mathcal{V}$-**Grph**, and allowed the $\mathcal{V}$-functor axioms to be written in a colax monad functor style which resulted in characterization of Lemma 7.3.3 for $\mathcal{V}$-**Cat**. We similarly proceed for arrows for $\mathcal{V}$-cocategories.

Definition 7.3.9. A *$\mathcal{V}$-cofunctor* $F_f:\mathcal{C}_X\to\mathcal{D}_Y$ between two $\mathcal{V}$-cocategories is a morphism of $\mathcal{V}$-graphs, consisting of a function $f:X\to Y$ between their sets of objects and arrows in $\mathcal{V}$

$$F_{x,z}:\mathcal{C}(x,z)\to\mathcal{D}(fx,fz) \tag{7.25}$$

for any two objects $x,z\in\text{ob}\mathcal{C}$, which satisfy the commutativity of

$$\begin{tikzcd}
\mathcal{C}(x,z) \arrow[r, "\Delta^C_{x,z}"] \arrow[dd, "F_{x,z}"'] & \sum\limits_{y\in X}\mathcal{C}(x,y)\otimes\mathcal{C}(y,z) \arrow[dr, "\sum\limits_y F_{x,y}\otimes F_{y,z}"] \arrow[dd, dotted] & \\
 & & \sum\limits_{fy\in Y}\mathcal{D}(fx,fy)\otimes\mathcal{D}(fy,fz) \arrow[dl, hook, "\iota"] \\
\mathcal{D}(fx,fz) \arrow[r, "\Delta^D_{fx,fz}"'] & \sum\limits_{w\in Y}\mathcal{D}(fx,w)\otimes\mathcal{D}(w,fz) &
\end{tikzcd} \tag{7.26}$$

$$\text{and}\qquad\begin{tikzcd}
\mathcal{C}(x,x) \arrow[r, "\epsilon^{\mathcal{C}}_{x,x}"] \arrow[d, "F_{x,x}"'] & I \\
\mathcal{D}(fx,fx). \arrow[ru, "\epsilon^{\mathcal{D}}_{fx,fx}"'] &
\end{tikzcd}$$

The above commutative diagrams express the compatibility with cocomposition and coidentities. Equivalently, we can view a $\mathcal{V}$-functor as a pair $(\phi,f):(C,X)\to(D,Y)$ between two comonads in $\mathcal{V}$-**Mat**, with $f:X\to Y$ a function and ϕ a 2-cell $C\Rightarrow f^*Df_*$ which satisfies the equalities

$$\begin{tikzcd}
X \arrow[r, "C", "\shortmid" marking] \arrow[rr, bend left=50, "C", "\shortmid" marking] \arrow[r, phantom, shift left=5ex, "\Downarrow\Delta" near end] \arrow[d, "f_*"', "\shortmid" marking] \arrow[rd, phantom, "\swarrow\hat{\phi}"] & X \arrow[r, "C", "\shortmid" marking] \arrow[d, "f_*", "\shortmid" marking] \arrow[rd, phantom, "\swarrow\hat{\phi}"] & X \arrow[d, "f_*", "\shortmid" marking] & & X \arrow[rr, "C", "\shortmid" marking] \arrow[d, "f_*"', "\shortmid" marking] \arrow[rrd, phantom, "\Downarrow\hat{\phi}"] & & X \arrow[d, "f_*", "\shortmid" marking] \\
Y \arrow[r, "D"', "\shortmid" marking] & Y \arrow[r, "D"', "\shortmid" marking] & Y & = & Y \arrow[rr, "D"', "\shortmid" marking] \arrow[rr, phantom, shift right=4ex, "\Downarrow\Delta"] \arrow[rd, bend right, "D"', "\shortmid" marking] & & Y \\
 & & & & & Y \arrow[ru, bend right, "D"', "\shortmid" marking] &
\end{tikzcd}\tag{7.27}$$

$$\begin{tikzcd}
X \arrow[r, "1_X"', "\shortmid" marking] \arrow[r, bend left=50, "C", "\shortmid" marking] \arrow[r, phantom, shift left=4ex, "\Downarrow\epsilon"] \arrow[d, "f_*"', "\shortmid" marking] \arrow[rd, dotted, "f_*" description, "\shortmid" marking] \arrow[d, phantom, shift left=4ex, "\cong"] & X \arrow[d, "f_*", "\shortmid" marking] \arrow[d, phantom, shift right=4ex, "\cong"] & = & X \arrow[r, "C", "\shortmid" marking] \arrow[d, "f_*"', "\shortmid" marking] \arrow[rd, phantom, "\Downarrow\hat{\phi}"] & X \arrow[d, "f_*", "\shortmid" marking] \\
Y \arrow[r, "1_Y"', "\shortmid" marking] & Y & & Y \arrow[r, "D", "\shortmid" marking] \arrow[r, bend right=50, "1_Y"', "\shortmid" marking] \arrow[r, phantom, shift right=4ex, "\Downarrow\epsilon"] & Y
\end{tikzcd}$$

for $\hat{\phi}:f_*C\Rightarrow Df_*$ the mate of ϕ 'on the one side'. These two ways of defining a $\mathcal{V}$-cofunctor are equivalent in the sense that there is a bijection between them. The components of $\hat{\phi}$ are given by

$$\sum_{x'\in f^{-1}y} I\otimes C(x',x)\to D(fx',fx)\otimes I$$

which for fixed x' are in bijection to (7.25). The equalities (7.27) written in terms of components then agree with the commutativity of (7.26) up to appropriate tensoring with I.

It is not hard to see that $\mathcal{V}$-cofunctors compose, also by viewing them as specific types of lax comonad functors dually to Remark 7.3.1. Therefore we obtain a category $\mathcal{V}$-**Cocat** of $\mathcal{V}$-cocategories and $\mathcal{V}$-cofunctors.

Dually to Lemma 7.3.2, we have the following.

LEMMA 7.3.10. *Let (C,X) be a $\mathcal{V}$-cocategory. If $f:X\to Y$ is a function, then the composite $\mathcal{V}$-matrix*

$$\begin{tikzcd}
Y \arrow[r, "f^*", "\shortmid" marking] & X \arrow[r, "C", "\shortmid" marking] & X \arrow[r, "f_*", "\shortmid" marking] & Y
\end{tikzcd}$$

is a comonoid in $\mathcal{V}$-$\mathbf{Mat}(Y,Y)$*, which implies that* (f_*Cf^*,Y) *is also a* $\mathcal{V}$*-cocategory.*

Proof. The comultiplication $\Delta' : f_*Cf^* \to f_*Cf^*f_*Cf^*$ and the counit $\epsilon' : f_*Cf^* \to 1_X$ are given by the composites

$$\begin{tikzcd}
Y \arrow[r, "f^*", "\shortmid"{marking}] & X \arrow[rrrr, bend left, "C", "\shortmid"{marking}] \arrow[rrrr, phantom, "\Downarrow\Delta"] \arrow[dr, "C", "\shortmid"{marking}] & & & & X \arrow[r, "f_*", "\shortmid"{marking}] & Y \\
& & X \arrow[rr, bend left, "1_X", "\shortmid"{marking}] \arrow[rr, phantom, "\Downarrow\breve{\eta}"] \arrow[dr, "f_*"', "\shortmid"{marking}] & & X \arrow[ur, "C", "\shortmid"{marking}] & & \\
& & & Y \arrow[ur, "f^*"', "\shortmid"{marking}] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
& X \arrow[r, bend left, "C", "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow\epsilon"] \arrow[r, bend right, "1_X"', "\shortmid"{marking}] & X \arrow[dr, "f_*", "\shortmid"{marking}] & \\
Y \arrow[ur, "f^*", "\shortmid"{marking}] \arrow[rrr, bend right, "1_Y"', "\shortmid"{marking}] \arrow[rrr, phantom, "\Downarrow\breve{\varepsilon}"] & & & Y,
\end{tikzcd}$$

where $\breve{\varepsilon}$ and $\breve{\eta}$ are the counit and unit of the adjunction $f_* \dashv f^*$ in $\mathcal{V}$-$\mathbf{Mat}$ and Δ and ϵ the comonoid structure maps of C. In terms of pasting oparation, the new comultiplication and counit can be written as

$$\Delta' = f_*\big((C\breve{\eta}C)\cdot\Delta\big)f^*,$$
$$\epsilon' = \breve{\varepsilon}\cdot(f_*\epsilon f^*).$$

The coassociativity and counit axioms follow immediately from the axioms of the comonoid $C : X \to\!\!\!\!\!\shortmid\;\, X$ and the the triangular identities for $\breve{\varepsilon}$ and $\breve{\eta}$. ☐

Once again, it can be deduced that the diagrams (7.27) a $\mathcal{V}$-cofunctor $F : (C,X) \to (D,Y)$ has to satisfy coincide with the ones for a comonoid arrow between f_*Cf^* and D. The following characterization is now established.

Lemma 7.3.11. *Objects in* $\mathcal{V}$-$\mathbf{Cocat}$ *are pairs*

$$(C,X) \in \mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X)) \times \mathbf{Set}$$

and morphisms are pairs $(\psi,f) : (C,X) \to (D,Y)$ *where*

$$\begin{cases} \psi : f_*Cf^* \to D & \textit{in } \mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(Y,Y)) \\ f : X \to Y & \textit{in } \mathbf{Set}. \end{cases}$$

Notice how, out of the two equivalent formulations for $\mathcal{V}$-graph morphisms of Definition 7.2.1, $\mathcal{V}$-functors are expressed via pairs (ϕ,f) and $\mathcal{V}$-cofunctors are expressed via pairs (ψ,f), where the 2-cells $\phi : G \Rightarrow f^*Hf_*$ and $\psi : f_*Gf^* \Rightarrow H$ are mates in $\mathcal{V}$-$\mathbf{Mat}$.

The category $\mathcal{V}$-$\mathbf{Cocat}$ obtains a monoidal structure when $\mathcal{V}$ is symmetric monoidal. For two $\mathcal{V}$-cocategories $\mathcal{C}_X$ and $\mathcal{D}_Y$, $\mathcal{C}\otimes\mathcal{D}$ is their tensor product as $\mathcal{V}$-graphs, *i.e.* has as set of objects the cartesian product $X \times Y$ and consists of the family of objects in $\mathcal{V}$

$$(\mathcal{C}\otimes\mathcal{D})\big((z,w),(x,y)\big) = \mathcal{C}(z,x)\otimes\mathcal{D}(w,y).$$

The cocomposition law is given by the composite

$$\begin{tikzcd}
\mathcal{C}(z,x)\otimes D(w,y) \arrow[r, dashed] \arrow[ddr, bend right, "\Delta^C_{z,x}\otimes\Delta^D_{w,y}"'] & \sum\limits_{(x',y')}\mathcal{C}(z,x')\otimes\mathcal{D}(w,y')\otimes\mathcal{C}(x',x)\otimes\mathcal{D}(y',y) \\
& \sum\limits_{(x',y')}\mathcal{C}(z,x')\otimes\mathcal{C}(x',x)\otimes\mathcal{D}(w,y')\otimes\mathcal{D}(y',y) \arrow[u, "s"'] \\
& \sum\limits_{x'}\mathcal{C}(z,x')\otimes\mathcal{C}(x',x)\otimes\sum\limits_{y'}\mathcal{D}(w,y')\otimes\mathcal{D}(y',y) \arrow[u, "\cong"']
\end{tikzcd}$$

and the coidentity element is

$$\mathcal{C}(x,x)\otimes\mathcal{D}(y,y)\xrightarrow{\epsilon^C_{x,x}\otimes\epsilon^D_{y,y}} I\otimes I\cong I.$$

The unit for this tensor product is the unit $\mathcal{V}$-graph $\mathcal{I}$ with obvious cocomposition and coidentities. Similarly we can define the tensor product of two $\mathcal{V}$-cofunctors between $\mathcal{V}$-cocategories, and also symmetry is inherited, hence $(\mathcal{V}\text{-}\mathbf{Cocat},\otimes,\mathcal{I})$ is a symmetric monoidal category.

Dually to Proposition 7.3.4, we now construct the 'cofree $\mathcal{V}$-cocategory' functor using the cofree comonoid construction. As discussed in Section 3.3, the existence of the cofree comonoid usually requires more assumptions on $\mathcal{V}$ than the free monoid, and the following is no exception.

Proposition 7.3.12. *Suppose $\mathcal{V}$ is a locally presentable monoidal category, such that $\otimes$ preserves colimits in both variables. Then, the evident forgetful functor*

$$\tilde{U}:\mathcal{V}\text{-}\mathbf{Cocat}\longrightarrow\mathcal{V}\text{-}\mathbf{Grph}$$

has a right adjoint $\tilde{R}$, which maps a $\mathcal{V}$-graph (G,Y) to the cofree comonoid (RG,Y) on $G\in\mathcal{V}\text{-}\mathbf{Mat}(Y,Y)$.

Proof. The forgetful functor $\tilde{U}$ maps any $\mathcal{V}$-cocategory (C,X) to the 'underlying' $\mathcal{V}$-graph (UC,X), where U is the forgetful functor from the category of comonoids of the monoidal category $(\mathcal{V}\text{-}\mathbf{Mat}(Y,Y),\circ,1_Y)$. By Corollary 7.1.2, U has a right adjoint

$$R:\mathcal{V}\text{-}\mathbf{Mat}(Y,Y)\longrightarrow\mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(Y,Y))$$

namely the cofree comonoid functor. By Lemma 7.3.11, the pair (RG,Y) where RG is the cofree comonoid on an endoarrow $G:Y\mathrel{-\mkern-6mu+\mkern-6mu\rightarrow}Y$ is in fact a $\mathcal{V}$-cocategory with set of objects Y. We claim that the mapping

$$\begin{aligned}\tilde{R}:\ \mathcal{V}\text{-}\mathbf{Grph}&\longrightarrow\mathcal{V}\text{-}\mathbf{Cocat}\\ (G,Y)&\longmapsto(RG,Y)\end{aligned}\tag{7.28}$$

gives rise to a right adjoint of the forgetful $\tilde{U}$. It is enough to show that for ε the counit of the cofree comonoid adjunction $U\dashv R$, the $\mathcal{V}$-graph arrow $\tilde{\varepsilon}=(\varepsilon,\mathrm{id}_Y)$: $\tilde{U}\tilde{R}(G,Y)\to(G,Y)$ is universal. This means that for any $\mathcal{V}$-cocategory $\mathcal{C}_X$ and any $\mathcal{V}$-graph morphism F from its underlying $\mathcal{V}$-graph $\tilde{U}(C,X)$ to (G,Y), there exists a unique $\mathcal{V}$-cofunctor $H : (C, X) \to (RG, Y)$ such that the diagram

$$\begin{tikzcd}
\tilde{U}(RG,Y) \arrow[rr, "\tilde{\varepsilon}"] & & (G,Y) \\
& \tilde{U}(C,X) \arrow[ul, dotted, "\tilde{U}H"] \arrow[ur, "F"'] &
\end{tikzcd} \tag{7.29}$$

commutes.

The $\mathcal{V}$-graph arrow F can be seen as a pair (ψ, f) where $f : X \to Y$ is the function on objects and $\psi : f_*Cf^* \to G$ is an arrow in $\mathcal{V}$-**Mat**(Y, Y). However, by Lemma 7.3.10 the composite f_*Cf^* is an object of **Comon**($\mathcal{V}$-**Mat**(Y, Y)), since C is a comonoid itself. Due to RG being the cofree comonoid on G, this ψ extends uniquely to a comonoid arrow $\chi : f_*Cf^* \to RG$ such that the diagram

$$\begin{tikzcd}
RG \arrow[rr, "\varepsilon"] & & G \\
& f_*Cf^* \arrow[ul, dotted, "U\chi"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

commutes in $\mathcal{V}$-**Mat**(Y, Y). Then, by Lemma 7.3.11 this 2-cell χ in **Comon**($\mathcal{V}$-**Mat**(Y, Y)) along with the function $f : X \to Y$ determines a $\mathcal{V}$-cofunctor $H : (C, X) \to (RG, Y)$, which satisfies the commutativity of (7.29). Therefore $\tilde{R}$ extends to a functor with mapping on objects as in (7.28), which establishes the 'cofree $\mathcal{V}$-cocategory' adjunction $\tilde{U} \dashv \tilde{R} : \mathcal{V}$-**Grph** $\to \mathcal{V}$-**Cocat**. ☐

At this point, properties of $\mathcal{V}$-**Cocat** cease to be straightforward dualizations of the ones of $\mathcal{V}$-**Cat**. As an example, in order to deduce results such as comonadicity of $\mathcal{V}$-**Cocat** over $\mathcal{V}$-**Grph**, we will later show that $\mathcal{V}$-**Cocat** is locally presentable via a different method, under the conditions for the existence of the cofree $\mathcal{V}$-cocategory functor $\tilde{R}$.

We close this section by the construction of colimits in $\mathcal{V}$-**Cocat**. In fact, this follows from the construction of colimits in $\mathcal{V}$-**Grph** in Proposition 7.2.2, with an induced extra structure on the colimiting cocone which amounts to a colimit of $\mathcal{V}$-cocategories.

PROPOSITION 7.3.13. *Suppose that $\mathcal{V}$ is a locally presentable monoidal category, such that $\otimes$ preserves colimits in both variables. The category $\mathcal{V}$-**Cocat** has all small colimits.*

PROOF. Consider a diagram in $\mathcal{V}$-**Cocat** given by

$$\begin{tikzcd}
D: \mathcal{J} \arrow[r] & \mathcal{V}\text{-}\mathbf{Cocat} \\
j \arrow[r, dotted, mapsto] \arrow[d, "\theta"'] & (C_j, X_j) \arrow[d, "(\psi_\theta, f_\theta)"] \\
k \arrow[r, dotted, mapsto] & (C_k, X_k)
\end{tikzcd}$$

for a small category $\mathcal{J}$. By Lemma 7.3.11, $f_\theta : X_j \to X_k$ is a function and ψ_θ is an arrow $(f_\theta)_* C_j (f_\theta)^* \to C_k$ in $\mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(X_k, X_k))$, i.e. a 2-cell in $\mathcal{V}$-**Mat**

$$\begin{tikzcd} X_j \arrow[r, "C_j", "\shortmid"{marking}] & X_j \arrow[d, "(f_\theta)_*", "\shortmid"{marking}] \\ X_k \arrow[u, "(f_\theta)^*", "\shortmid"{marking}] \arrow[r, "C_k"', "\shortmid"{marking}] \arrow[ur, phantom, "\Downarrow\psi_\theta"] & X_k \end{tikzcd}$$

satisfying the usual comonoid morphism properties. We can first construct the colimit of the underlying $\mathcal{V}$-graphs of this diagram as in Proposition 7.2.2. We then obtain a colimiting cocone

$$\big((C_j, X_j) \xrightarrow{(\lambda_j, \tau_j)} (C, X) \,|\, j \in \mathcal{J}\big) \tag{7.30}$$

in $\mathcal{V}$-**Grph**, where $(\tau_j : X_j \to X \,|\, j \in \mathcal{J})$ is the colimit of the sets of objects of the $\mathcal{V}$-cocategories in **Set**, and $(\lambda_j : (\tau_j)_* C_j (\tau_j)^* \to C \,|\, j \in \mathcal{J})$ is the colimiting cocone of the diagram K as in (7.17) in the cocomplete $\mathcal{V}\text{-}\mathbf{Mat}(X, X)$.

Notice that $K : \mathcal{J} \to \mathcal{V}\text{-}\mathbf{Mat}(X, X)$ in fact lands inside $\mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(X, X))$: Lemma 7.3.10 ensures that $\mathcal{V}$-matrices of the form $f_* C f^*$ for any comonoid C inherit a comonoid structure, and also the composite arrows (7.18) where the middle 2-cell is now the comonoid arrow ψ_θ ensure that $K\theta$ are comonoid morphisms. Since by Corollary 7.1.2 the category of comonoids is comonadic over $\mathcal{V}\text{-}\mathbf{Mat}(X, X)$, the respective forgetful functor creates all colimits, therefore $C : X \nrightarrow X$ obtains a unique comonoid structure. Moreover, the legs of the cocone

$$\begin{tikzcd} X_j \arrow[r, "C_j", "\shortmid"{marking}] & X_j \arrow[d, "(\tau_j)_*", "\shortmid"{marking}] \\ X \arrow[u, "(\tau_j)^*", "\shortmid"{marking}] \arrow[r, "C"', "\shortmid"{marking}] \arrow[ur, phantom, "\Downarrow\lambda_j"] & X \end{tikzcd}$$

are comonoid arrows, hence together with the functions τ_j they form $\mathcal{V}$-cofunctors. Thus the colimit (7.30) lifts in $\mathcal{V}$-**Cocat**. □

7.4. Enrichment of $\mathcal{V}$-categories in $\mathcal{V}$-cocategories

We now wish to extend the results presented in Section 6.1, where the existence of the universal measuring comonoid and the induced enrichment of monoids in comonoids were established. Similarly to the previous development, we aim to identify an *action* of the symmetric monoidal closed category $\mathcal{V}$-**Cocat** on the ordinary category $\mathcal{V}$-**Cat** (or better its opposite), with a parametrized adjoint which will turn out to be the 'enriched-hom' functor of a ($\mathcal{V}$-**Cocat**)-enriched category with underlying category $\mathcal{V}$-**Cat**. The relevant theory which underlies this process is contained in Section 4.3.

Suppose that $\mathcal{V}$ is a cocomplete symmetric monoidal closed category with products. Recall that there exists a lax functor of bicategories

$$\mathrm{Hom} : (\mathcal{V}\text{-}\mathbf{Mat})^{\mathrm{co}} \times \mathcal{V}\text{-}\mathbf{Mat} \longrightarrow \mathcal{V}\text{-}\mathbf{Mat}$$

defined as in (7.5). Then the functor between the hom-categories (of endoarrows) $\mathrm{Hom}_{(X,Y),(X,Y)}$ induces the internal hom ${}^{g}\mathrm{Hom} : \mathcal{V}\text{-}\mathbf{Grph}^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{Grph} \to \mathbf{Grph}$ of $\mathcal{V}$-graphs as described in Proposition 7.2.3, via

$$\mathrm{Hom}((G,X),(H,Y))(k,s) := \prod_{x,x' \in X} [G(x',x), H(kx',sx)]$$

for all $k, s \in Y^X$. Moreover, by Lemma 3.3.3, every lax functor between bicategories induces a functor between monoids of hom-categories of endoarrows. For the lax functor Hom, we obtain

$$\mathbf{Mon}(\mathrm{Hom}_{(X,Y),(X,Y)})\colon \mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X))^{\mathrm{op}} \times \mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(Y,Y)) \to \mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(Y^X,Y^X)) \tag{7.31}$$

which is just the restriction of $\mathrm{Hom}_{(X,Y),(X,Y)}$ on the category

$$\begin{aligned}\mathbf{Mon}\Big((\mathcal{V}\text{-}\mathbf{Mat}^{\mathrm{co}}\times\mathcal{V}\text{-}\mathbf{Mat})((X,Y),(X,Y))\Big) &\cong \mathbf{Mon}\Big(\mathcal{V}\text{-}\mathbf{Mat}(X,X)^{\mathrm{op}}\times\mathcal{V}\text{-}\mathbf{Mat}(Y,Y)\Big)\\ &\cong \mathbf{Mon}\Big(\mathcal{V}\text{-}\mathbf{Mat}(X,X)^{\mathrm{op}}\Big)\times\mathbf{Mon}\Big(\mathcal{V}\text{-}\mathbf{Mat}(Y,Y)\Big)\\ &\cong \mathbf{Comon}\Big(\mathcal{V}\text{-}\mathbf{Mat}(X,X)\Big)^{\mathrm{op}}\times\mathbf{Mon}\Big(\mathcal{V}\text{-}\mathbf{Mat}(Y,Y)\Big).\end{aligned}$$

Since a $\mathcal{V}$-cocategory $\mathcal{C}_X = (C,X)$ has the structure of a comonoid in the monoidal $(\mathcal{V}\text{-}\mathbf{Mat}(X,X), \circ, 1_X)$ and a $\mathcal{V}$-category $\mathcal{B}_Y = (B,Y)$ has the structure of a monoid in $(\mathcal{V}\text{-}\mathbf{Mat}(Y,Y), \circ, 1_Y)$, we deduce that $\mathbf{Mon}(\mathrm{Hom}_{(X,Y),(X,Y)})$ is in fact the object mapping of a functor

$$K : \mathcal{V}\text{-}\mathbf{Cocat}^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat} \tag{7.32}$$

which is the restriction of the functor ${}^{g}\mathrm{Hom}$ on the product of $\mathcal{V}$-cocategories and $\mathcal{V}$-categories. This concretely means that whenever we have a $\mathcal{V}$-cocategory $\mathcal{C}_X$ and a $\mathcal{V}$-category $\mathcal{B}_Y$, the $\mathcal{V}$-graph $K(\mathcal{C}_X, \mathcal{B}_Y) \equiv \mathrm{Hom}(\mathcal{C}, \mathcal{B})_{Y^X}$ obtains the structure of a $\mathcal{V}$-category.

Explicitly, for each triple of functions $k, s, t \in Y^X$, the composition law $M : K(\mathcal{C},\mathcal{B})(k,s) \otimes K(\mathcal{C},\mathcal{B})(s,t) \to K(\mathcal{C},\mathcal{B})(k,t)$ for $\mathcal{K}(\mathcal{C},\mathcal{B})$ is an arrow

$$\prod_{a,a} [\mathcal{C}(a',a), \mathcal{B}(ka',sa)] \otimes \prod_{b,b'} [\mathcal{C}(b',b), \mathcal{B}(sb',tb)] \to \prod_{c,c'} [\mathcal{C}(c',c), \mathcal{B}(kc',tc)].$$

This is defined via its adjunct under the usual tensor-hom adjunction

$$\begin{tikzcd}
\prod\limits_{a,a'} [\mathcal{C}(a',a),\mathcal{B}(ka',sa)]\otimes \prod\limits_{b,b'} [\mathcal{C}(b',b),\mathcal{B}(sb',tb)]\otimes\mathcal{C}(c',c) \arrow[r, dashed] \arrow[d, "1\otimes\Delta_{c',c}"'] & \mathcal{B}(kc',tc) \\
\prod\limits_{a,a'} [\mathcal{C}(a',a),\mathcal{B}(ka',sa)]\otimes \prod\limits_{b,b'} [\mathcal{C}(b',b),\mathcal{B}(sb',tb)]\otimes\sum\limits_{c''}\mathcal{C}(c',c'')\otimes\mathcal{C}(c'',c) \arrow[d, "s"'] & \\
\sum\limits_{c''}\prod\limits_{a,a'} [\mathcal{C}(a',a),\mathcal{B}(ka',sa)]\otimes\mathcal{C}(c',c'')\otimes \prod\limits_{b,b'} [\mathcal{C}(b',b),B(sb',tb)]\otimes\mathcal{C}(c'',c) \arrow[d, "\pi_{c',c''}\otimes1\otimes\pi_{c'',c}\otimes1"'] & \\
\sum\limits_{c''} [\mathcal{C}(c',c''),\mathcal{B}(kc',sc'')]\otimes\mathcal{C}(c',c'')\otimes[\mathcal{C}(c'',c),\mathcal{B}(sc'',tc)]\otimes\mathcal{C}(c'',c) \arrow[r, "\mathrm{ev}\otimes\mathrm{ev}"'] & \sum\limits_{c''}\mathcal{B}(kc',sc'')\otimes\mathcal{B}(sc'',tc) \arrow[uuu, "M_{kc',tc}"']
\end{tikzcd}$$

for fixed c, c'. The identities for each object $s \in Y^X$ are arrows

$$\eta_k : I \to K(\mathcal{C}, \mathcal{B})(k,k) = \prod_{a,a' \in X} [\mathcal{C}(a',a), \mathcal{B}(sa',sa)] \tag{7.33}$$

which correspond uniquely for fixed $a = a' \in X$ to the composite

$$\begin{tikzcd}
I \otimes \mathcal{C}(a,a) \arrow[rrr, dashed] \arrow[dr, "1\otimes\epsilon_{a,a}"'] & & & \mathcal{B}(sa,sa). \\
& I \otimes I \arrow[r, "r_I"'] & I \arrow[ur, "\eta_{sa,sa}"'] &
\end{tikzcd}$$

At the diagrams above, Δ and ϵ are the cocomposition and coidentites of $\mathcal{C}$ and M, η the composition and identities of $\mathcal{B}$. For $a \neq a'$, the arrow (7.33) corresponds to

$$I \otimes \mathcal{C}(a',a) \xrightarrow{1\otimes\epsilon_{a',a}} 0 \xrightarrow{!} \mathcal{B}(sa',sa).$$

Moreover, it can be checked that for a $\mathcal{V}$-cofunctor $F_f : \mathcal{C}'_{X'} \to \mathcal{C}_X$ and a $\mathcal{V}$-functor $G_g : \mathcal{B}_Y \to \mathcal{B}'_{Y'}$, the $\mathcal{V}$-graph arrow

$${}^g\mathrm{Hom}(F,G)_{g^f} : {}^g\mathrm{Hom}(\mathcal{C},\mathcal{B})_{Y^X} \to {}^g\mathrm{Hom}(\mathcal{C}',\mathcal{B}')_{Y'^{X'}}$$

as defined in (7.19) is in fact a $\mathcal{V}$-functor between the $\mathcal{V}$-categories, *i.e.* respects the compositions and identities described above. Therefore we deduce that the functor K is well defined.

PROPOSITION 7.4.1. *Suppose that $\mathcal{V}$ is a cocomplete symmetric monoidal closed category with products. The functor K (7.32) is an action, and so is its opposite functor*

$$\begin{tikzcd}
K^{\mathrm{op}} : \mathcal{V}\textbf{-Cocat} \times \mathcal{V}\textbf{-Cat}^{\mathrm{op}} \arrow[r] & \mathcal{V}\textbf{-Cat}^{\mathrm{op}} \\
(\ \mathcal{C}_X\ ,\ \mathcal{B}_Y\ ) \arrow[r, mapsto] & \mathrm{Hom}^{\mathrm{op}}(\mathcal{C},\mathcal{B})_{Y^X}.
\end{tikzcd}$$

PROOF. By Lemma 4.3.2, the internal hom functor in any symmetric monoidal closed category $\mathcal{V}$ constitutes an action of $\mathcal{V}^{\mathrm{op}}$ on $\mathcal{V}$. Thus for the symmetric monoidal closed category of $\mathcal{V}$-graphs, the functors

$${}^g\mathrm{Hom} : \mathcal{V}\textbf{-Grph}^{\mathrm{op}} \times \mathcal{V}\textbf{-Grph} \to \mathcal{V}\textbf{-Grph}$$

as well as ${}^g\mathrm{Hom}^{\mathrm{op}}$ are actions. As stressed earlier, K is the restriction of ${}^g\mathrm{Hom}$ on $\mathcal{V}\textbf{-Cocat}^{\mathrm{op}} \times \mathcal{V}\textbf{-Cat}$, hence there exists isomorphisms

$$\mathrm{Hom}(\mathcal{C} \otimes \mathcal{D}, \mathcal{A}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{C}, \mathrm{Hom}(\mathcal{D},\mathcal{A}))$$
$$\mathrm{Hom}(\mathcal{I},\mathcal{D}) \xrightarrow{\sim} \mathcal{D}$$

for any $\mathcal{V}$-cocategories $\mathcal{C}_X$, $\mathcal{D}_Y$ and $\mathcal{V}$-category $\mathcal{A}_Z$, initially in $\mathcal{V}$**-Grph**. Notice that $\otimes$ and $\mathcal{I}$ of the monoidal $\mathcal{V}$**-Cocat** are inherited from $\mathcal{V}$**-Grph**, and Hom is the object function of both ${}^g\mathrm{Hom}$ and K.

Since $\tilde{S} : \mathcal{V}\textbf{-Cat} \to \mathcal{V}\textbf{-Grph}$ is conservative, these isomorphisms are reflected into $\mathcal{V}$**-Cat**, and the coherence diagrams still commute. Therefore K is an action, and in particular its opposite functor K^{op} is an action of the symmetric monoidal category $\mathcal{V}$**-Cocat** on the category $\mathcal{V}\textbf{-Cat}^{\mathrm{op}}$. □

What is left to show is that this action K^{op} has a parametrized adjoint, which will induce the enrichment of the category on which the monoidal category acts. In order to prove the existence of the adjoint in question, we need some preliminary results which further clarify the structure of $\mathcal{V}$-**Cocat**.

First of all, we can apply the techniques from Propositions 3.3.5 and 3.4.2 regarding the expression of the categories $\mathbf{Comon}(\mathcal{V})$ and $\mathbf{Comod}_{\mathcal{V}}(C)$ as an equifier, so that we obtain the following result.

PROPOSITION 7.4.2. *Suppose that $\mathcal{V}$ is a locally presentable monoidal category, such that $(-\otimes-)$ preserves colimits on both sides. Then, the category $\mathcal{V}$-*$\mathbf{Cocat}$ *is a locally presentable category.*

PROOF. Define an endofunctor on the category of $\mathcal{V}$-graphs by

$$\begin{tikzcd}
F\colon & \mathcal{V}\text{-}\mathbf{Grph} \arrow[r] & \mathcal{V}\text{-}\mathbf{Grph} \\
& (G,X) \arrow[r, mapsto, dotted] \arrow[d, "{(\psi,f)}"'] & (G\circ G,X)\times(1_X,X) \arrow[d, "{F(\psi,f)}"] \\
& (H,Y) \arrow[r, mapsto, dotted] & (H\circ H,Y)\times(1_Y,Y).
\end{tikzcd}$$

The mapping on arrows, for a 2-cell $\psi : f_*Gf^* \Rightarrow H$, is explicitly

$$\begin{tikzcd}
X \arrow[r, "G", "|" marking] \arrow[dr, phantom, "\Downarrow\psi"] & X \arrow[r, "1_X", "|" marking] \arrow[d, "f_*"', "|" marking] \arrow[r, phantom, "\Downarrow\check{\eta}"', shift right=4] & X \arrow[r, "G", "|" marking] \arrow[dr, phantom, "\Downarrow\psi"] & X \arrow[d, "f_*", "|" marking] & \times & X \arrow[rr, "1_X", "|" marking] \arrow[rrd, "f_*", "|" marking] & & X \arrow[d, "f_*", "|" marking] \arrow[dl, phantom, "\Downarrow\cong"] \\
Y \arrow[u, "f^*", "|" marking] \arrow[r, "H"', "|" marking] & Y \arrow[ru, bend right, "f^*"', "|" marking] \arrow[rr, "H"', "|" marking] & & Y & & Y \arrow[u, "f^*", "|" marking] \arrow[rr, "1_Y"', "|" marking] \arrow[ru, phantom, "\Downarrow\check{\varepsilon}"] & & Y
\end{tikzcd}$$

where the left unitor λ of the bicategory $\mathcal{V}$-**Mat** is suppressed.

The category of coalgebras $\mathbf{Coalg}F$ for this endofunctor has as objects $\mathcal{V}$-graphs (C,X) equipped with a morphism $\alpha : C \to C\circ C\times 1_X$, *i.e.* two $\mathcal{V}$-graph arrows

$$\alpha_1 : (C,X)\to(C\circ C,X) \quad \text{and} \quad \alpha_2 : (C,X)\to(1_X,X).$$

A morphism $(C,\alpha)\to(D,\beta)$ is a $\mathcal{V}$-graph morphism $(\psi,f) : (C,X)\to(D,Y)$ which is compatible with α and β, *i.e.* satisfy the equalities

$$\begin{tikzcd}
X \arrow[r, "C", "|" marking] \arrow[rrr, bend left=50, "C", "|" marking] \arrow[rrr, phantom, bend left=25, "\Downarrow\alpha_1"] \arrow[dr, phantom, "\Downarrow\psi"] & X \arrow[r, "1_X", "|" marking] \arrow[d, "f_*"', "|" marking] \arrow[r, phantom, "\Downarrow\check{\eta}"', shift right=4] & X \arrow[r, "C", "|" marking] \arrow[dr, phantom, "\Downarrow\psi"] & X \arrow[d, "f_*", "|" marking] & & X \arrow[rr, "C", "|" marking] \arrow[drr, phantom, "\Downarrow\psi"] & & X \arrow[d, "f_*", "|" marking] \\
Y \arrow[u, "f^*", "|" marking] \arrow[r, "D"', "|" marking] & Y \arrow[ru, bend right, "f^*"', "|" marking] \arrow[rr, "D"', "|" marking] & & Y & = & Y \arrow[u, "f^*", "|" marking] \arrow[rr, "D", "|" marking] \arrow[rd, "D"', "|" marking] \arrow[rr, phantom, "\Downarrow\beta_1"', shift right=4] & & Y \\
& & & & & & Y \arrow[ru, "D"', "|" marking] &
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "1_X"'] \arrow[r, bend left=50, "C"] \arrow[r, phantom, bend left=25, "\Downarrow\alpha_2"] \arrow[dr, dotted, "f_*" description] & X \arrow[d, "f_*"] \arrow[dl, phantom, "\cong"] & = & X \arrow[r, "C"] \arrow[dr, phantom, "\Downarrow\psi"] & X \arrow[d, "f_*"] \\
Y \arrow[u, "f^*"] \arrow[r, "1_Y"'] \arrow[ur, phantom, "\Downarrow\check{\varepsilon}"'] & Y & & Y \arrow[u, "f^*"] \arrow[r, "D"] \arrow[r, bend right=50, "1_Y"'] \arrow[r, phantom, bend right=25, "\Downarrow\beta_2"] & Y.
\end{tikzcd}$$

Notice that the category **Coalg**F contains $\mathcal{V}$-**Cocat** as a full subcategory: the morphisms are precisely the same, by comparing the above diagrams with (7.27) where $\hat{\phi}$ is a mate of ψ, and objects are $\mathcal{V}$-graphs equipped with cocomposition and coidentities arrows that don't necessarily satisfy coassociativity and counit axioms.

Since $\mathcal{V}$-**Cocat** is a cocomplete category by Proposition 7.3.13, we claim that it is furthermore accessible, thus a locally presentable category. It is enough to express $\mathcal{V}$-**Cocat** as an equifier of a family of pairs of natural transformations between accessible functors, *i.e.* functors between accessible categories that preserve filtered colimits.

First of all, we have to show that the endofunctor F preserves all filtered colimits. Take a colimiting cocone

$$\big((G_j, X_j) \xrightarrow{(\lambda_j, \tau_j)} (G, X) \,|\, j \in \mathcal{J}\big)$$

in $\mathcal{V}$-**Grph** for a diagram like (7.16) for a small filtered category $\mathcal{J}$, constructed as in Proposition 7.2.2, *i.e.* $(\tau_j : X_j \to X)$ is a colimiting cocone in **Set** and $(\lambda_j : (\tau_j)_* C_j (\tau_j)^* \to C)$ is a colimiting cocone in $\mathcal{V}$-$\mathbf{Mat}(X, X)$. We require its image under F

$$F(\lambda_j, \tau_j) : (G_j \circ G_j, X_j) \times (1_{X_j}, X_j) \to (G \circ G, X) \times (1_X, X) \tag{7.34}$$

to be a colimiting cocone in $\mathcal{V}$-**Grph**.

For the first part of the diagram, we can immediately deduce that

$$(\tau_j)_* \circ G_j \circ (\tau_j)^* \circ (\tau_j)_* \circ G_j \circ (\tau_j)^* \xrightarrow{\lambda_j * \lambda_j} G \circ G$$

is a colimit in $(\mathcal{V}\text{-}\mathbf{Mat}(X, X), \circ, 1_X)$, as the tensor product (horizontal composite) of two colimiting cocones. We claim that pre-composing this with the unit

$$1 * \check{\eta} * 1 : (\tau_j)_* \circ G_j \circ 1_{X_j} \circ G_j \circ (\tau_j)^* \to (\tau_j)_* \circ G_j \circ (\tau_j)^* \circ (\tau_j)_* \circ G_j \circ (\tau_j)^*$$

still gives a colimiting cocone. Indeed, if we take components in $\mathcal{V}$ of the respective 2-cells in $\mathcal{V}$-**Mat**, this comes down to showing that the inclusion

$$\sum_{z \in X_j}^{\substack{\tau_j u = x' \\ \tau_j w = x}} G_j(u, z) \otimes G_j(z, w) \hookrightarrow \sum_{\tau_j a = \tau_j b}^{\substack{\tau_j u = x' \\ \tau_j w = x}} G_j(u, a) \otimes G_j(b, w)$$

for any two fixed $x, x' \in X$, where $u, w, a, b \in X_j$, does not alter the colimit. One way of showing this is by considering the following discrete opfibrations over the filtered shape $\mathcal{J}$:

$$\begin{aligned}\mathcal{L} &= \{(j,a,b)\,|\,j\in\mathcal{J}, a,b\in X_j, \tau_j a=\tau_j b\}\\ \mathcal{M} &= \{(j,z)\,|\,j\in\mathcal{J}, z\in X_j\}\end{aligned}$$

where for example the arrows $(j,a,b)\to(j',a',b')$ in $\mathcal{L}$ are determined by arrows $\theta : j\to j'$ in $\mathcal{J}$ such that $a'=f_\theta(a)$ and $b'=f_\theta(b)$ (the function $f_\theta : X_j\to X_{j'}$ is the image of the diagram (7.16) in **Set**). We can now define diagrams of shape $\mathcal{L}$ and $\mathcal{M}$ in $\mathcal{V}$

$$\begin{tikzcd} L: & \mathcal{L} \arrow[r] & \mathcal{V} & M: & \mathcal{M} \arrow[r] & \mathcal{V} \\ & (j,a,b) \arrow[r, mapsto] & G_j(u,a)\otimes G_j(b,w) & & (j,z) \arrow[r, mapsto] & G_j(u,z)\otimes G_j(z,w) \end{tikzcd}$$

and appropriately on morphisms. The colimits for these diagrams in $\mathcal{V}$, taking into account that the fibres are discrete categories, are

$$\begin{aligned}\operatorname{colim} L &\cong \operatorname*{colim}_j \sum_{\tau_j a=\tau_j b} G_j(u,a)\otimes G_j(b,w)\\ \operatorname{colim} M &\cong \operatorname*{colim}_j \sum_{z\in X_j} G_j(u,z)\otimes G_j(z,w).\end{aligned}$$

Finally, notice that there exists a functor $T:\mathcal{M}\to\mathcal{L}$ mapping each (j,z) to (j,z,z) and making the triangle

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "T"] \arrow[dr, "M"'] & & \mathcal{L} \arrow[dl, "L"] \\ & \mathcal{V} & \end{tikzcd}$$

commute. Due to the construction of filtered colimits in **Set**, it is not hard to show that the slice category $\big((j,z,w)\downarrow T\big)$ is non-empty and connected. Hence T is a final functor and we can restrict the diagram on $\mathcal{L}$ to $\mathcal{M}$ without changing the colimit, as claimed.

For the second part of the diagram, it is enough to show that

$$\begin{tikzcd} & X_j \arrow[dr, "(\tau_j)_*", "|" marking] \arrow[d, phantom, "\Downarrow\check{\varepsilon}"] & \\ X \arrow[ur, "(\tau_j)^*", "|" marking] \arrow[rr, bend right, "1_X"', "|" marking] & {} & Y \end{tikzcd}$$

is a colimiting cocone in $\mathcal{V}$-**Mat**(X,X), for the diagram mapping each j to

$$\begin{tikzcd} X \arrow[r, "(\tau_j)^*", "|" marking] & X_j \arrow[r, "1_{X_j}", "|" marking] & X_j \arrow[r, "(\tau_j)_*", "|" marking] & X \end{tikzcd}$$

as in (7.32). This can be established by first verifying that $\check{\varepsilon}$ is a cocone, and then that it has the required universal property.

We have thus shown that the cocone (7.34) is indeed colimiting, hence F is a finitary functor as required. This part of the proof is due to Ignacio Lopez Franco.

Since $\mathcal{V}$-**Grph** is locally presentable and the endofunctor F preserves filtered colimits, **Coalg**F is a locally presentable category by the basic facts for endofunctor coalgebra categories in Section 3.3. Also the forgetful functor $\overline{V}$: $\mathbf{Coalg}F \to \mathcal{V}$-**Grph** creates all colimits. Now consider the following pairs of natural transformations between functors from $\mathbf{Coalg}F$ to $\mathcal{V}$-**Grph**:

$$\phi^1, \psi^1 : \overline{V} \Rightarrow FF\overline{V}, \quad \phi^2, \psi^2 : \overline{V} \Rightarrow (- \circ 1_X)\overline{V}, \quad \phi^3, \psi^3 : \overline{V} \Rightarrow \overline{V}(- \circ 1_X)$$

given by the components

$$\begin{tikzcd}
\phi^1_{(C,X)}: X \arrow[rrr, bend left=30, "C"] \arrow[dr, "C"'] \arrow[drr, bend left=20, "C"] \arrow[dr, phantom, bend left=25, "\Downarrow\alpha_1"] & & \arrow[d, phantom, "\Downarrow\alpha_1"] & X, \\
& X \arrow[r, "C"'] & X \arrow[ur, "C"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
\psi^1_{(C,X)}: X \arrow[rrr, bend left=30, "C"] \arrow[dr, "C"'] \arrow[r, phantom, "\Downarrow\alpha_1"] & & \arrow[d, phantom, "\Downarrow\alpha_1"] & X \\
& X \arrow[r, "C"'] \arrow[urr, bend left=20, "C"] & X \arrow[ur, "C"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\phi^2_{(C,X)}: X \arrow[rr, bend left=30, "C"] \arrow[dr, bend left=30, "C"] \arrow[dr, bend right=20, "1_X"'] \arrow[dr, phantom, "\Downarrow\alpha_2"] & \arrow[d, phantom, "\Downarrow\alpha_1"] & X, \\
& X \arrow[ur, "C"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
\psi^2_{(C,X)}: X \arrow[rr, bend left=30, "C"] \arrow[dr, "1_X"'] & \cong & X \\
& X \arrow[ur, "C"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\phi^3_{(C,X)}: X \arrow[rr, bend left=30, "C"] \arrow[dr, "C"'] & \Downarrow\alpha_1 & X, \\
& X \arrow[ur, bend left=30, "C"] \arrow[ur, bend right=20, "1_X"'] \arrow[ur, phantom, "\Downarrow\alpha_2"] &
\end{tikzcd}
\quad
\begin{tikzcd}
\psi^3_{(C,X)}: X \arrow[rr, bend left=30, "C"] \arrow[dr, "C"'] & \cong & X. \\
& X \arrow[ur, "1_X"'] &
\end{tikzcd}$$

It is now clear that the full subcategory of $\mathbf{Coalg}F$ spanned by those objects (C, X) which satisfy $\phi^i_{(C,X)} = \psi^i_{(C,X)}$ is precisely the category of $\mathcal{V}$-cocategories,

$$\mathbf{Eq}((\phi^i, \psi^i)_{i=1,2,3}) = \mathcal{V}\text{-}\mathbf{Cocat}$$

as in Definition 7.3.8. Since all categories and functors involved are accessible, $\mathcal{V}$-**Cocat** is accessible too. ☐

The fact that $\mathcal{V}$-**Cocat** is a locally presentable category is very useful for the proof of existence of various adjoints, as seen below.

Proposition 7.4.3. *Suppose $\mathcal{V}$ is a locally presentable monoidal category such that $\otimes$ preserves colimits in both entries. The forgetful functor $\tilde{U}$: $\mathcal{V}$-**Cocat** $\to$ $\mathcal{V}$-**Grph** is comonadic.*

Proof. By Proposition 7.3.12 the forgetful $\tilde{U}$ has a right adjoint, namely the cofree $\mathcal{V}$-cocategory functor $\tilde{R}$. By adjusting the arguments of Proposition 3.3.5, consider the following commutative triangle

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Cocat} \arrow[r, hook, "\iota"] \arrow[dr, "\tilde{U}"'] & \mathbf{Coalg}G \arrow[d, "\overline{V}"] \\
& \mathcal{V}\text{-}\mathbf{Grph}
\end{tikzcd}$$

where the top functor is the inclusion of the full subcategory in the functor coalgebra category as described above, and the respective forgetful functors discard the structures maps α of the coalgebras. We already know that $\mathbf{Coalg}F$ is comonadic over

$\mathcal{V}$-**Grph**, hence $\overline{V}$ creates equalizers of split pairs, so it is enough to show that the inclusion ι also creates equalizers of split pairs, since we already have $\tilde{U} \dashv \tilde{R}$. Both $\mathcal{V}$-**Cocat** and $\mathcal{V}$-**Grph** are locally presentable categories so in particular complete, and it is easy to see that ι preserves and reflects, thus creates, all limits. Hence $\tilde{U}$ satisfy the conditions of Precise Monadicity Theorem and the result follows. □

PROPOSITION 7.4.4. *Suppose that $\mathcal{V}$ is a locally presentable symmetric monoidal closed category. Then the category of $\mathcal{V}$-cocategories is symmetric monoidal closed as well.*

PROOF. The symmetric monoidal structure of $\mathcal{V}$-**Cocat** was described in the previous section and is given by a functor of two variables

$$-\otimes- : \mathcal{V}\text{-}\mathbf{Cocat}^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{Cocat} \to \mathcal{V}\text{-}\mathbf{Cocat}.$$

The functor $(-\otimes\mathcal{D}_Y)$ for a fixed $\mathcal{V}$-cocategory $\mathcal{D}_Y$ evidently has a right adjoint: the following commutative diagram

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Cocat} \arrow[r, "(-\otimes\mathcal{D}_Y)"] \arrow[d, "\tilde{U}"'] & \mathcal{V}\text{-}\mathbf{Cocat} \arrow[d, "\tilde{U}"] \\
\mathcal{V}\text{-}\mathbf{Grph} \arrow[r, "(-\otimes\tilde{U}\mathcal{D}_Y)"'] & \mathcal{V}\text{-}\mathbf{Grph}
\end{tikzcd}$$

shows it is cocontinuous, since the comonadic $\tilde{U}$ creates all colimits and the bottom arrow preserves them by the adjunction $(-\otimes\mathcal{G}_Y) \dashv {}^{g}\mathrm{Hom}(\mathcal{G}_Y, -)$ for any $\mathcal{V}$-graph $\mathcal{G}_Y$ (Proposition 7.2.3). Also $\mathcal{V}$-**Cocat** is a locally presentable category, hence cocomplete with a small dense subcategory. Thus by Theorem 3.0.1 for example, we have an adjunction

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Cocat} \arrow[r, shift left, "-\otimes\mathcal{D}_Y"] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathbf{Cocat} \arrow[l, shift left, "{}^{g}\mathrm{HOM}(\mathcal{D}_Y,-)"]
\end{tikzcd} \tag{7.35}$$

which exhibits the uniquely induced bifunctor

$${}^{g}\mathrm{HOM} : \mathcal{V}\text{-}\mathbf{Cocat}^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{Cocat} \longrightarrow \mathcal{V}\text{-}\mathbf{Cocat}$$

as the internal hom of $\mathcal{V}$-**Cocat**. □

At this point, we possess all the necessary tools in order to show the existence of an adjoint of the action K^{op} as outlined earlier, as well as demonstrate the enrichment of $\mathcal{V}$-categories in $\mathcal{V}$-cocategories.

PROPOSITION 7.4.5. *The functor $K^{\mathrm{op}} : \mathcal{V}\text{-}\mathbf{Cocat} \times \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}}$ has a parametrized adjoint*

$$T : \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{V}\text{-}\mathbf{Cocat}, \tag{7.36}$$

given by adjunctions $K(-,\mathcal{B}_Y)^{\mathrm{op}} \dashv T(-,\mathcal{B}_Y)$ for every $\mathcal{V}$-category $\mathcal{B}_Y$.

PROOF. By Proposition 7.4.2, the domain $\mathcal{V}$-**Cocat** of $K(-,\mathcal{B})^{\mathrm{op}}$ is locally presentable, hence cocomplete with a small dense subcategory, namely the presentable objects. Now consider the following diagram

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Cocat} \arrow[r, "K(-{,}\mathcal{B}_Y)^{\mathrm{op}}"] \arrow[d, "\tilde{U}"'] & \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \arrow[d, "\tilde{S}"] \\
\mathcal{V}\text{-}\mathbf{Grph} \arrow[r, "{}^{g}\mathrm{Hom}(-{,}\tilde{S}\mathcal{B}_Y)^{\mathrm{op}}"'] & \mathcal{V}\text{-}\mathbf{Grph}^{\mathrm{op}}
\end{tikzcd}$$

which commutes by definition of K, and the left and right legs create all colimits by Propositions 7.3.5 and 7.4.3. The bottom arrow preserves all colimits by ${}^{g}\mathrm{Hom}(-,\mathcal{G}_Y)^{\mathrm{op}} \dashv {}^{g}\mathrm{Hom}(-,\mathcal{G}_Y)$ for any internal hom functor in a monoidal closed category, thus the functor $K(-,\mathcal{B})^{\mathrm{op}}$ is cocontinuous. By Kelly's adjoint functor theorem 3.0.1, there are adjunctions

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Cocat} \arrow[r, shift left, "K(-{,}\mathcal{B}_Y)^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \arrow[l, shift left, "T(-{,}\mathcal{B}_Y)"]
\end{tikzcd}$$

for all $\mathcal{V}$-categories $\mathcal{B}_Y$. This suffices to uniquely make T into a functor of two variables (7.36), which is by definition the parametrized adjoint of K^{op}. $\square$

The functor T, which is a generalization of the universal measuring comonoid functor P (6.3) in the 'many-object' context, is called *generalized Sweedler hom.* Morever, it can also be deduced that the functor $K(\mathcal{C}_X,-)^{\mathrm{op}}$ has a right adjoint for any $\mathcal{V}$-cocategory $\mathcal{C}_X$, or equivalently its opposite functor has a left adjoint. The following diagram

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Cat} \arrow[r, "K(\mathcal{C}_X{,}-)"] \arrow[d, "\tilde{S}"'] & \mathcal{V}\text{-}\mathbf{Cat} \arrow[d, "\tilde{S}"] \\
\mathcal{V}\text{-}\mathbf{Grph} \arrow[r, "{}^{g}\mathrm{Hom}(\tilde{U}\mathcal{C}_X{,}-)"'] & \mathcal{V}\text{-}\mathbf{Grph}
\end{tikzcd}$$

commutes, where $\tilde{S}$ is the monadic forgetful functor and the locally presentable category $\mathcal{V}$-**Cat** has all coequalizers. Thus by Dubuc's Adjoint Triangle Theorem in [**Dub68**], the existence of a left adjoint $(\mathcal{C}_X \otimes -) \dashv {}^{g}\mathrm{Hom}(\mathcal{C}_X,-)$ for any (underlying) $\mathcal{V}$-graph $\mathcal{C}_X$ in the symmetric monoidal closed $\mathcal{V}$-**Grph** implies the existence of a left adjoint $(\mathcal{C}_X \triangleright -)$ of the top functor. The induced functor of two variables

$$\triangleright : \mathcal{V}\text{-}\mathbf{Cocat} \times \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}$$

is called the *generalized Sweedler product*, since it is an extension of the respective functor (6.7).

The conditions of Corollaries 4.3.4 and 4.3.5 are now satisfied, for the symmetric monoidal category closed $\mathcal{V}$-**Cocat** which acts on the opposite of the category $\mathcal{V}$-**Cat** via the action K^{op}.

THEOREM 7.4.6. *Suppose $\mathcal{V}$ is a symmetric monoidal closed category which is locally presentable, and T is the generalized Sweedler hom functor.*

(1) The opposite category of $\mathcal{V}$-categories $\mathcal{V}$-$\mathbf{Cat}^{\mathrm{op}}$ is enriched in the category of $\mathcal{V}$-cocategories $\mathcal{V}$-$\mathbf{Cocat}$, with hom-objects

$$\mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}}(\mathcal{A}_X, \mathcal{B}_Y) = T(\mathcal{B}_Y, \mathcal{A}_X)$$

where the ($\mathcal{V}$-$\mathbf{Cocat}$)-enriched category with underlying category $\mathcal{V}$-$\mathbf{Cat}^{\mathrm{op}}$ is denoted by the same name.

(2) The category $\mathcal{V}$-$\mathbf{Cat}$ is a tensored and cotensored ($\mathcal{V}$-$\mathbf{Cocat}$)-enriched category, with hom-objects

$$\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A}_X, \mathcal{B}_Y) = T(\mathcal{A}_X, \mathcal{B}_Y),$$

cotensor product $K(\mathcal{C}, \mathcal{B})_{YZ}$ and tensor product $\mathcal{C}_Z \triangleright \mathcal{A}_X$, for any $\mathcal{V}$-cocategory $\mathcal{C}_Z$ and any $\mathcal{V}$-categories $\mathcal{A}_X, \mathcal{B}_Y$.

7.5. Graphs, categories and cocategories as (op)fibrations

This section presents a different approach to establishing the enrichment of $\mathcal{V}$-categories in $\mathcal{V}$-cocategories. In the section above, the result follows from the existence of an adjoint T which constitues the enriched hom-functor, as a straightforward application of an adjoint functor theorem (Proposition 7.4.5). This is possible basically due to local presentability of $\mathcal{V}$-**Cocat**. However, the categories $\mathcal{V}$-**Grph**, $\mathcal{V}$-**Cat** and $\mathcal{V}$-**Cocat** also have a structure which places them in a fibrational context, allowing the application of the theory of fibrations of Chapter 5. In particular, we will show how we can alternatively obtain this adjoint T as an application of Theorem 5.3.7 regarding adjunctions between fibrations.

First of all, we are going to exhibit in detail the fibrational structure of the categories involved, a well-known fact at least for $\mathcal{V}$-categories over sets. We initially assume that $\mathcal{V}$ is a cocomplete monoidal category, such that the tensor product preserves colimits on both sides.

PROPOSITION 7.5.1. *The category $\mathcal{V}$-**Grph** of small $\mathcal{V}$-graphs is a bifibration over* **Set**.

PROOF. Due to the correspondence between fibrations and pseudofunctors studied in Theorem 5.2.1, it is enough to define certain indexed categories, *i.e.* pseudofunctors $\mathscr{M} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$ and $\mathscr{F} : \mathbf{Set} \to \mathbf{Cat}$ which give rise to a fibration and opfibration with total category isomorphic to $\mathcal{V}$-**Grph**, via the Grothendieck construction.

Define the pseudofunctor $\mathscr{M}$ as follows:

$$\begin{tikzcd}
\mathscr{M} : \mathbf{Set}^{\mathrm{op}} \arrow[r] & \mathbf{Cat} \\
X \arrow[r, mapsto, dotted] \arrow[d, "f"'] & \mathcal{V}\text{-}\mathbf{Mat}(X,X) \\
Y \arrow[r, mapsto, dotted] & \mathcal{V}\text{-}\mathbf{Mat}(Y,Y), \arrow[u, "\mathscr{M}f"']
\end{tikzcd} \tag{7.37}$$

where the functor $\mathscr{M}f$ is given by the mapping

$$\begin{tikzcd}(Y \arrow[r, "H", "|" marking] & Y) \arrow[r, dotted, mapsto] & (X \arrow[r, "f_*", "|" marking] & Y \arrow[r, "H", "|" marking] & Y \arrow[r, "f^*", "|" marking] & X)\end{tikzcd}$$

on objects and

$$\begin{tikzcd}(Y \arrow[r, bend left, "H", "|" marking, ""{name=U, below}] \arrow[r, bend right, "H'"', "|" marking, ""{name=D}] & Y) \arrow[r, dotted, mapsto] & (X \arrow[r, "f_*", "|" marking] & Y \arrow[r, bend left, "H", "|" marking, ""{name=U2, below}] \arrow[r, bend right, "H'"', "|" marking, ""{name=D2}] & Y \arrow[r, "f^*", "|" marking] & X) \arrow[from=U, to=D, Rightarrow, "\sigma"] \arrow[from=U2, to=D2, Rightarrow, "\sigma"]\end{tikzcd}$$

on arrows. In other words, $\mathscr{M}f = (f^* \circ - \circ f_*)$ is the functor 'pre-composition with f_* and post-composition with f^*', where the induced $\mathcal{V}$-matrices f_* and f^* are defined as in (7.11). In terms of components, the family $\{H(y',y)\}_{y,y'\in Y}$ of objects in $\mathcal{V}$ which defines the $\mathcal{V}$-matrix H, is mapped to the family

$$\{\big((\mathscr{M}f)H\big)(x',x)\}_{x,x'\in X} = \{I \otimes H(fx',fx) \otimes I\}_{fx,fx'\in Y}$$

and the family $\{\sigma_{y',y} : H(y',y) \to H'(y',y)\}_{y',y}$ of arrows in $\mathcal{V}$ which define the 2-cell σ, is mapped to the family

$$\big((\mathscr{M}f)\sigma\big)_{x',x} : I \otimes H(fx',fx) \otimes I \xrightarrow{1\otimes\sigma_{fx',fx}\otimes 1} I \otimes H'(fx',fx) \otimes I$$

for all $x',x \in X$.

In order to show that the above data determine a pseudofunctor $\mathscr{M}$, we need the existence of certain natural isomorphisms satisfying coherence conditions as in Definition 2.1.3. For every triple of sets X,Y,Z, there is a natural isomorphism δ with components

$$\begin{tikzcd} & \mathcal{V}\text{-}\mathbf{Mat}(Y,Y) \arrow[dr, "\mathscr{M}f"] \arrow[d, phantom, "\Downarrow\delta^{g,f}"] & \\ \mathcal{V}\text{-}\mathbf{Mat}(Z,Z) \arrow[ur, "\mathscr{M}g"] \arrow[rr, bend right, "\mathscr{M}(g\circ f)"'] & {} & \mathcal{V}\text{-}\mathbf{Mat}(X,X)\end{tikzcd}$$

for any $f : X \to Y$ and $g : Y \to Z$, satisfying the commutativity of (2.5). Explicitely, each $\delta^{g,f}$ has components, for each $\mathcal{V}$-matrix $J : Z \nrightarrow Z$, the invertible arrows

$$\delta^{g,f}_J : (\mathscr{M}f \circ \mathscr{M}g)J \xrightarrow{\sim} \mathscr{M}(g \circ f)J$$

in $\mathcal{V}$-$\mathbf{Mat}(X,X)$ which are the composite 2-cells

$$\begin{tikzcd} X \arrow[r, "f_*", "|" marking] \arrow[drr, bend right, "(gf)_*"', "|" marking] & Y \arrow[r, "g_*", "|" marking] \arrow[d, phantom, "\Downarrow\zeta^{g,f}"] & Z \arrow[r, "J", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow 1_J"] & Z \arrow[r, "g^*", "|" marking] \arrow[d, equal] & Y \arrow[r, "f^*", "|" marking] \arrow[d, phantom, "\Downarrow\xi^{g,f}"] & X \\ & & Z \arrow[r, "J"', "|" marking] & Z \arrow[urr, bend right, "(gf)^*"', "|" marking] & & \end{tikzcd} \tag{7.38}$$

where the isomorphisms ζ and ξ are defined in Lemma 7.1.3. This 2-isomorphism

$$\delta^{g,f} = \xi^{g,f} * 1_J * \zeta^{g,f}$$

is given by the family of invertible arrows

$$(\delta^{g,f}_J)_{x',x} : I \otimes I \otimes J(gfx',gfx) \otimes I \otimes I \xrightarrow{r_I \otimes 1 \otimes r_I} I \otimes J(gfx',gfx) \otimes I$$

in $\mathcal{V}$, and the coherence axiom is satisfied by the properties of ξ and ζ (see Lemma 7.1.4). Moreover, for any set X there is a natural isomorphism γ with components the natural transformations

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Mat}(X,X) \arrow[r, bend left, "\mathbf{1}_{\mathcal{V}\text{-}\mathbf{Mat}(X,X)}", ""{name=U, below}] \arrow[r, bend right, "\mathscr{M}(\mathrm{id}_X)"', ""{name=D, above}] & \mathcal{V}\text{-}\mathbf{Mat}(X,X) \arrow[from=U, to=D, phantom, "\Downarrow \gamma^X"]
\end{tikzcd}$$

where id_X is the identity function on any set X and $\mathbf{1}$ is the identity functor. Explicitly, γ^X has as components invertible arrows in $\mathcal{V}\text{-}\mathbf{Mat}(X,X)$

$$\gamma^X_G : \mathbf{1}_{\mathcal{V}\text{-}\mathbf{Mat}(X,X)}G \xrightarrow{\sim} \mathscr{M}(\mathrm{id}_X)G$$

for any $\mathcal{V}$-matrix $G : X \relbar\joinrel\mapstochar\joinrel\rightarrow X$, which are the composite 2-cells

$$\begin{tikzcd}
\gamma^X_G :\ X \arrow[rrr, bend left=30, "G", "\shortmid" marking] \arrow[dr, "(\mathrm{id}_X)_*"', "\shortmid" marking] & \Downarrow \rho_G^{-1} & & X. \\
& X \arrow[r, "G"', "\shortmid" marking] \arrow[urr, bend left, "G", "\shortmid" marking] & X \arrow[ur, bend right, "(\mathrm{id}_X)^*"', "\shortmid" marking] \arrow[u, phantom, "\Downarrow \lambda_G^{-1}"] &
\end{tikzcd} \tag{7.39}$$

By recalling that $(\mathrm{id}_X)^* = (\mathrm{id}_X)_* = 1_X$ by (7.15), this isomorphism

$$\gamma^X_G = (\lambda_G^{-1} 1_X) \cdot (\rho_G^{-1})$$

consists of the family of invertible arrows

$$(\gamma^X_G)_{x',x} : G(x',x) \xrightarrow{l^{-1}} I \otimes G(x',x) \xrightarrow{1 \otimes r^{-1}} I \otimes G(x',x) \otimes I$$

in $\mathcal{V}$. It can be verified that the axioms (2.6) are satisfied, therefore $\mathscr{M}$ is a pseudofunctor.

The Grothendieck category $\mathfrak{G}\mathscr{M}$ for this pseudofunctor has as objects pairs (G,X), where X is a set and G is an object in the category $\mathscr{M}X = \mathcal{V}\text{-}\mathbf{Mat}(X,X)$, and as arrows $(\phi, f) : (G,X) \to (H,Y)$ pairs

$$\begin{cases} G \xrightarrow{\phi} (\mathscr{M}f)H & \text{in } \mathscr{M}X \\ X \xrightarrow{f} Y & \text{in } \mathbf{Set} \end{cases} = \begin{cases} G \xrightarrow{\phi} f^* \circ H \circ f_* & \text{in } \mathcal{V}\text{-}\mathbf{Mat}(X,X) \\ X \xrightarrow{f} Y & \text{in } \mathbf{Set}. \end{cases}$$

By Definition 7.2.1, this category is isomorphic to $\mathcal{V}$-**Grph**, in the sense that there is a one-to-one correspondence between the objects, which can actually be identified, and the hom-sets. Thus $\mathscr{M}$ gives rise to a fibration $P_{\mathscr{M}} : \mathfrak{G}\mathscr{M} \to \mathbf{Set}$ which is isomorphic to the forgetful functor $Q : \mathcal{V}\text{-}\mathbf{Grph} \to \mathbf{Set}$, *i.e.*

$$\begin{tikzcd}
\mathfrak{G}\mathscr{M} \arrow[rr, "\cong"] \arrow[dr, "P_{\mathscr{M}}"'] & & \mathcal{V}\text{-}\mathbf{Grph} \arrow[dl, "Q"] \\
& \mathbf{Set} &
\end{tikzcd}$$

commutes by definition of the functors involved, hence Q is a fibration.

Now, define a covariant indexed category $\mathscr{F}$ as follows:

$$\begin{tikzcd}
\mathscr{F} : \mathbf{Set} \arrow[r] & \mathbf{Cat} \\
X \arrow[r, mapsto, dotted] \arrow[d, "f"'] & \mathcal{V}\text{-}\mathbf{Mat}(X,X) \arrow[d, "\mathscr{F}f"] \\
Y \arrow[r, mapsto, dotted] & \mathcal{V}\text{-}\mathbf{Mat}(Y,Y)
\end{tikzcd} \tag{7.40}$$

where the mapping on objects is the same as for the pseudofunctor $\mathscr{M}$ above, and $\mathscr{F}f$ is the mapping

$$\begin{tikzcd}
(X \arrow[r, bend left, "G", "|"{marking}] \arrow[r, bend right, "G'"', "|"{marking}] \arrow[r, phantom, "\Downarrow\tau"] & X) \arrow[r, mapsto, dotted] & (Y \arrow[r, "f^*", "|"{marking}] & X \arrow[r, bend left, "G", "|"{marking}] \arrow[r, bend right, "G'"', "|"{marking}] \arrow[r, phantom, "\Downarrow\tau"] & X \arrow[r, "f_*", "|"{marking}] & Y)
\end{tikzcd}$$

on objects and on arrows, *i.e.* $\mathscr{F}f = (f_* \circ - \circ f^*)$. In terms of components, the family $\{G(x',x)\}_{x,x'\in X}$ of objects in $\mathcal{V}$ which define the $\mathcal{V}$-matrix G, is mapped to the family

$$\{\mathscr{F}f(G)(y',y)\}_{y,y'\in Y} = \{\sum_{\substack{fx'=y'\\ fx=y}} I\otimes G(x',x)\otimes I\}_{y,y'\in Y}$$

and the family $\{\tau_{x,x'} : G(x',x) \to G'(x',x)\}_{x,x'}$ of arrows in $\mathcal{V}$ which defines the 2-cell τ, is mapped to the family of arrows

$$\mathscr{F}f(\tau)_{y',y} : \sum I\otimes G(x',x)\otimes I \xrightarrow{\sum 1\otimes\sigma_{x',x}\otimes 1} \sum I\otimes G'(x',x)\otimes I,$$

where the sums are over x, x' such that $fx' = y'$, $fx = fy$, based on the computations of Section 7.1. Again, there exist natural isomorphisms δ, γ with components

$$\delta^{g,f} : \mathscr{F}g \circ \mathscr{F}f \Rightarrow \mathscr{F}(g\circ f) : \mathcal{V}\text{-}\mathbf{Mat}(X,X) \to \mathcal{V}\text{-}\mathbf{Mat}(Z,Z)$$

$$\gamma^X : \mathbf{1}_{\mathcal{V}\text{-}\mathbf{Mat}(X,X)} \Rightarrow \mathscr{F}(\mathrm{id}_X) : \mathcal{V}\text{-}\mathbf{Mat}(X,X) \to \mathcal{V}\text{-}\mathbf{Mat}(X,X)$$

which satisfy the properties (2.5) and (2.6) from the definition of a pseudofunctor. In fact, they are essentially the same as in the case of $\mathscr{M}$, *i.e.* δ now has components the invertible composite 2-cells

$$\begin{tikzcd}[column sep=small]
 & & Y \arrow[dr, bend left, "f^*", "|"{marking}] & & & Y \arrow[dr, bend left, "g_*", "|"{marking}] & \\
\delta^{f,g}_G : & Z \arrow[ur, bend left, "g^*", "|"{marking}] \arrow[rr, bend right, "(gf)^*"', "|"{marking}] & \Downarrow\xi^{g,f} & X \arrow[r, bend left=40, "G", "|"{marking}] \arrow[r, bend right=40, "G"', "|"{marking}] \arrow[r, phantom, "\Downarrow 1_G"] & X \arrow[ur, bend left, "f_*", "|"{marking}] \arrow[rr, bend right, "(gf)_*"', "|"{marking}] & \Downarrow\zeta^{g,f} & Z,
\end{tikzcd} \tag{7.41}$$

which are formed like (7.38) but composing with ζ and ξ in the reverse order, and γ is the same as in (7.39). Therefore $\mathscr{F}$ is a pseudofunctor, and by Theorem 5.2.2 it gives rise to an opfibration

$$U_{\mathscr{F}} : \mathfrak{G}\mathscr{F} \to \mathbf{Set}.$$

The Grothendieck category in this case coincides with the isomorphic characterization of $\mathcal{V}$-**Grph** in Definition 7.2.1, with the 'second version' form of arrows. Hence $U_{\mathscr{F}}$ is again isomorphic to the forgetful $Q : \mathcal{V}\text{-}\mathbf{Grph} \to \mathbf{Set}$, endowing it with the structure of an opfibration. Thus $\mathcal{V}$-**Grph** is a bifibration over **Set**. □

Notice that we could immediately deduce that the fibration Q is a bifibration via Remark 5.1.1. The reindexing functor $\mathscr{M}f = f_* \circ \text{-} \circ f^*$ does have a left adjoint $f^* \circ \text{-} \circ f_*$, by the natural bijection between 2-cells of the form (7.14). We explicitly constructed the opfibration above in order to employ it later.

An immediate consequence of viewing the category of $\mathcal{V}$-graphs as a bifibration is that we can discuss fibred and opfibred limits and colimits (see Section 5.3).

Corollary 7.5.2. *The bifibration $Q : \mathcal{V}\text{-}\mathbf{Grph} \to \mathbf{Set}$ has all fibred limits and all opfibred colimits, when $\mathcal{V}$ is complete and cocomplete respectively.*

Proof. By Corollary 5.3.11 and its dual, an (op)fibration with (co)complete base category has all (op)fibred (co)limits if and only if the total category has all (co)limits and the fibration strictly preserves them. In this case, the base of the bifibration is the complete and cocomplete category of sets, and since the total category $\mathcal{V}$-**Grph** is (co)complete when $\mathcal{V}$ is and the forgetful functor Q preserves limits and colimits 'on the nose' by construction, the result follows. $\square$

Moreover, by Proposition 5.3.9 we can now deduce that the reindexing functors $(f^* \circ \text{-} \circ f_*)$ and $(f_* \circ \text{-} \circ f^*)$ preserve all limits and colimits between the complete and cocomplete fibres $\mathcal{V}\text{-}\mathbf{Mat}(X, X)$. The latter was evident by Proposition 7.1.1.

The construction of the two pseudofunctors $\mathscr{M}$ and $\mathscr{F}$ which exhibit $\mathcal{V}$-**Grph** as a bifibred category over **Set** clarify the way in which the categories $\mathcal{V}$-**Cat** and $\mathcal{V}$-**Cocat** are themselves fibred and opfibred respectively over **Set**.

Proposition 7.5.3. *The category $\mathcal{V}$-**Cat** of small $\mathcal{V}$-categories is a fibration over* **Set**.

Proof. Similarly to the above proof, it will suffice to construct an indexed category $\mathscr{L} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$ such that the category $\mathcal{V}$-**Cat** is isomorphic to the Grothendieck category of the fibration $P_{\mathscr{L}}$.

Define the pseudofunctor $\mathscr{L}$ as follows: a set X is mapped to the category

$$\mathscr{L}X = \mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(X, X))$$

of monoids of the monoidal category of endoarrows $(\mathcal{V}\text{-}\mathbf{Mat}(X, X), \circ, 1_X)$, and a function between sets $f : X \to Y$ is mapped contravariantly to the functor

$$\begin{tikzcd}
\mathscr{L}f :\ \mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(Y,Y)) \arrow[r] & \mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X)) \\
(B,\mu,\eta) \arrow[r, mapsto, dotted] \arrow[d, "\sigma"'] & (f^*Bf_*, \mu', \eta') \arrow[d, "f^*\sigma f_*"] \\
(E,\mu,\eta) \arrow[r, mapsto, dotted] & (f^*Ef_*, \mu', \eta').
\end{tikzcd}$$

As described in detail in Lemma 7.3.2, the induced monoid f^*Bf_* has multiplication $\mu' = f^*[\mu \cdot (B\check{\epsilon}B)]f_*$ and unit $\eta' = (f^*\eta f_*) \cdot \check{\eta}$, where $\check{\epsilon}$ and $\check{\eta}$ are the counit and unit of the adjunction $f_* \dashv f^*$, and also $(f^*\sigma f_*)$ can easily be checked to commute with the appropriate monoid structure maps. Evidently, this functor $\mathscr{L}f$ is just $\mathscr{M}f = (f^* \circ - \circ f_*)$ defined in (7.37), restricted between the respective categories of monoids.

Again, we need to identify natural transformations γ and δ satisfying certain coherence axioms, for $\mathscr{L}$ to be a pseudofunctor according to Definition 2.1.3. In this case, these will have components natural isomorphisms

$$\delta^{g,f}:\mathscr{L}f\circ\mathscr{L}g\Rightarrow\mathscr{L}(g\circ f):\mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(Z,Z))\to\mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X))$$
$$\gamma^X:\mathbf{1}_{\mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X))}\Rightarrow\mathscr{L}(\mathrm{id}_X):\mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X))\to\mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X))$$

for $X\xrightarrow{f}Y\xrightarrow{g}Z$ in **Set**, where id_X is the identity function. We can define $\delta^{g,f}$ and γ^X to be natural transformations given exactly as the ones for the pseudofunctor $\mathscr{M}$, as in (7.38) and (7.39). The domains and codomains of these composite 2-cells are by default monoids in the appropriate endoarrow hom-categories of $\mathcal{V}$-matrices, and it can be verified via computations that the invertible arrows $\delta^{g,f}_J$ and γ^X_A for monoids $J:Z\looparrowright Z$ and $A:X\looparrowright X$ commute with the respective multiplications and units of the monoids involved. Moreover, the diagrams (2.5, 2.6) commute because they do for all $\mathcal{V}$-matrices, by pseudofunctoriality of $\mathscr{M}$. Therefore $\mathscr{L}$ is indeed a pseudofunctor.

If we construct the Grothendieck category for $\mathscr{L}:\mathbf{Set}^{\mathrm{op}}\to\mathbf{Cat}$, with objects pairs (A,X) where $A\in\mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X))$ for a set X, and arrows $(A,X)\to(B,Y)$ pairs

$$\begin{cases} A\xrightarrow{\phi}f^*Bf_* & \text{in } \mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X))\\ X\xrightarrow{f}Y & \text{in } \mathbf{Set},\end{cases}$$

it is evident by Lemma 7.3.3 that $\mathfrak{G}\mathscr{L}\cong\mathcal{V}\text{-}\mathbf{Cat}$. Moreover, both forgetful functors to **Set** have the same effect on objects and on arrows, namely separating the set-part of the data. Hence

$$P:\mathcal{V}\text{-}\mathbf{Cat}\longrightarrow\mathbf{Set}$$

is a fibration, isomorphic to $P_{\mathscr{L}}$ arising via the Grothendieck construction. □

Corollary 7.5.4. *The fibration* $P:\mathcal{V}\text{-}\mathbf{Cat}\to\mathbf{Set}$ *has all fibred limits when* $\mathcal{V}$ *is complete.*

Proof. Since the fibration P has as base category the complete category **Set**, in order for P to have all fibred limits it suffices for $\mathcal{V}$-**Cat** to be complete and for the forgetful P to preserve all limits strictly, again by Corollary 5.3.11. Corollary 7.3.6 ensures that $\mathcal{V}$-**Cat** has all limits and since a limit of $\mathcal{V}$-graphs has as underlying set precisely the limit of the sets, the result follows. □

Finally, in order to establish that $\mathcal{V}$-**Cocat** is opfibred over **Set**, we are going to use the pseudofunctor $\mathscr{F}$ defined as in (7.40).

Proposition 7.5.5. *The category* $\mathcal{V}$-**Cocat** *of small* $\mathcal{V}$*-cocategories is an opfibration over* **Set**.

Proof. We will once more construct a covariant indexed category $\mathscr{K}:\mathbf{Set}\to\mathbf{Cat}$, for which the Grothendieck construction gives a category isomorphic to $\mathcal{V}$-**Cocat** along with the forgetful functor to sets, mapping every $\mathcal{V}$-cocategory to its set of objects.

Define $\mathscr{K}$ as follows: a set X is mapped to the category of comonoids in the monoidal category $(\mathcal{V}\text{-}\mathbf{Mat}(X,X),\circ,1_X)$, and a function $f:X\to Y$ is mapped to the functor

$$\mathscr{K}f:\mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X))\to\mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(Y,Y))$$

which precomposes with f^* and postcomposes with f_* both $\mathcal{V}$-matrices and 2-cells. Explicitly, the functor $\mathscr{K}f$ is defined on objects by

$$(C,\Delta,\epsilon)\mapsto(f_*Cf^*,\Delta',\epsilon')$$

where $\Delta'=f_*[(C\check{\eta}C)\cdot\Delta]f^*$ and $\epsilon'=\check{\epsilon}\cdot(f_*\epsilon f^*)$ as described in detail in Lemma 7.3.10, and on arrows

$$(C\overset{\tau}{\Longrightarrow}D)\mapsto(f_*Cf^*\overset{f_*\tau f^*}{\Longrightarrow}f_*Df^*)$$

where $f_*\tau f^*$ can easily be verified to commute with the respective counits and comultiplications. Again, notice that $\mathscr{K}f$ is in fact the restriction of $\mathscr{F}f$ (7.40) to the categories of comonoids. The above mappings define a pseudofunctor $\mathscr{K}$, since the two natural transformations γ and δ in this case, with components natural isomorphisms

$$\begin{tikzcd}
\mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X)) \arrow[r, bend left, "\mathscr{K}g\circ\mathscr{K}f"] \arrow[r, bend right, "\mathscr{K}(g\circ f)"'] \arrow[r, phantom, "\Downarrow\delta^{g,f}"] & \mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(Z,Z))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X)) \arrow[r, bend left, "\mathbf{1}_{\mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X))}"] \arrow[r, bend right, "\mathscr{K}(\mathrm{id}_X)"'] \arrow[r, phantom, "\Downarrow\gamma^X"] & \mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X))
\end{tikzcd}$$

consist of the invertible composite 2-cells as in (7.41) and (7.39) for the pseudofunctor $\mathscr{F}$. Their domains and codomains are by construction comonoids in the appropriate categories of $\mathcal{V}$-matrices, and they satisfy the properties of comonoid morphisms. Hence δ and γ are well-defined, and the diagrams (2.5, 2.6) commute by pseudofunctoriality of $\mathscr{F}$.

The Grothendieck category $\mathfrak{G}\mathscr{K}$ for this pseudofunctor has as objects pairs (C,X) where $C\in\mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X))$ for a set X, and as arrows $(C,X)\to(D,Y)$ pairs

$$\begin{cases} f_*Cf^*\xrightarrow{\psi}D & \text{in } \mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(Y,Y)) \\ X\xrightarrow{f}Y & \text{in } \mathbf{Set}. \end{cases}$$

By Lemma 7.3.11, this is isomorphic to the category $\mathcal{V}$-**Cocat**. As a result, the forgetful functor

$$W:\mathcal{V}\text{-}\mathbf{Cocat}\longrightarrow\mathbf{Set}$$

is an opfibration, isomorphic to $U_{\mathscr{K}}$ arising via the Grothendieck construction since they have the same effect on objects and on morphisms. $\square$

Corollary 7.5.6. *The opfibration* $W : \mathcal{V}\text{-}\mathbf{Cocat} \to \mathbf{Set}$ *has all opfibred colimits, when* $\mathcal{V}$ *is locally presentable.*

Proof. The base category of this opfibration is again the cocomplete category of sets, and also the total category $\mathcal{V}$-**Cocat** has all colimits which by construction are strictly preserved by the forgetful functor, see Proposition 7.3.13. Therefore by the dual of Corollary 5.3.11, the opfibration W has all opfibred colimits. ☐

Remark. For the definition of the two pseudofunctors which give rise to $\mathcal{V}$-categories and $\mathcal{V}$-cocategories as their Grothendieck categories, the functors $\mathscr{M}f = f^* \circ - \circ f_*$ and $\mathscr{F}f = f_* \circ - \circ f^*$ as in (7.37), (7.40) were employed. Lemmas 7.3.2 and 7.3.10 suggested already that these two functors may 'lift' to the respective categories of monoids and comonoids. This can be further clarified if we observe that both these functors have the structure of a lax/colax monoidal functor respectively, between the monoidal hom-categories of endomorphisms in $\mathcal{V}$-**Mat**. For example, for a function $f : X \to Y$ and two $\mathcal{V}$-matrices $B, B' : Y \,\mathrlap{\to}{\,\shortmid}\, Y$, the lax monoidal structure map

$$\phi_{B,B'} : f^* \circ B \circ f_* \circ f^* \circ B' \circ f_* \Rightarrow f^* \circ B \circ B' \circ f_*$$

of $\mathscr{M}f$ has components the composite 2-cells

$$\begin{tikzcd}
 & & & X \arrow[dr, bend left, "f_*", "\shortmid"{marking}] & & & \\
X \arrow[r, "f_*", "\shortmid"{marking}] & Y \arrow[r, "B'", "\shortmid"{marking}] & Y \arrow[ur, bend left, "f^*", "\shortmid"{marking}] \arrow[rr, "1_Y"', "\shortmid"{marking}] \arrow[rrr, bend right=35, "B"', "\shortmid"{marking}] & \Downarrow\breve{\varepsilon} & Y \arrow[r, "B", "\shortmid"{marking}] & Y \arrow[r, "f^*", "\shortmid"{marking}] & X. \\
 & & & & \Downarrow\overset{\varrho}{\cong} & &
\end{tikzcd}$$

Similarly for ϕ_0, and also for the functor $\mathscr{F}f$. Therefore, these lax/colax monoidal functors induce functors between the categories of monoids and comonoids of $\mathcal{V}$-$\mathbf{Mat}(X, X)$ in a straightforward way, as in (3.16).

The fibre categories for the bifibration, fibration and opfibration Q, P and W respectively are

$$\begin{aligned}
\mathcal{V}\text{-}\mathbf{Grph}_X &= \mathcal{V}\text{-}\mathbf{Mat}(X, X) \\
\mathcal{V}\text{-}\mathbf{Cat}_X &= \mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(X, X)) \\
\mathcal{V}\text{-}\mathbf{Cocat}_X &= \mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(X, X)).
\end{aligned}$$

Notice that, even if the total categories of $\mathcal{V}$-categories and $\mathcal{V}$-cocategories have a monoidal structure as seen in Section 7.3, their fibres are not monoidal categories, since the monoidal $(\mathcal{V}\text{-}\mathbf{Mat}(X, X), \circ, 1_X)$ fails to be symmetric or braided.

We now turn back to the primary question of the existence of a right adjoint for the functor

$$K(-, \mathcal{B}_Y)^{\mathrm{op}} : \mathcal{V}\text{-}\mathbf{Cocat} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}}$$

coming from K (7.32), which in reality is the internal hom functor ${}^{g}\mathrm{Hom}$ of the monoidal closed category of small $\mathcal{V}$-graphs restricted on $\mathcal{V}$-cocategories and $\mathcal{V}$-categories, as explained in detail in Section 7.4. The plan is to now obtain this adjoint via the theory of fibrations and in particular from Theorem 5.3.7, and then the enrichment of $\mathcal{V}$-categories in $\mathcal{V}$-cocategories will follow in the exact same way as in the end of previous section.

Lemma 7.5.7. *The diagram*

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Cocat} \arrow[r, "K(-{,}(B{,}Y))^{\mathrm{op}}"] \arrow[d, "W"'] & \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \arrow[d, "P^{\mathrm{op}}"] \\
\mathbf{Set} \arrow[r, "Y^{(-)^{\mathrm{op}}}"'] & \mathbf{Set}^{\mathrm{op}}
\end{tikzcd}$$

exhibits $\left(K(-,(B,Y))^{\mathrm{op}}, Y^{(-)^{\mathrm{op}}}\right)$ *as an opfibred 1-cell between the opfibrations* W *and* P^{op}*.*

Proof. It is straightforward to verify that the above diagram commutes, since the set of objects of the internal hom is by construction the exponential of the underlying sets of objects of the $\mathcal{V}$-cocategory and the $\mathcal{V}$-category, and similarly for morphisms (see Proposition 7.2.3). It remains to show that $K(-,(B,Y))^{\mathrm{op}}$ is a cocartesian functor, or equivalently that the contravariant $K(-,(B,Y))$ maps cocartesian liftings to cartesian liftings.

Using the canonical choice of cocartesian liftings for any opfibration obtained via the Grothendieck construction (see Theorem 5.2.1), consider a cocartesian lifting of (C,X) along the function $f : X \to Z$ with respect to the opfibration $W : \mathcal{V}$-**Cocat** $\to$ **Set**:

$$\begin{tikzcd}
C \arrow[r, "1_{f_*Cf^*}"] \arrow[d, dotted] & f_*Cf^* \arrow[d, dotted] & \text{in } \mathcal{V}\text{-}\mathbf{Cocat} \\
X \arrow[r, "f"'] & Z & \text{in } \mathbf{Set}.
\end{tikzcd}$$

Notice that the pair notation for objects in the total category is suppressed, since the respective set of objects of the $\mathcal{V}$-cocategories is clear from the picture. The image of this arrow under $K(-,(B,Y))$ gives

$$\begin{tikzcd}
\mathrm{Hom}((f_*Cf^*,Z),(B,Y)) \arrow[r, "{[[1_{f_*Cf^*},1_B]]}"] \arrow[d, dotted] & \mathrm{Hom}((C,X),(B,Y)) \arrow[d, dotted] & \text{in } \mathcal{V}\text{-}\mathbf{Cat} \\
Y^Z \arrow[r, "Y^f"'] & Y^X & \text{in } \mathbf{Set}
\end{tikzcd}$$

by definition of the functor ${}^g\mathrm{Hom}$, and the 2-cell in $\mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(Y^Z,Y^Z))$

$$\begin{tikzcd}
Y^Z \arrow[r, "\mathrm{Hom}(f_*Cf^*{,}B)", "\mid" marking] \arrow[d, "(Y^f)_*"', "\mid" marking] \arrow[dr, phantom, "\Downarrow{[[1_{f_*Cf^*},1_B]]}"] & Y^Z \\
Y^X \arrow[r, "\mathrm{Hom}(C{,}B)"', "\mid" marking] & Y^X \arrow[u, "(Y^f)^*"', "\mid" marking]
\end{tikzcd}$$

as in (7.20) explicitly consists of arrows $[[1_{f_*Cf^*}, 1_B]]_{k,s}$

$$\prod_{z,z'}[\sum_{\substack{fx=z\\fx'=z'}} I\otimes C(x',x)\otimes I, B(kz',sz)] \to I\otimes \prod_{x,x'}[C(x',x), B(kfx',sfx)]\otimes I$$

in $\mathcal{V}$ for all $k,s\in Y^Z$. On the other hand, the canonical cartesian lifting of $(\mathrm{Hom}(C,B), Y^X)$ along the function Y^f with respect to the fibration $P:\mathcal{V}\text{-}\mathbf{Cat}\to\mathbf{Set}$ is

$$\begin{tikzcd}
(Y^f)^*\mathrm{Hom}(C,B)(Y^f)_* \arrow[r, "1_{(Y^f)^*\mathrm{Hom}(C,B)(Y^f)_*}"] \arrow[d, dotted] & \mathrm{Hom}(C,B) \arrow[d, dotted] \\
Y^Z \arrow[r, "Y^f"'] & Y^X.
\end{tikzcd}$$

By comparing this cartesian arrow with the image under $K(-,(B,Y))$ above, it is enough to show that $[[1_{f^*Cf_*}, 1_B]]$ is isomorphic to the identity arrow in the fibre $\mathcal{V}\text{-}\mathbf{Cat}_{Y^Z}=\mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(Y^Z,Y^Z))$. We have natural isomorphisms

$$\begin{aligned}
\prod_{z,z'}[\sum_{\substack{fx=z\\fx'=z'}} I\otimes C(x',x)\otimes I, B(kz',sz)] &\cong \prod_{z,z'}\prod_{\substack{fx=z\\fx'=z'}}[I\otimes C(x',x)\otimes I, B(kz',sz)]\\
\cong \prod_{x',x}[I\otimes C(x',x)\otimes I, B(kfx',sfx)] &\cong I\otimes\prod_{x,x'}[C(x',x), B(kfx',sfx)]\otimes I
\end{aligned}$$

since sum commutes with $\otimes$ and $[-,A]$ maps colimits to limits for any monoidal closed category $\mathcal{V}$. By applying r and l to move the I's appropriately, we deduce that the result holds. □

Lemma 7.5.8. *Suppose that $\mathcal{V}$ is a locally presentable symmetric monoidal closed category, and ε is the counit of the exponential adjunction*

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, shift left, "Y^{(-)^{\mathrm{op}}}"] \arrow[r, phantom, "\perp"] & \mathbf{Set}^{\mathrm{op}}. \arrow[l, shift left, "Y^{(-)}"]
\end{tikzcd} \tag{7.42}$$

For any $\mathcal{V}$-category $\mathcal{B}_Y$ and any set Z, the composite functor

$$\mathcal{V}\text{-}\mathbf{Cocat}_{Y^Z} \xrightarrow{K(-,\mathcal{B}_Y)^{\mathrm{op}}} \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}}_{Y^{Y^Z}} \xrightarrow{(\varepsilon_Z)_!} \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}}_Z$$

has a right adjoint $T_0(-,\mathcal{B}_Y)$.

Proof. We can rewrite the above composite as

$$\begin{tikzcd}
\mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(Y^Z,Y^Z)) \arrow[r, "\mathbf{Mon}(\mathrm{Hom}(-,(B,Y)))^{\mathrm{op}}"] \arrow[dr, dashed] & \mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(Y^{Y^Z},Y^{Y^Z}))^{\mathrm{op}} \arrow[d, "\mathscr{L}\varepsilon_Z"] \\
 & \mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(Z,Z))^{\mathrm{op}}
\end{tikzcd}$$

where the top functor was already given by (7.31) but is now viewed as the induced 'functor between the fibres' from $K(-,\mathcal{B})$, as in (5.8). By Corollary 7.1.2, the category of comonoids $\mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(Y^Z,Y^Z))$ of the locally presentable monoidal category $\mathcal{V}\text{-}\mathbf{Mat}(Y^Z,Y^Z)$ is also locally presentable. As such, it is in particular cocomplete and has a small dense subcategory. Moreover, the following commutative diagram

$$\begin{tikzcd}
\mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X)) \arrow[r, "\mathbf{Mon}(\mathrm{Hom}_X(-,(B,Y))^{\mathrm{op}}"] \arrow[d, "U"'] & \mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(Y^X,Y^X))^{\mathrm{op}} \arrow[d, "S^{\mathrm{op}}"] \\
\mathcal{V}\text{-}\mathbf{Mat}(X,X) \arrow[r, "\mathrm{Hom}_X(-,(B,Y))^{\mathrm{op}}"'] & \mathcal{V}\text{-}\mathbf{Mat}(Y^X,Y^X)^{\mathrm{op}}
\end{tikzcd}$$

for a fixed $\mathcal{V}$-category (B,Y) shows that the top arrow $K_X(-,(B,Y))$ is cocontinuous for any set X. This is the case because the functors U and S^{op} are comonadic by Corollary 7.1.2 and the bottom arrow is the cocontinuous internal hom ${}^{g}\mathrm{Hom}(-,B)^{\mathrm{op}}$ restricted between the cocomplete fibres. Finally, Proposition 5.3.9 ensures that all reindexing functors for the fibration P are continuous, since $P : \mathcal{V}\text{-}\mathbf{Cat} \to \mathbf{Set}$ has all fibred limits by Corollary 7.5.4. So the ones for the opfibration P^{op} are cocontinuous, and in particular so is $(\varepsilon_Z)_!$. Thus, by Kelly's theorem 3.0.1, the composite functor $(\varepsilon_Z)_! \circ K_{YZ}(-,\mathcal{B}_Y)$ has a right adjoint

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Cocat}_{YZ} \arrow[r, shift left, "(\varepsilon_Z)_! \circ K(-,\mathcal{B}_Y)^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathbf{Cat}_Z^{\mathrm{op}}. \arrow[l, shift left, "T_0(-,\mathcal{B}_Y)"]
\end{tikzcd}$$

□

At this point, all the assumptions of Lemma 5.3.6 are satisfied, so we can apply it in this setting to obtain the enriched hom-functor T, evidently isomorphic to (7.36) of the previous section.

PROPOSITION 7.5.9. *The functor between the total categories*

$$K^{\mathrm{op}} : \mathcal{V}\text{-}\mathbf{Cocat} \times \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}}$$

has a parametrized adjoint

$$T : \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{V}\text{-}\mathbf{Cocat}$$

which makes the following diagram serially commute:

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Cocat} \arrow[r, shift left, "K(-,\mathcal{B}_Y)^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] \arrow[d, "W"'] & \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \arrow[l, shift left, "T(-,\mathcal{B}_Y)"] \arrow[d, "P^{\mathrm{op}}"] \\
\mathbf{Set} \arrow[r, shift left, "Y^{(-)^{\mathrm{op}}}"] \arrow[r, phantom, "\perp"] & \mathbf{Set}^{\mathrm{op}}. \arrow[l, shift left, "Y^{(-)}"]
\end{tikzcd}$$

PROOF. By Lemma 7.5.7, we have an opfibred 1-cell $(K(-,\mathcal{B}_Y)^{\mathrm{op}}, Y^{(-)^{\mathrm{op}}})$ between the opfibrations $W : \mathcal{V}\text{-}\mathbf{Cocat} \to \mathbf{Set}$ and $P^{\mathrm{op}} : \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \to \mathbf{Set}^{\mathrm{op}}$. Also there is an adjunction $Y^{(-)^{\mathrm{op}}} \dashv Y^{(-)}$ between the base categories, since the exponential is the internal hom in the cartesian monoidal closed **Set**. Lastly, by Lemma 7.5.8 the composite functor between the fibre categories

$$K_{YZ}(-,\mathcal{B}_Y) \circ (\varepsilon_Z)_! : \mathcal{V}\text{-}\mathbf{Cocat}_{YZ} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}_Z^{\mathrm{op}}$$

has a right adjoint $T_Z(-,\mathcal{B}_Y)$ for any fixed set Z.

Therefore, by Theorem 5.3.7 the functor $K(-,\mathcal{B}_Y)^{\mathrm{op}}$ has a right adjoint $T(-,\mathcal{B}_Y)$ between the total categories

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Cocat} \arrow[r, shift left, "K(-{,}\mathcal{B}_Y)^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}}, \arrow[l, shift left, "T(-{,}\mathcal{B}_Y)"]
\end{tikzcd} \tag{7.43}$$

with $(K(-,\mathcal{B}_Y)^{\mathrm{op}}, Y^{(-)^{\mathrm{op}}}) \dashv (T(-,\mathcal{B}_Y), Y^{(-)})$ in $\mathbf{Cat^2}$, *i.e.* (W, P^{op}) is a map of adjunctions. The adjunction (7.43) for any $\mathcal{V}$-category $\mathcal{B}_Y$ makes T into a functor of two variables such that the natural isomorphism of the adjunction is natural in all three variables, *i.e.* T is the parametrized adjoint of K^{op}. □

Notice that the above proof of existence of the adjoint T between the total categories automatically provides us with the underlying set of objects of the $\mathcal{V}$-cocategory $T(\mathcal{A}_X,\mathcal{B}_Y)$, namely Y^X. On the contrary, Proposition 7.4.5 did not establish this piece of data in a straightforward way. We could also explicitly construct T on arrows, using the formulas provided in Section 5.3.

7.6. $\mathcal{V}$-modules and $\mathcal{V}$-comodules

In these last two sections of the chapter, the aim is to generalize the existence of the universal measuring comodule, which induces an enrichment of the global category of modules in the global category of comodules as seen in Section 6.3. This follows the idea of the ($\mathcal{V}$-**Cocat**)-enrichment of $\mathcal{V}$-**Cat** as the many-object generalization of the enrichment of monoids in comonoids in $\mathcal{V}$ of Section 6.1.

We are going to closely follow the development of the previous chapter in defining the global category of $\mathcal{V}$-enriched modules and the global category of $\mathcal{V}$-enriched comodules. On that level, by employing the theory of fibrations and opfibrations once again, we will determine the objects that induce the enrichment in question.

In Section 4.2, a brief account of the bicategory of $\mathcal{V}$-bimodules was given, with emphasis on the one-sided modules of $\mathcal{V}$-categories. In the current setting of the bicategory of $\mathcal{V}$-matrices, we can reformulate Definition 4.2.1 of a left $\mathcal{A}$-module for a $\mathcal{V}$-category $\mathcal{A}$ in a way that will clarify how $\mathcal{V}$-modules are a special case of modules for a monad in a bicategory as in Section 2.2. Motivated by Remark 2.2.4, we are here interested in categories of modules in the bicategory $\mathcal{V}$-**Mat** with fixed domain the singleton set $1 = \{*\}$, *i.e.* the initial object in **Set**. The monads in this bicategory are of course $\mathcal{V}$-categories $A : X \twoheadrightarrow X$.

For the following definitions, the assumptions on $\mathcal{V}$ are initially the ones required for the formation of $\mathcal{V}$-**Mat**, *i.e.* existence of sums which are preserved by the tensor product on both sides.

Definition 7.6.1. The category of *left $\mathcal{A}$-modules* for a $\mathcal{V}$-category $\mathcal{A}_X$, *i.e* a monad (A, X), is the category of left A-modules with domain the singleton set in the bicategory $\mathcal{V}$-**Mat**, *i.e.* the category of Eilenberg-Moore algebras for the (ordinary) monad 'post-composition with A' on the hom-category $\mathcal{V}\text{-}\mathbf{Mat}(1,X)$

$$\mathcal{V}\text{-}{}_{\mathcal{A}}\mathbf{Mod}=\mathcal{V}\text{-}\mathbf{Mat}(1,X)^{\mathcal{V}\text{-}\mathbf{Mat}(1,A)}.$$

Explicitly, the objects are $\mathcal{V}$-matrices $\Psi:1 \twoheadrightarrow X$ given by a family $\{\Psi(x)\}_{x\in X}$ of objects in $\mathcal{V}$, equipped with an action $\mu: A\circ\Psi\Rightarrow\Psi$ with components

$$\mu_x:\sum_{x'\in X}A(x,x')\otimes\Psi(x')\to\Psi(x)$$

such that the diagrams

$$\begin{tikzcd}
\sum\limits_{x''}(\sum\limits_{x'}A(x,x')\otimes A(x',x''))\otimes\Psi(x'') \arrow[rr, "\alpha"] \arrow[d, "\sum M_{x,x''}\otimes 1"'] & & \sum\limits_{x'}A(x,x')\otimes(\sum\limits_{x''}A(x',x'')\otimes\Psi(x'')) \arrow[d, "\sum 1\otimes\mu_{x'}"] \\
\sum\limits_{x''}A(x,x'')\otimes\Psi(x'') \arrow[dr, "\mu_x"'] & & \sum\limits_{x'}A(x,x')\otimes\Psi(x') \arrow[dl, "\mu_x"] \\
& \Psi(x), &
\end{tikzcd}$$

$$\begin{tikzcd}
\sum\limits_{x\in X}A(x,x)\otimes\Psi(x) \arrow[rr, "\mu_x"] & & \Psi(x) \\
& I\otimes\Psi(x) \arrow[ul, "\eta_x\otimes 1"] \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

commute. M and η are the composition law and identities for $\mathcal{A}$, and α, λ are the associator and left unitor of the bicategory $\mathcal{V}$-$\mathbf{Mat}$. Morphisms between two left $\mathcal{A}$-modules Ψ and Ψ' are 2-cells $\sigma:\Psi\Rightarrow\Psi'$ in $\mathcal{V}$-$\mathbf{Mat}$ compatible with the actions, *i.e.* families of arrows

$$\sigma_x:\Psi(x)\to\Psi'(x)$$

in $\mathcal{V}$ for all $x\in X$, making the diagram

$$\begin{tikzcd}
\sum\limits_{x'}A(x,x')\otimes\Psi(x') \arrow[r, "\mu_x^{\Psi}"] \arrow[d, "\sum 1\otimes\sigma_{x'}"'] & \Psi(x) \arrow[d, "\sigma_x"] \\
\sum\limits_{x'}A(x,x')\otimes\Psi'(x') \arrow[r, "\mu_x^{\Psi'}"'] & \Psi'(x)
\end{tikzcd}$$

commute.

This is essentially Definition 4.2.1, with a slight variation in the notation due to the different convention used for composition of $\mathcal{V}$-matrices. It directly follows from Definition 2.2.3 for $\mathcal{K}=\mathcal{V}$-$\mathbf{Mat}$, where the axioms (2.13, 2.14) for the appropriate 2-cells

$$\begin{tikzcd}
& X \arrow[dr, bend left, "A", "\shortmid"{marking}] & \\
1 \arrow[ur, bend left, "\Psi", "\shortmid"{marking}] \arrow[rr, bend right, "\Psi"', "\shortmid"{marking}] & \Downarrow\mu & X,
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, bend left=40, "\Psi", "\shortmid"{marking}, ""{name=U, below}] \arrow[r, bend right=40, "\Psi'"', "\shortmid"{marking}, ""{name=D, above}] \arrow[from=U, to=D, phantom, "\Downarrow\sigma"] & X
\end{tikzcd}$$

expressing the action and left t-modules morphisms, coincide with the above diagrams for their components in $\mathcal{V}$. Notice also how in Section 4.2, a left $\mathcal{A}$-module was denoted by $\Psi : \mathcal{A} \nrightarrow \mathcal{I}$, not to be confused with the actual $\mathcal{V}$-matrix $\Psi : 1 \nrightarrow X$ which encodes its data, where $\mathcal{I}$ is the unit category and 1 is the singleton set.

Similarly, we can define the category of *right $\mathcal{B}$-modules* for a $\mathcal{V}$-category $\mathcal{B}_Y$, *i.e.* a monad $B : Y \nrightarrow Y$, to be the category of right B-modules with codomain 1

$$\mathcal{V}\text{-}\mathbf{Mod}_{\mathcal{B}} \equiv \mathcal{V}\text{-}\mathbf{Mat}(Y,1)^{\mathcal{V}\text{-}\mathbf{Mat}(B,1)}$$

and also the more general category of $(\mathcal{A}_X, \mathcal{B}_Y)$-bimodules as the category of algebras for the monad 'pre-composition with B and post-composition with A'

$$\mathcal{V}\text{-}{}_{\mathcal{A}}\mathbf{Mod}_{\mathcal{B}} \equiv \mathcal{V}\text{-}\mathbf{Mat}(Y,X)^{\mathcal{V}\text{-}\mathbf{Mat}(B,A)}$$

which gives the hom-category of a bicategory of $\mathcal{V}$-enriched bimodules $\mathcal{V}$-**BMod**. This way of presenting of enriched bimodules is also included in [**BCSW83**]. We note that this bicategorical structure as well as the one that the enriched bicomodules later possibly form are not central for the current development.

In a completely dual way, we now proceed to the study of the notion of a $\mathcal{V}$-enriched comodule for a $\mathcal{V}$-cocategory. The definitions of the various cases of comodules for comonads in bicategories can again be found in Section 2.2, and in particular for $\mathcal{K} = \mathcal{V}$-**Mat**, a comonad is a $\mathcal{V}$-cocategory $C : X \nrightarrow X$.

DEFINITION 7.6.2. The category of *left $\mathcal{C}$-comodules* for a $\mathcal{V}$-cocategory (C, X) is the category of left C-comodules with fixed domain the singleton set in the bicategory $\mathcal{V}$-**Mat**

$$\mathcal{V}\text{-}{}_{\mathcal{C}}\mathbf{Comod} = \mathcal{V}\text{-}\mathbf{Mat}(1,X)^{\mathcal{V}\text{-}\mathbf{Mat}(1,C)}.$$

Objects are $\mathcal{V}$-matrices $\Phi : 1 \nrightarrow X$ given by a family of objects $\{\Phi(x)\}_{x\in X}$ in $\mathcal{V}$, equipped with the coaction $\delta : C \circ \Phi \Rightarrow \Phi$, a 2-cell in $\mathcal{V}$-**Mat** with components

$$\delta_x : \Phi(x) \to \sum_{x'\in X} C(x,x') \otimes \Phi(x')$$

satisfying the commutativity of the following diagrams:

$$\begin{tikzcd}
 & \Phi(x) \arrow[dl, "\delta_x"'] \arrow[dr, "\delta_x"] & \\
\sum\limits_{x''} C(x,x'') \otimes \Phi(x'') \arrow[d, "\sum \Delta_{x,x''}\otimes 1"'] & & \sum\limits_{x'} C(x,x') \otimes \Phi(x') \arrow[d, "\sum 1\otimes\delta_{x'}"] \\
\sum\limits_{x''}(\sum\limits_{x'} C(x,x') \otimes C(x',x'')) \otimes \Phi(x'') \arrow[rr, "\alpha"] & & \sum\limits_{x'} C(x,x') \otimes (\sum\limits_{x''} C(x',x'') \otimes \Phi(x'')),
\end{tikzcd}$$

$$\begin{tikzcd}
\Phi(x) \arrow[rr, "\delta_x"] \arrow[dr, "\lambda^{-1}"'] & & \sum\limits_{x} C(x,x) \otimes \Phi(x) \arrow[dl, "\epsilon_x\otimes 1"] \\
 & I \otimes \Phi(x). &
\end{tikzcd}$$

Δ and ϵ are the cocomposition law and coidentities for $\mathcal{C}$. Morphisms between two left $\mathcal{C}$-comodules Φ and Φ' are 2-cells $\tau : \Phi \Rightarrow \Phi'$ in $\mathcal{V}$-**Mat** which are compatible with the coactions, *i.e.* families of arrows

$$\tau_x : \Phi(x) \to \Phi'(x)$$

in $\mathcal{V}$ for all $x \in X$, which satisfy the commutativity of

$$\begin{tikzcd}
\Phi(x) \arrow[r, "\delta_x^\Phi"] \arrow[d, "\tau_x"'] & \sum\limits_{x'} C(x,x') \otimes \Phi(x') \arrow[d, "\sum 1 \otimes \tau_{x'}"] \\
\Phi'(x) \arrow[r, "\delta_x^{\Phi'}"'] & \sum\limits_{x'} C(x,x') \otimes \Phi'(x').
\end{tikzcd}$$

In an analogous way, we can define the category of *right $\mathcal{D}_Y$-comodules* for a $\mathcal{V}$-cocategory to be the category of right D-comodules with codomain 1

$$\mathcal{V}\text{-}\mathbf{Comod}_{\mathcal{D}} = \mathcal{V}\text{-}\mathbf{Mat}(Y,1)^{\mathcal{V}\text{-}\mathbf{Mat}(D,1)}$$

and also more generally the category of *left $\mathcal{C}_X$/right $\mathcal{D}_Y$-bicomodules* as the category of coalgebras for the monad 'pre-composition with D and post-composition with C'

$$\mathcal{V}\text{-}{}_{\mathcal{C}}\mathbf{Mod}_{\mathcal{D}} = \mathcal{V}\text{-}\mathbf{Mat}(Y,X)^{\mathcal{V}\text{-}\mathbf{Mat}(D,C)}.$$

By Proposition 7.1.1, the hom-categories $\mathcal{V}\text{-}\mathbf{Mat}(X,Y) = \mathcal{V}^{Y\times X}$ of the bicategory $\mathcal{V}$-**Mat** have various useful properties, which may be transferred to the categories defined above. For example, $\mathcal{V}\text{-}{}_{\mathcal{A}}\mathbf{Mod}$ and $\mathcal{V}\text{-}{}_{\mathcal{C}}\mathbf{Comod}$ which are monadic and comonadic by definition, have all limits/colimits that $\mathcal{V}$ has, and those colimits/limits that are preserved by the monad/comonad. Also, they inherit local presentability, as explained below.

PROPOSITION 7.6.3. *Suppose $\mathcal{V}$ is a cocomplete monoidal category such that the tensor product preserves colimits in both variables.*

- *(1) The category of left $\mathcal{A}$-modules for a $\mathcal{V}$-category $\mathcal{A}_X$ is cocomplete and locally presentable when $\mathcal{V}$ is.*
- *(2) The category of left $\mathcal{C}$-comodules for a $\mathcal{V}$-cocategory $\mathcal{C}_X$ is cocomplete and locally presentable when $\mathcal{V}$ is.*

PROOF. (1) The ordinary monad $\mathcal{V}\text{-}\mathbf{Mat}(1,A)$ which post-composes every $\mathcal{V}$-matrix $S : 1 \to\!\!\!\!\!\mapsto X$ with the monad $A : X \to\!\!\!\!\!\mapsto X$ preserves colimits, since composition of $\mathcal{V}$-matrices commutes with all colimits in general.

In particular, $A \circ -$ preserves filtered colimits, therefore $\mathcal{V}\text{-}{}_{\mathcal{A}}\mathbf{Mod}$ is finitary monadic over $\mathcal{V}\text{-}\mathbf{Mat}(1,X)$, which is locally presentable when $\mathcal{V}$ is. By Theorem 3.4.3, categories of finitary algebras of locally presentable categories are also locally presentable, hence the result follows.

(2) The category $\mathcal{V}\text{-}{}_{\mathcal{C}}\mathbf{Comod}$ has all colimits since they are created from those in the cocomplete $\mathcal{V}\text{-}\mathbf{Mat}(1,X)$. The endofunctor

$$F_C : \mathcal{V}\text{-}\mathbf{Mat}(1,X) \xrightarrow{C\circ -} \mathcal{V}\text{-}\mathbf{Mat}(1,X)$$

which gives rise to that comonad is again finitary, so for a locally presentable $\mathcal{V}$, Theorem 3.4.3 applies. ☐

REMARK.

(*i*) We can also express the axioms which define the objects and the arrows in $\mathcal{V}\text{-}_{\mathcal{C}}\mathbf{Comod}$ by the diagrams

$$\begin{tikzcd}
\Phi \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "\alpha"'] & C\circ\Phi \arrow[d, Rightarrow, "1\circ\alpha"] \\
C\circ\Phi \arrow[r, Rightarrow, "\Delta\circ 1"'] & C\circ C\circ\Phi,
\end{tikzcd}
\qquad
\begin{tikzcd}
\Phi \arrow[r, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "1_\Phi"'] & C\circ\Phi \arrow[d, Rightarrow, "\epsilon\circ 1"] \\
& \Phi
\end{tikzcd}$$

for a $\mathcal{V}$-matrix Φ with domain 1 equipped with $\alpha : \Phi \Rightarrow C\circ\Phi$, and

$$\begin{tikzcd}
\Phi \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "k"'] & C\circ\Phi \arrow[d, Rightarrow, "1\circ k"] \\
\Psi \arrow[r, Rightarrow, "\beta"'] & C\circ\Psi
\end{tikzcd}$$

for a 2-cell $k : \Phi \Rightarrow \Psi$. This could create the impression that $\mathcal{V}\text{-}_{\mathcal{C}}\mathbf{Comod}$ is an ordinary category of comodules for a comonoid, here $C \in \mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X))$, in some monoidal category. However, that would require everything to take place in the context of the fixed monoidal category $(\mathcal{V}\text{-}\mathbf{Mat}(X,X),\circ,1_X)$, therefore the comodules category would be

$$\mathbf{Comod}_{\mathcal{V}\text{-}\mathbf{Mat}(X,X)}(C) = \mathcal{V}\text{-}\mathbf{Mat}(X,X)^{\mathcal{V}\text{-}\mathbf{Mat}(X,C)}$$

by Proposition 3.4.1. In our terminology, this is the category of left C-comodules with fixed domain X in the bicategory $\mathcal{V}$-**Mat**, rather than just the ones with domain $1 = \{*\}$, like $\mathcal{V}\text{-}_{\mathcal{C}}\mathbf{Comod}$ was defined. The same applies to the categories of modules for a $\mathcal{V}$-category $A \in \mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X))$.

From this point of view, we could formulate all the above definitions in a more abstract way: left $\mathcal{A}_X$-modules could be $\mathcal{V}$-matrices $\Psi : Y \nrightarrow X$ with arbitrary domain set Y, given by a family of objects $\{\Psi(x,y)\}_{(x,y)\in X\times Y}$ in $\mathcal{V}$ and a left action from A given by arrows

$$\mu_{x.y} : \sum_{x'\in X} A(x,x')\otimes\Psi(x',y) \to \Psi(x,y)$$

satisfying appropriate axioms. This is also how $\mathcal{V}$-bimodules are defined. Nevertheless, for the purposes of this thesis we are interested in $\mathcal{V}$-modules/comodules given by families indexed only over the set of objects of the underlying $\mathcal{V}$-category/cocategory.

(*ii*) Notice that establishing local presentability for particular categories of interest has been of varied difficulty, depending on their further structure. For example, for the categories $\mathbf{Comod}_{\mathcal{V}}(C)$ (Proposition 3.4.2) and $\mathcal{V}\text{-}_{\mathcal{C}}\mathbf{Comod}$ the result was straightforward because they were both evidently finitary comonadic over locally presentable categories. On the other hand, for $\mathbf{Comon}(\mathcal{V})$ and $\mathcal{V}$-**Cocat** we first had to verify local presentability (Propositions 3.3.5 and 7.4.2), and comonadicity followed afterwards. Notably, expressing a category as an equifier of a family of natural transformations of accessible functors between accessible categories has been the underlying key technique in all cases.

We now consider global categories of enriched modules and comodules, *i.e.* (left) $\mathcal{V}$-modules and (left) $\mathcal{V}$-comodules for which the $\mathcal{V}$-category and $\mathcal{V}$-cocategory which acts or co-acts is not fixed as above, but varies. The definitions below are motivated by the concepts in Section 6.2.

Definition 7.6.4. The *global category of left $\mathcal{V}$-modules* $\mathcal{V}$-**Mod** is defined as follows. Objects are left $\mathcal{A}$-modules Ψ for an arbitrary $\mathcal{V}$-category $\mathcal{A}_X$, denoted by $\Psi_{\mathcal{A}}$, and a morphism $\kappa_F : \Psi_{\mathcal{A}} \to \Xi_{\mathcal{B}}$ between a $\mathcal{A}_X$-module Ψ and a $\mathcal{B}_Y$-module Ξ consists of a $\mathcal{V}$-functor $F_f : \mathcal{A}_X \longrightarrow \mathcal{B}_Y$ and a family of arrows in $\mathcal{V}$ $\kappa_x : \Psi(x) \longrightarrow \Xi(fx)$ for all objects $x \in X$ of $\mathcal{A}$, such that the diagram

$$\begin{tikzcd}
\sum\limits_{x'} A(x,x') \otimes \Psi(x') \arrow[rr, "\mu^{\Psi}_{x}"] \arrow[d, "\sum 1\otimes\kappa_x"'] & & \Psi(x) \arrow[d, "\kappa_x"] \\
\sum\limits_{x'} A(x,x') \otimes \Xi(fx') \arrow[r, "\sum F_{x,x'}\otimes 1"'] & \sum\limits_{x'} B(fx,fx') \otimes \Xi(fx') \arrow[r, "\mu^{\Xi}_{fx}"'] & \Xi(fx)
\end{tikzcd} \tag{7.44}$$

commutes. The arrows μ^{Ψ} and μ^{Ξ} are the left $\mathcal{A}$ and $\mathcal{B}$ actions on Ψ and Ξ respectively.

Dually, the *global category of left $\mathcal{V}$-comodules* $\mathcal{V}$-**Comod** has as objects left $\mathcal{C}$-comodules for an arbitrary $\mathcal{V}$-cocategory $\mathcal{C}_X$, denoted by $\Phi_{\mathcal{C}}$, and a morphism $s_G : \Phi_{\mathcal{C}} \to \Omega_{\mathcal{D}}$ consists of a $\mathcal{V}$-cofunctor $G_g : \mathcal{C}_X \to \mathcal{D}_Y$ and a family of arrows in $\mathcal{V}$ $\nu_x : \Phi(x) \to \Omega(gx)$ for all $x \in X$, such that the diagram

$$\begin{tikzcd}
\Phi(x) \arrow[r, "\delta^{\Phi}_{x}"] \arrow[dd, "\nu_x"'] & \sum\limits_{x'} C(x,x') \otimes \Phi(x') \arrow[r, "\sum G_{x,x'}\otimes 1"] & \sum\limits_{x'} D(gx,gx') \otimes \Phi(x') \arrow[d, "\sum 1\otimes\nu_x"] \\
& & \sum\limits_{x'} D(gx,gx') \otimes \Omega(gx') \arrow[d, tail, "\iota"] \\
\Omega(gx) \arrow[rr, "\delta^{\Omega}_{gx}"'] & & \sum\limits_{y\in Y} D(gx,y) \otimes \Omega(y)
\end{tikzcd} \tag{7.45}$$

commutes. The arrows δ^{Φ} and δ^{Ω} are the corresponding coactions, and ι is the inclusion into a larger sum.

Notice the similarities between the diagrams (7.44), (7.45) that morphisms between $\mathcal{V}$-modules and $\mathcal{V}$-comodules over different $\mathcal{V}$-categories and $\mathcal{V}$-cocategories have to satisfy, with the respective diagrams from Definition 6.2.1. This was of course expected, since $\mathcal{V}$-**Mod** and $\mathcal{V}$-**Comod** are to be thought of as the many-object generalizations of the global categories **Mod** and **Comod**.

Both global categories of $\mathcal{V}$-enriched modules and comodules have the structure of a (symmetric) monoidal category, when $\mathcal{V}$ is symmetric monoidal. For a left $\mathcal{A}_X$-module Ψ and a left $\mathcal{B}_Y$-module Ξ, their tensor product is a $\mathcal{V}$-matrix

$$\Psi \otimes \Xi : \ 1 \longrightarrow\!\!\!\!\!\!\!+\;\; X \times Y \tag{7.46}$$

given by the family of objects in $\mathcal{V}$

$$(\Psi \otimes \Xi)(x, y) := \Psi(x) \otimes \Xi(y)$$

equipped with a left $(\mathcal{A} \otimes \mathcal{B})_{X \times Y}$ action (since $\mathcal{V}$-**Cat** is monoidal) a 2-cell $\mu : (\mathcal{A} \otimes \mathcal{B}) \circ (\Psi \otimes \Xi) \Rightarrow \Psi \otimes \Xi$, with components arrows in $\mathcal{V}$

$$\mu_{(x,y)} : \sum_{(x',y') \in X \times Y} (\mathcal{A} \otimes \mathcal{B})((x, y), (x', y')) \otimes (\Psi \otimes \Xi)(x', y') \to (\Psi \otimes \Xi)(x, y)$$

which are explicitly the composites

$$\begin{tikzcd}
A(x,x') \otimes B(y,y') \otimes \Psi(x') \otimes \Xi(y') \arrow[r, "1 \otimes s \otimes 1"] \arrow[dr, dashed] & A(x,x') \otimes \Psi(x') \otimes B(y,y') \otimes \Xi(y') \arrow[d, "\mu^{\Psi}_x \otimes \mu^{\Xi}_y"] \\
& \Psi(x) \otimes \Xi(y)
\end{tikzcd}$$

for all $x, x' \in X$ and $y, y' \in Y$. The axioms for an $\mathcal{A} \otimes \mathcal{B}$-action are satisfied by the axioms for μ^{Ψ} and μ^{Ξ}. Dually, if Φ is a left $\mathcal{C}_X$-comodule and Ω is a left $\mathcal{D}_Y$-module, their tensor product is a $\mathcal{V}$-matrix $\Phi \otimes \Omega$ as (7.46) given by $(\Phi \otimes \Omega)(x, y) = \Phi(x) \otimes \Omega(y)$, with left $(\mathcal{C} \otimes \mathcal{D})_{X \times Y}$-action consisting of the composite arrows

$$\begin{tikzcd}
\Phi(x) \otimes \Omega(y) \arrow[r, "\delta^{\Phi}_x \otimes \delta^{\Omega}_y"] \arrow[dr, dashed] & \sum\limits_{x' \in X} C(x,x') \otimes \Phi(x') \otimes \sum\limits_{y' \in Y} D(y,y') \otimes \Omega(y') \arrow[d, "1 \otimes s \otimes 1"] \\
& \sum\limits_{\substack{x' \in X \\ y' \in Y}} C(x,x') \otimes D(y,y') \otimes \Phi(x') \otimes \Omega(y').
\end{tikzcd}$$

Notice that the right arrow incorporates an isomorphism due to $\otimes$ preserving sums. It is not hard to check that we can extend the definition of a tensor product to $\mathcal{V}$-module and comodule morphisms, and also symmetry from $\mathcal{V}$ is clearly inherited. The monoidal unit in both cases is again the unit $\mathcal{V}$-matrix $\mathcal{I} : 1 \longrightarrow\!\!\!\!\!+\; 1$, with trivial $\mathcal{I}$-action from the unit $\mathcal{V}$-(co)category.

There are obvious forgetful functors from these global categories to $\mathcal{V}$-categories and $\mathcal{V}$-cocategories

$$N : \mathcal{V}\text{-}\mathbf{Mod} \to \mathcal{V}\text{-}\mathbf{Cat}$$

$$H : \mathcal{V}\text{-}\mathbf{Comod} \to \mathcal{V}\text{-}\mathbf{Cocat}$$

which map any left $\mathcal{A}$-module Ψ_A and $\mathcal{C}$-comodule $\Phi_{\mathcal{C}}$ to the $\mathcal{V}$-category $\mathcal{A}$ and $\mathcal{V}$-cocategory $\mathcal{C}$ respectively, and the morphisms to the underlying $\mathcal{V}$-functor and $\mathcal{V}$-cofunctor. These functors will turn out to be a fibration and an opfibration, allowing us to once again employ Theorem 5.3.7 regarding adjunctions between fibrations, in order to establish an enrichment of $\mathcal{V}$-**Mod** in $\mathcal{V}$-**Comod**.

Similarly to the $\mathcal{V}$-categories and $\mathcal{V}$-cocategories development, we will first formulate isomorphic characterizations of these two categories which will clarify the fibrational and opfibrational structure later. Lemmas 7.3.3 and 7.3.11 justify the form of the $\mathcal{V}$-functors and $\mathcal{V}$-cofunctors used below.

LEMMA 7.6.5. *Suppose that $\Xi : 1 \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ is a left $\mathcal{B}$-module and $F : (A, X) \xrightarrow{(\phi,f)} (B, Y)$ is a $\mathcal{V}$-functor. Then, the composite $\mathcal{V}$-matrix*

$$1 \overset{\Xi}{\relbar\joinrel\mapstochar\joinrel\rightarrow} Y \overset{f^*}{\relbar\joinrel\mapstochar\joinrel\rightarrow} X$$

has the structure of a left $\mathcal{A}$-module. Moreover, this mapping gives rise to a functor

$$(f^* \circ -) : \mathcal{V}\text{-}_{\mathcal{B}}\mathbf{Mod} \longrightarrow \mathcal{V}\text{-}_{\mathcal{A}}\mathbf{Mod}.$$

PROOF. The induced left $\mathcal{A}$-action μ' on $f^*\Xi$ is the composite 2-cell

$$\begin{tikzcd}
 & Y \arrow[r, "f^*"] \arrow[dr, "B"'] \arrow[d, phantom, "\Downarrow\mu"] & X \arrow[dr, "A"] \arrow[d, phantom, "\Downarrow\hat{\phi}"] & \\
1 \arrow[ur, "\Xi"] \arrow[rr, bend right=10, "\Xi"'] & \; & Y \arrow[r, "f^*"'] & X
\end{tikzcd}$$

where $\hat{\phi} : f_*A \Rightarrow Bf_*$ corresponds bijectively to $\phi : A \Rightarrow f^*Bf_*$ via mates. In terms of pasting operations, this is the composite 2-cell

$$\mu' : \ Af^*\Xi \xRightarrow{\hat{\phi}\Xi} f^*B\Xi \xRightarrow{f^*\mu} f^*\Xi.$$

The fact that μ' satisfies the axioms for an A-action for a monad $A : X \relbar\joinrel\mapstochar\joinrel\rightarrow X$ follows from the axioms of the $\mathcal{V}$-functor $F = (\phi, f)$ and the left B-action μ on Ξ. Also, it is easy to check that if $\sigma : \Xi \to \Xi'$ is a left $\mathcal{B}$-module morphism, then

$$\begin{tikzcd}
1 \arrow[r, bend left=40, "\Xi"] \arrow[r, bend right=40, "\Xi'"'] \arrow[r, phantom, "\Downarrow\sigma"] & Y \arrow[r, "f^*"] & X
\end{tikzcd}$$

is a left $\mathcal{A}$-module morphism. In terms of components, the family $\{\Xi(y)\}_{y\in Y}$ of objects in $\mathcal{V}$ is mapped to the family

$$\{(f^* \circ \Xi)(x)\}_{x\in X} = \{I \otimes \Xi(fx)\}_{x\in X}$$

and the family $\sigma_y : \Xi(y) \to \Xi'(y)$ of arrows in $\mathcal{V}$ is mapped to

$$(f^*\sigma)_x : I \otimes \Xi(fx) \xrightarrow{1\otimes\sigma_{fx}} I \otimes \Xi'(fx).$$

Compatibility with composition and identities for this functor follow from properties of vertical and horizontal composition of 2-cells. $\square$

Notice that the above lemma, like other results of this section, does not only hold for left modules with fixed domain the singleton set 1, but for modules with arbitrary domain. Similarly, for right modules with fixed codomain, if we replace $(f^* \circ \text{-})$ with $(\text{-} \circ f_*)$ we get an analogous functor. Dually, we can consider left $\mathcal{V}$-comodules.

Lemma 7.6.6. *If $\Phi : 1 \to\!\!\!\!\!+\, X$ is a left $\mathcal{C}$-comodule and $G : (C, X) \xrightarrow{(\psi,g)} (D, Y)$ is a $\mathcal{V}$-cofunctor, the composite $\mathcal{V}$-matrix*

$$1 \xrightarrow[]{\Phi} X \xrightarrow[]{g_*} Y$$

obtains the structure of a left $\mathcal{D}$-comodule. This mapping gives rise to a functor

$$(g_* \circ -) : \mathcal{V}\text{-}_{\mathcal{C}}\mathbf{Comod} \longrightarrow \mathcal{V}\text{-}_{\mathcal{D}}\mathbf{Comod}.$$

Proof. The induced $\mathcal{D}$-coaction δ' on $g_*\Phi$ is the composite 2-cell

$$\begin{tikzcd}
1 \arrow[rr, "\Phi"] \arrow[dr, "\Phi"'] & {} \arrow[d, phantom, "\Downarrow\delta"] & X \arrow[r, "g_*"] \arrow[d, phantom, "\Downarrow\hat{\psi}"] & Y \\
& X \arrow[ur, "C"'] \arrow[r, "g_*"'] & Y \arrow[ur, "D"'] &
\end{tikzcd}$$

where again $\hat{\psi}$ is the mate of ψ ‘on the one side’. This is the pasted composite

$$\delta' : \; g_*\Phi \xRightarrow{g_*\delta} g_*C\Phi \xRightarrow{\hat{\psi}\Phi} Dg_*\Phi \;,$$

and the D-coaction axioms are satisfied by the axioms for δ and the $\mathcal{V}$-cofunctor $G = (\psi, g)$. Moreover, if $\tau : \Phi \to \Phi'$ is a left $\mathcal{C}$-comodule morphism, post-composing it with g_* produces a 2-cell which satisfies the axioms for a left $\mathcal{D}$-comodule. In terms of components, the functor $(g_* \circ -)$ maps the family $\{\Phi(x)\}_{x\in X}$ of objects in $\mathcal{V}$ to

$$\{(g_* \circ \Phi)(y)\}_{y\in Y} = \{\sum_{y=fx} I \otimes \Phi(x)\}_{y\in Y}$$

and the family $\tau_x : \Phi(x) \to \Phi(x')$ of arrows in $\mathcal{V}$ to

$$(g_*\tau)_y : \sum_{y=fx} I \otimes \Phi(x) \xrightarrow{\sum 1\otimes\tau_x} \sum_{y=fx} I \otimes \Phi'(x).$$

This mapping is a functor since it preserves composition and identities for evident reasons. □

We can now give the following characterizations of the global categories of $\mathcal{V}$-modules and $\mathcal{V}$-comodules.

Lemma 7.6.7. *The objects of $\mathcal{V}$-**Mod** are pairs $(\Psi, \mathcal{A}_X) \in \mathcal{V}\text{-}_{\mathcal{A}}\mathbf{Mod} \times \mathcal{V}\text{-}\mathbf{Cat}$ and morphisms are (in bijection with) pairs $(\kappa, F_f) : (\Psi, \mathcal{A}_X) \to (\Xi, \mathcal{B}_Y)$ where*

$$\begin{cases} \Psi \xrightarrow{\kappa} f^* \circ \Xi & \text{in } \mathcal{V}\text{-}_{\mathcal{A}}\mathbf{Mod} \\ F : (A, X) \xrightarrow{(\phi,f)} (B, Y) & \text{in } \mathcal{V}\text{-}\mathbf{Cat}. \end{cases}$$

Evidently the objects of this description are exactly the same as in Definition 7.6.4, whereas the morphisms satisfy

$$\begin{tikzcd}
 & & X \arrow[dr, "A"] & & & & X \arrow[dr, "A"] & \\
1 \arrow[urr, bend left, "\Psi"] \arrow[r, "\Xi"'] \arrow[drr, bend right, "\Xi"'] & Y \arrow[ur, "f^*"'] \arrow[dr, "B"'] & \Downarrow\hat{\phi} & X & = & 1 \arrow[ur, "\Psi"] \arrow[rr, "\Psi" description] \arrow[dr, "\Xi"'] & & X. \\
 & & Y \arrow[ur, "f^*"'] & & & & Y \arrow[ur, "f^*"'] &
\arrow[from=2-2, to=1-2, phantom, "\Downarrow\kappa"]
\arrow[from=2-2, to=3-2, phantom, "\Downarrow\mu"]
\arrow[from=1-7, to=2-7, phantom, "\Downarrow\mu"]
\arrow[from=2-7, to=3-7, phantom, "\Downarrow\kappa"]
\end{tikzcd}$$

where the multiplication of $f^* \circ \Xi$ is given by Lemma 7.6.5. If translated in terms of components $\kappa_x : \Psi(x) \to I \otimes \Xi(fx)$, the above is equivalent to the commutative diagram (7.44), again 'up to tensoring with I in the left'. This implies that there is a bijection between these two forms of the morphisms.

Lemma 7.6.8. *The objects of* $\mathcal{V}$-**Comod** *are pairs* $(\Phi, \mathcal{C}_X) \in \mathcal{V}\text{-}_{\mathcal{C}}\mathbf{Comod} \times \mathcal{V}\text{-}\mathbf{Cocat}$ *and morphisms are pairs* $(\nu, G_g) : (\Phi, \mathcal{C}_X) \to (\Omega, \mathcal{D}_Y)$ *where*

$$\begin{cases} g_* \circ \Phi \xrightarrow{\nu} \Omega & \text{in } \mathcal{V}\text{-}_{\mathcal{C}}\mathbf{Comod} \\ G : (C, X) \xrightarrow{(\psi, g)} (D, Y) & \text{in } \mathcal{V}\text{-}\mathbf{Cocat}. \end{cases}$$

We are now in position to illustrate the fibrational and opfibrational structure of the categories of enriched modules and comodules. Similarly to Section 7.5, the idea is to define appropriate pseudofunctors, which will then give rise via the Grothendieck construction to (op)fibrations isomorphic to the forgetful functors N and T. The fibre categories will evidently be the categories of left modules/comodules for a fixed $\mathcal{V}$-category/cocategory.

Proposition 7.6.9. *The global category of* $\mathcal{V}$*-modules* $\mathcal{V}$-**Mod** *is fibred over the category of* $\mathcal{V}$*-categories* $\mathcal{V}$-**Cat**.

Proof. Define an indexed category $\mathscr{H}$ as follows:

$$\begin{tikzcd}
\mathscr{H} : \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \arrow[r] & \mathbf{Cat} \\
(A, X) \arrow[r, mapsto, dotted] \arrow[d, "{(\phi, f)}"'] & \mathcal{V}\text{-}_{\mathcal{A}}\mathbf{Mod} \\
(B, Y) \arrow[r, mapsto, dotted] & \mathcal{V}\text{-}_{\mathcal{B}}\mathbf{Mod} \arrow[u, "{\mathscr{H}(\phi, f)}"']
\end{tikzcd}$$

where $\mathscr{H}(\phi, f) = (f^* \circ \text{-})$ as described in Lemma 7.6.5, *i.e.* post-composition with the $\mathcal{V}$-matrix f^* induced from the object mapping f of the $\mathcal{V}$-functor. For any two composable $\mathcal{V}$-functors $F_f : (A, X) \to (B, Y)$ and $G_g : (B, Y) \to (E, Z)$, there is a natural isomorphism

$$\begin{tikzcd}
 & \mathcal{V}\text{-}_{\mathcal{B}}\mathbf{Mod} \arrow[dr, "\mathscr{H}F"] \arrow[d, phantom, "\Downarrow\delta^{F,G}"] & \\
\mathcal{V}\text{-}_{\mathcal{E}}\mathbf{Mod} \arrow[ur, "\mathscr{H}G"] \arrow[rr, bend right, "\mathscr{H}(G \circ F)"'] & {} & \mathcal{V}\text{-}_{\mathcal{A}}\mathbf{Mod}
\end{tikzcd}$$

with components invertible arrows in ${}_{\mathcal{A}}\mathbf{Mod}$

$$\begin{tikzcd}
 & & Y \arrow[dr, "f^*"] & \\
\delta_\Psi^{G,F} :\ 1 \arrow[r, "\Psi"] & Z \arrow[ur, "g^*"] \arrow[rr, bend right, "(gf)^*"'] \arrow[rr, phantom, "\Downarrow\xi^{g,f}"] & & X
\end{tikzcd}$$

where ξ is like in (7.12). These 2-cells consist of families of isomorphisms in $\mathcal{V}$

$$(\delta_\Psi^{G,F})_x \ : I\otimes I\otimes\Psi(gfx) \xrightarrow{r_I\otimes 1} I\otimes\Psi(gfx)$$

which trivially commute with the induced $\mathcal{A}$-actions of the modules $f^*g^*\Psi$ and $(gf)^*\Psi$. Also, for any $\mathcal{V}$-category (A,X), there is a natural isomorphism

$$\begin{tikzcd}
\mathcal{V}\text{-}{}_{\mathcal{A}}\mathbf{Mod} \arrow[rr, bend left, "\mathbf{1}_{\mathcal{V}\text{-}{}_{\mathcal{A}}\mathbf{Mod}}"] \arrow[rr, bend right, "\mathscr{H}(\mathbf{1}_{\mathcal{A}})"'] \arrow[rr, phantom, "\Downarrow\gamma^A"] & & \mathcal{V}\text{-}{}_{\mathcal{A}}\mathbf{Mod}
\end{tikzcd}$$

with components invertible arrows

$$\begin{tikzcd}
\gamma_\Psi^A :\ 1 \arrow[rr, bend left, "\Psi"] \arrow[dr, "\Psi"'] \arrow[rr, phantom, "\Downarrow\lambda^{-1}"] & & X \\
 & X \arrow[ur, "1_X"'] &
\end{tikzcd}$$

where $(\mathrm{id}_X)^* = 1_X$ is the underlying function of the identity functor $\mathbf{1}_{\mathcal{A}}$ and λ is the left unitor of the bicategory $\mathcal{V}$-$\mathbf{Mat}$, thus consist of isomorphisms

$$(\gamma_\Psi^A)_x \ : \Psi(x) \xrightarrow{l^{-1}} I\otimes\Psi(x),$$

again trivially being left $\mathcal{A}$-module morphisms. The natural transformations δ and γ with components the above isomorphisms can be verified to satisfy the conditions 2.5 and 2.6, therefore $\mathscr{H}$ is a well-defined pseudofunctor.

By Theorem 5.2.1, the Grothendieck category $\mathfrak{G}\mathscr{H}$ has as objects pairs $(\Psi,\mathcal{A}_X)$ where $\mathcal{A}_X$ is in $\mathcal{V}$-$\mathbf{Cat}$ and Ψ is in $\mathcal{V}$-${}_{\mathcal{A}}\mathbf{Mod}$, and morphisms $(\Psi,\mathcal{A}_X)\to(\Xi,\mathcal{B}_Y)$ are pairs

$$\begin{cases} \Psi\to(\mathscr{H}F)\Xi & \text{in } \mathscr{H}\mathcal{B}_Y \\ F:(A,X)\to(B,Y) & \text{in } \mathcal{V}\text{-}\mathbf{Cat} \end{cases}$$

which, by definition of the functor $\mathscr{H}F$, coincide with the isomorphic formulation of left $\mathcal{V}$-module morphisms as in Lemma 7.6.7, hence $\mathfrak{G}\mathscr{H}\cong\mathcal{V}$-$\mathbf{Mod}$. Moreover, the forgetful functor $N:\mathcal{V}\text{-}\mathbf{Mod}\to\mathcal{V}\text{-}\mathbf{Cat}$ which keeps the $\mathcal{V}$-category and $\mathcal{V}$-functor part of structure, has essentially the same effect as the fibration

$$P_{\mathscr{H}} : \mathfrak{G}\mathscr{H} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}$$

so $N\cong P_{\mathscr{H}}$ exhibits N as a fibration itself. □

Proposition 7.6.10. *The global category of (left) $\mathcal{V}$-comodules $\mathcal{V}$-*$\mathbf{Comod}$ *is opfibred over the category of $\mathcal{V}$-cocategories $\mathcal{V}$-*$\mathbf{Cocat}$.

Proof. Define a (covariant) indexed category as follows:

$$\begin{tikzcd}
\mathscr{S}: & \mathcal{V}\text{-}\mathbf{Cocat} \arrow[r] & \mathbf{Cat} \\
& (C,X) \arrow[r, mapsto, dotted] \arrow[d, "{(\psi,f)}"'] & \mathcal{V}\text{-}_{\mathcal{C}}\mathbf{Comod} \arrow[d, "{\mathscr{S}(\psi,f)}"] \\
& (D,Y) \arrow[r, mapsto, dotted] & \mathcal{V}\text{-}_{\mathcal{D}}\mathbf{Comod}
\end{tikzcd}$$

where $\mathscr{S}(\psi,f)=(f_*\circ\text{-})$ as in Lemma 7.6.6. For any two composable $\mathcal{V}$-cofunctors $F_f:(C,X)\to(D,Y)$ and $G_g:(D,Y)\to(E,Z)$, we have a natural isomorphism $\delta^{G,F}:\mathscr{S}G\circ\mathscr{S}F\Rightarrow\mathscr{S}(G\circ F)$ with components the composite 2-cells

$$\begin{tikzcd}
\delta_{\Phi}^{G,F}: \; 1 \arrow[r, "\Phi", "\shortmid"{marking}] & X \arrow[r, bend left, "f_*", "\shortmid"{marking}] \arrow[rr, bend right, "(gf)_*"', "\shortmid"{marking}] & Y \arrow[d, phantom, "\Downarrow\zeta^{g,f}"] \arrow[r, bend left, "g_*", "\shortmid"{marking}] & Z \\
& & {} &
\end{tikzcd}$$

in $\mathcal{V}$-$_{\mathcal{E}}\mathbf{Comod}$, consisting of the families of arrows in $\mathcal{V}$

$$(\delta_{\Phi}^{G,F})_z\;:\sum_{\substack{z=gy\\y=fx}} I\otimes I\otimes\Phi(x)\xrightarrow{\sum r_I\otimes 1}\sum_{z=gfx} I\otimes\Phi(x)$$

which trivially commute with the respective $\mathcal{E}$-coactions. Moreover, for any $\mathcal{V}$-cocategory $(\mathcal{C},X)$, we have a natural isomorphism $\gamma^{C}:\mathbf{1}_{\mathcal{V}\text{-}_{\mathcal{C}}\mathbf{Comod}}\Rightarrow\mathscr{S}(\mathbf{1}_{\mathcal{C}})$ with components the same invertible arrows $\lambda^{\text{-}1}$ as in the previous proof. The natural isomorphisms δ and γ can be checked to satisfy the appropriate axioms 2.5 and 2.6, so $\mathscr{S}$ is a well-defined pseudofunctor. Via Grothendieck construction, it gives rise to an opfibration

$$U_{\mathscr{S}}:\mathfrak{G}\mathscr{S}\longrightarrow\mathcal{V}\text{-}\mathbf{Cocat}$$

which maps a pair $(\Psi,\mathcal{C}_X)$ where $\Psi\in\mathcal{V}$-$_{\mathcal{C}}\mathbf{Comod}$ to its $\mathcal{V}$-cocategory $\mathcal{C}_X$, and

$$\begin{cases}(\mathscr{S}F)\Phi\to\Omega & \text{in } \mathscr{S}\mathcal{C}_X\\ F:(C,X)\to(D,Y) & \text{in } \mathcal{V}\text{-}\mathbf{Cocat}\end{cases}$$

to the $\mathcal{V}$-functor F. By Lemma 7.6.8 it is now evident that $U_{\mathscr{S}}\cong H$, hence the forgetful functor $H:\mathcal{V}\text{-}\mathbf{Comod}\to\mathcal{V}\text{-}\mathbf{Cocat}$ is an opfibration. □

Corollary 7.6.11. *The opfibration H has all opfibred colimits, hence $\mathcal{V}$-**Comod** has all colimits and H strictly preserves them.*

Proof. The fibre categories of the opfibration H are the cocomplete categories $\mathcal{V}$-$_{\mathcal{C}}\mathbf{Comod}$ for each $\mathcal{V}$-cocategory $\mathcal{C}_X$, and the reindexing functors $(f_*\circ\text{-})$ for any $\mathcal{V}$-cofunctor F_f preserve colimits (as composition of $\mathcal{V}$-matrices always does). Therefore, Proposition 5.3.9 ensures that H is opfibred cocomplete, so by Corollary 5.3.11 and cocompleteness of $\mathcal{V}$-**Cocat**, the result follows. □

Remark. In this section, emphasis was given to the study of left-sided $\mathcal{V}$-modules and $\mathcal{V}$-comodules, whereas in Section 6.2 where the 'one-object case' global categories **Mod** and **Comod** where defined, the distinction between left and right was mostly omitted due to symmetry in $\mathcal{V}$. In fact, in a very similar manner we could have defined *global categories of right $\mathcal{V}$-modules* and *$\mathcal{V}$-comodules.* Then by slightly changing the reindexing functors (replacing post- with pre-composition, and lower with upper stars), we would end up with a fibrational characterization as above.

However, in this case there does not exist an isomorphism between right and left enriched modules and comodules as before, which would allow us to regard the different (fibre and total) categories as essentially the same. Explicitly, for $\mathbf{Mod}_{\mathcal{V}}(A)$ with $\mathcal{V}$ symmetric, a left A-module (M,μ) for a monoid A always gives rise to a right A-action μ' on M via

$$\begin{tikzcd}
A\otimes M \arrow[r, "\mu"] & M \\
M\otimes A \arrow[u, "s"', "\wr|"] \arrow[ur, "\mu'"'] &
\end{tikzcd}$$

and all appropriate axioms are satisfied. On the other hand, the left $\mathcal{A}$-action for a $\mathcal{V}$-category $\mathcal{A}_X$ on a $\mathcal{V}$-module Ψ is given by arrows in $\mathcal{V}$

$$\mathcal{A}(x,x')\otimes\Psi(x')\to\Psi(x)$$

for all $x,x'\in X$, which are not in bijective correspondence with arrows which would define a right $\mathcal{A}$-action on Ψ, of the form

$$\Psi(x')\otimes\mathcal{A}(x',x)\to\Psi(x)$$

for all x,x', even if $\mathcal{V}$ is symmetric. This is because the elements of the indexing set of the family of objects of Ψ in the formula would agree with the second, rather than the first entry of the hom-sets of $\mathcal{A}$ in the above formula.

7.7. Enrichment of $\mathcal{V}$-modules in $\mathcal{V}$-comodules

Similarly to Sections 6.3 and 7.4, we are now going to work our way through the data which induce an enrichment of the global category of enriched modules $\mathcal{V}$-$\mathbf{Mod}$ in the global category of enriched comodules $\mathcal{V}$-$\mathbf{Comod}$.

Suppose that $\mathcal{V}$ is a symmetric monoidal closed category, with products and coproducts. Recall that the lax functor $\mathrm{Hom}:\mathcal{V}\text{-}\mathbf{Mat}^{\mathrm{co}}\times\mathcal{V}\text{-}\mathbf{Mat}\to\mathcal{V}\text{-}\mathbf{Mat}$ as in (7.5) provides a functor between the hom-categories

$$\mathrm{Hom}_{(Y,W),(X,Z)}:\mathcal{V}\text{-}\mathbf{Mat}(Z,X)^{\mathrm{op}}\times\mathcal{V}\text{-}\mathbf{Mat}(W,Y)\to\mathcal{V}\text{-}\mathbf{Mat}(W^Z,Y^X)$$

which maps a pair of $\mathcal{V}$-matrices $(S:Z\relbar\joinrel\mapsto X,\ T:W\relbar\joinrel\mapsto Y)$ to $\mathrm{Hom}(S,T)$ given by the family of objects in $\mathcal{V}$

$$\mathrm{Hom}(S,T)(k,m)=\prod_{\substack{x\in X\\ z\in Z}}[S(x,z),T(mx,kz)]$$

for all $k\in W^Z$ and $m\in Y^X$. Moreover, in Section 7.4 we made use of the induced functor $\mathbf{Mon}(\mathrm{Hom}_{(X,Y),(X,Y)})$ as in (7.31), between the categories of comonoids and monoids of the endoarrow hom-category. This gave rise to the functor

$$K:\mathcal{V}\text{-}\mathbf{Cocat}^{\mathrm{op}}\times\mathcal{V}\text{-}\mathbf{Cat}\to\mathcal{V}\text{-}\mathbf{Cat}$$

between $\mathcal{V}$-(co)categories, *i.e.* the $\mathcal{V}$-matrix $K(\mathcal{C}_X, \mathcal{B}_Y) = \mathrm{Hom}(\mathcal{C}, \mathcal{B})_{Y^X}$ obtains the structure a $\mathcal{V}$-category.

Now, by Proposition 2.2.10, we know that for any lax functor $\mathscr{F}$ between bicategories $\mathcal{K}, \mathcal{L}$ and any monad t in $\mathcal{K}$, there is an induced functor $\mathbf{Mod}(\mathscr{F}_{A,B})$ between the category of left t-modules in $\mathcal{K}$ and left $\mathscr{F}t$-modules in $\mathcal{L}$. If we apply this in the current setting, the induced functor is $\mathbf{Mod}(\mathrm{Hom}_{(Y,W),(X,Z)})$

$$\Big(\mathcal{V}\text{-}\mathbf{Mat}(Z,X)^{\mathcal{V}\text{-}\mathbf{Mat}(Z,C)}\Big)^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{Mat}(W,Y)^{\mathcal{V}\text{-}\mathbf{Mat}(W,B)} \longrightarrow \mathcal{V}\text{-}\mathbf{Mat}(W^Z,Y^X)^{\mathcal{V}\text{-}\mathbf{Mat}(W^Z,\mathrm{Hom}(C,B))}$$

for (C, X) a $\mathcal{V}$-cocategory and (B, Y) a $\mathcal{V}$-category, for any sets X, Y, Z, W. This is the case, because a monad in the domain category of the lax functor Hom is a pair (C, B) where C is a monad in $\mathcal{V}$-$\mathbf{Mat}^{\mathrm{co}}$, *i.e.* a comonad in $\mathcal{V}$-$\mathbf{Mat}$, and B is a monad in $\mathcal{V}$-$\mathbf{Mat}$. Also the domain of the above induced functor is isomorphic to

$$\Big((\mathcal{V}\text{-}\mathbf{Mat}^{\mathrm{co}} \times \mathcal{V}\text{-}\mathbf{Mat})((Z,W),(X,Y))\Big)^{(\mathcal{V}\text{-}\mathbf{Mat}^{\mathrm{co}} \times \mathcal{V}\text{-}\mathbf{Mat})((Z,W),(C,B))}$$

since $\mathcal{V}$-$\mathbf{Mat}^{\mathrm{co}}(Z, X) = \mathcal{V}$-$\mathbf{Mat}(Z, X)^{\mathrm{op}}$ and the category of algebras for the monad (in fact, opposite comonad) $\mathcal{V}$-$\mathbf{Mat}(Z, C)^{\mathrm{op}}$ on this category is precisely the opposite category of coalgebras

$$\big(\mathcal{V}\text{-}\mathbf{Mat}(Z, X)^{\mathcal{V}\text{-}\mathbf{Mat}(Z,C)}\big)^{\mathrm{op}}.$$

In particular, if we choose $Z{=}W{=}1$ to be the singleton set, we obtain the functor

$$\mathbf{Mod}(\mathrm{Hom}_{(1,1),(X,Y)}) \colon \Big(\mathcal{V}\text{-}\mathbf{Mat}(1,X)^{(\text{-}\circ C)}\Big)^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{Mat}(1,Y)^{(\text{-}\circ B)} \to \mathcal{V}\text{-}\mathbf{Mat}(1,Y^X)^{(\text{-}\circ \mathrm{Hom}(C,B))}$$

where the 'pre-composition' monads and comonads are just the endofunctors $\mathcal{V}$-$\mathbf{Mat}(1, C)$, $\mathcal{V}$-$\mathbf{Mat}(1, B)$ and $\mathcal{V}$-$\mathbf{Mat}(1, \mathrm{Hom}(C, B))$ respectively. We denote this functor by

$$\begin{tikzcd} \bar{K}_{(X,Y)} : \mathcal{V}\text{-}{}_{\mathcal{C}}\mathbf{Comod}^{\mathrm{op}} \times \mathcal{V}\text{-}{}_{\mathcal{B}}\mathbf{Mod} \arrow[r] & \mathcal{V}\text{-}{}_{\mathrm{Hom}(\mathcal{C},\mathcal{B})}\mathbf{Mod} \\ (\, \Phi \,,\, \Psi \,) \arrow[r, mapsto] & \mathrm{Hom}(\Phi, \Psi) \end{tikzcd}$$

using Definitions 7.6.1 and 7.6.2 for the categories involved. This concretely means that whenever Φ is a left $\mathcal{C}_X$-comodule and Ψ is a left $\mathcal{B}_Y$-module, the $\mathcal{V}$-matrix

$$\mathrm{Hom}(\Phi_{\mathcal{C}}, \Psi_{\mathcal{B}}) : 1 \relbar\joinrel\mapstochar\joinrel\rightarrow Y^X$$

obtains the structure of a left $\mathrm{Hom}(\mathcal{C}, \mathcal{B})$-module, where $\mathrm{Hom}(\mathcal{C}, \mathcal{B}) : Y^X \mapstochar\rightarrow Y^X$ is a monad in $\mathcal{V}$-$\mathbf{Mat}$ as mentioned above. Explicitly, the left $\mathrm{Hom}(\mathcal{C}, \mathcal{B})$-action

$$\mu_s : \sum_{t \in Y^X} \mathrm{Hom}(\mathcal{C}, \mathcal{B})(s, t) \otimes \mathrm{Hom}(\Phi, \Psi)(t) \to \mathrm{Hom}(\Phi, \Psi)(s)$$

for all $s \in Y^X$ is given by a family of arrows in $\mathcal{V}$

$$\sum_{t \in Y^X} \prod_{a,a' \in X} [\mathcal{C}(a', a), \mathcal{B}(sa', ta)] \otimes \prod_{b \in X} [\Phi(b), \Psi(tb)] \to \prod_{c \in X} [\Phi(c), \Psi(sc)]$$

which, for fixed $t \in Y^X$ and $c \in X$, corresponds bijectively under the usual tensor-hom adjunction to the composite

$$\begin{tikzcd}
\prod_{a,a'}[\mathcal{C}(a',a),\mathcal{B}(sa',ta)]\otimes\prod_b[\Phi(b),\Psi(tb)]\otimes\Phi(c) \arrow[d, "1\otimes\delta_c"'] \arrow[r, dashed] & \Psi(sc) \\
\prod_{a,a'}[\mathcal{C}(a',a),\mathcal{B}(sa',ta)]\otimes\prod_b[\Phi(b),\Psi(tb)]\otimes\sum_{c'}\mathcal{C}(c,c')\otimes\Phi(c') \arrow[d, "\wr\|"'] & \\
\sum_{c'}\prod_b[\Phi(b),\Psi(tb)]\otimes\mathcal{C}(c,c')\otimes\prod_b[\Phi(b),\Psi(tb)]\otimes\Phi(c') \arrow[d, "\sum\pi_{c,c'}\otimes 1\otimes\pi_{c'}\otimes 1"'] & \\
\sum_{c'}[\mathcal{C}(c,c'),\mathcal{B}(sc,tc')]\otimes\mathcal{C}(c,c')\otimes[\Phi(c'),\Psi(tc')]\otimes\Phi(c') \arrow[r, "\sum\mathrm{ev}\otimes\mathrm{ev}"'] & \sum_{c'}\mathcal{B}(sc,tc')\otimes\Psi(tc') \arrow[uuu, "\mu_{sc}"']
\end{tikzcd}$$

where δ is the left $\mathcal{C}$-coaction on Φ and μ is the left $\mathcal{A}$-action on Φ. Notice that for this formula to work, both the $\mathcal{V}$-module and the $\mathcal{V}$-comodule have to be left-sided. Also, by Proposition 2.2.10 again, this induced functor between the categories of modules is by construction such that the diagram

$$\begin{tikzcd}
\mathcal{V}\text{-}{}_{\mathcal{C}}\mathbf{Comod}^{\mathrm{op}}\times\mathcal{V}\text{-}{}_{\mathcal{B}}\mathbf{Mod} \arrow[r, "\bar{K}_{(X,Y)}"] \arrow[d] & \mathcal{V}\text{-}{}_{\mathrm{Hom}(\mathcal{C},\mathcal{B})}\mathbf{Mod} \arrow[d] \\
\mathcal{V}\text{-}\mathbf{Mat}(1,X)^{\mathrm{op}}\times\mathcal{V}\text{-}\mathbf{Mat}(1,Y) \arrow[r, "\mathrm{Hom}_{(1,1),(X,Y)}"'] & \mathcal{V}\text{-}\mathbf{Mat}(1,Y^X).
\end{tikzcd} \tag{7.47}$$

commutes. The left and right arrows are the respective monadic forgetful functors from the categories of algebras to the base categories, for $\mathcal{C}_X$ a $\mathcal{V}$-cocategory and $\mathcal{B}_Y$ a $\mathcal{V}$-category.

As done earlier for the functor K (7.32), we can now define a functor between the global categories of left $\mathcal{V}$-modules and $\mathcal{V}$-comodules

$$\bar{K}:\mathcal{V}\text{-}\mathbf{Comod}^{\mathrm{op}}\times\mathcal{V}\text{-}\mathbf{Mod}\longrightarrow\mathcal{V}\text{-}\mathbf{Mod} \tag{7.48}$$

given by $\bar{K}_{(X,Y)}$ on objects. For any left $\mathcal{V}$-module morphism $\kappa_F:\Psi_{\mathcal{B}}\to\Psi'_{\mathcal{B}'}$ and left $\mathcal{V}$-comodule morphism $\nu_G:\Phi'_{\mathcal{C}'}\to\Phi_{\mathcal{C}}$ as in Definition 7.6.4, define a morphism

$$\bar{K}(\nu,\kappa):\mathrm{Hom}(\Phi,\Psi)_{\mathrm{Hom}(\mathcal{C},\mathcal{B})}\longrightarrow\mathrm{Hom}(\Phi',\Psi')_{\mathrm{Hom}(\mathcal{C}',\mathcal{B}')}$$

in the global category $\mathcal{V}$-**Mod** as follows: it consists of the $\mathcal{V}$-functor

$$K(G,F)_{f^g}:\mathrm{Hom}(\mathcal{C},\mathcal{B})_{Y^X}\to\mathrm{Hom}(\mathcal{C}',\mathcal{B}')_{Y'^{X'}}$$

between the $\mathcal{V}$-categories which act on the $\mathcal{V}$-modules, and the family of arrows

$$\bar{K}(\nu,\kappa)_s:\mathrm{Hom}(\Phi,\Psi)(s)\longrightarrow\mathrm{Hom}(\Phi',\Psi')(f^g(s))\equiv\prod_x[\Phi(x),\Psi(sx)]\longrightarrow\prod_{x'}[\Phi'(x'),\Psi'(fsgx')]$$

which correspond uniquely, for a fixed $x'\in X$, to the composite morphism

$$\begin{tikzcd}
\prod_{x\in X}[\Phi(x),\Psi(sx)]\otimes\Phi'(x') \arrow[d, "\nu_{x'}"'] \arrow[rr, dashed] & & \Psi'(fsgx') \\
\prod_{x\in X}[\Phi(x),\Psi(sx)]\otimes\Phi(gx') \arrow[r, "\pi_{gx'}\otimes 1"'] & {[\Phi(gx'),\Psi(sgx')]\otimes\Phi(gx')} \arrow[r, "\mathrm{ev}"'] & \Psi(sgx') \arrow[u, "\kappa_{sgx'}"']
\end{tikzcd}$$

under the tensor-hom adjunction in the monoidal closed $\mathcal{V}$. It can be verified via computations that these arrows satisfy the commutativity of (7.44) thus $\bar{K}(\nu,\kappa)$ is a well-defined $\mathcal{V}$-module morphism.

Following the same approach as for earlier results, we would now like to exhibit this functor $\bar{K}$ as an action, whose adjoint will induce the suggested enrichment. Before we continue in this direction, we introduce a category whose properties will further clarify the current setting. In fact, the following structure serves very similar purposes as $\mathcal{V}$-graphs, which were used as the 'base case' for $\mathcal{V}$-**Cat** and $\mathcal{V}$-**Cocat**. If we conceive of $\mathcal{V}$-**Grph** as the category of all endo-1-cells of the bicategory $\mathcal{V}$-**Mat**, the following is the category of all 1-cells with fixed domain the singleton set 1.

Consider a category $\mathcal{C}$ with objects all $\mathcal{V}$-matrices of the form $S : 1 \nrightarrow X$ for any set X, *i.e.* families of objects $\{S(x)\}_{x\in X}$ in $\mathcal{V}$, where a morphism ν from S with codomain X to T with codomain Y

$$\nu_f : \ (1 \xrightarrow{S} X\ ) \to \ (1 \xrightarrow{T} Y\ )$$

consists of a function $f : X \to Y$ and arrows $\nu_x : S(x) \to T(fx)$ in $\mathcal{V}$ for all $x \in X$. Moreover, this category is in fact bifibred over **Set**, with reindexing functors those used in Propositions 7.6.9 and 7.6.10. However, this fact is not fundamental at this point since the (op)fibrations N and H have already been established, so details are not provided.

Under this section's assumptions on $\mathcal{V}$, the category $\mathcal{C}$ is a symmetric monoidal category, with the family of objects in $\mathcal{V}$

$$\{(S\otimes T)(x,y)\}_{(x,y)\in X\times Y} = \{S(x)\otimes T(x)\}_{\substack{x\in X\\ y\in Y}}$$

determining the tensor product $S\otimes T : 1 \nrightarrow X \times Y$ of $\mathcal{V}$-matrices S and T with codomains X and Y accordingly. In a sense, this is where the tensor products of $\mathcal{V}$-**Mod** and $\mathcal{V}$-**Comod** come from. Moreover, $\mathcal{C}$ is a monoidal closed category: for all $\mathcal{V}$-matrices S, T and R with codomains X, Y and Z respectively, there is a bijective correspondence between arrows

$$\begin{array}{ccc} (S\otimes T)_{X\times Y} \xrightarrow{\qquad\qquad} R_Z & \qquad & \text{in } \mathcal{C} \\ \hline S_X \xrightarrow{\qquad\quad} \mathrm{Hom}(T,R)_{Z^Y} & \qquad & \text{in } \mathcal{C} \end{array}$$

where $\mathrm{Hom}(T,R)$ is the mapping on objects of the functor $\mathrm{Hom}_{(1,1),(Y,Z)}$ as in (7.6). Indeed, any arrow $\kappa : S\otimes T \to M$ in $\mathcal{C}$, given by a function $f : X\times Y \to Z$ and arrows $\kappa_{(x,y)} : S(x)\otimes T(y) \to R(f(x,y))$ in $\mathcal{V}$ corresponds bijectively, under the tensor-hom adjunction in $\mathcal{V}$, to

$$S(x) \to [T(y), R(f(x,y))]$$

for all $x\in X$, $y \in Y$. Having in mind that by cartesian closedness in **Set**, $f(x,y) = \bar{f}_x(y)$ for the corresponding function $\bar{f} : X \to Z^Y$, the above is a family of arrows

$$S(x) \to \prod_{y\in Y}[T(x), R(\bar{f}_x y)]$$

for all $x \in X$, which together with $\bar{f}$ uniquely determine an arrow $S \to \mathrm{Hom}(T, R)$ in $\mathcal{C}$ as expected. This is natural in S, therefore $\mathrm{Hom}_{(1,1),(Y,Z)}(T, -)$ is the object function of a right adjoint of $- \otimes T$ which induces a functor of two variables

$$^c\mathrm{Hom}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{C}$$

namely the internal hom of $\mathcal{C}$. This is obviously very similar to the proof of Proposition 7.2.3. It is also evident that $\mathcal{V}$ has all small limits, and the proof is almost identical with that of completence of $\mathcal{V}$-**Grph** in Section 7.2.

Notice that the global categories $\mathcal{V}$-**Mod** and $\mathcal{V}$-**Comod** are (non-full) subcategories of this $\mathcal{C}$, like $\mathcal{V}$-**Cat** and $\mathcal{V}$-**Cocat** were subcategories of $\mathcal{V}$-**Grph**. Their objects are objects of $\mathcal{C}$ with extra structure. In particular, the functor $\bar{K}$ defined earlier is a restriction of $^c\mathrm{Hom}(-,-)$ to the appropriate subcategory of $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$.

We are now going to employ this category $\mathcal{C}$ in order to obtain comonadicity of $\mathcal{V}$-**Comod** and monadicity of $\mathcal{V}$-**Mod** over appropriate categories, similarly to Propositions 6.2.3 and 6.2.5 of the previous chapter.

Proposition 7.7.1. *The global category of $\mathcal{V}$-modules is monadic over the pullback category $\mathcal{C} \times_{\mathbf{Set}} \mathcal{V}$-**Cat** and the global category of $\mathcal{V}$-comodules is comonadic over the pullback category $\mathcal{C} \times_{\mathbf{Set}} \mathcal{V}$-**Cocat**.*

Proof. Consider the functor

$$\begin{tikzcd}
U : \mathcal{V}\text{-}\mathbf{Mod} \arrow[r] & \mathcal{C} \times_{\mathbf{Set}} \mathcal{V}\text{-}\mathbf{Cat} \\
\Psi_{\mathcal{A}} \arrow[r, dotted] \arrow[d, "\kappa_F"'] & (\Psi_X\ ,\ \mathcal{A}_X) \arrow[d, "(\kappa_f, F_f)"] \\
\Xi_{\mathcal{B}} \arrow[r, dotted] & (\Xi_Y\ ,\ \mathcal{B}_Y)
\end{tikzcd}$$

which 'separates' the $\mathcal{V}$-matrix with domain 1 from the $\mathcal{V}$-category which acts on it. This is well-defined: the pullback category is formed as in

$$\begin{tikzcd}
\mathcal{C} \times_{\mathbf{Set}} \mathcal{V}\text{-}\mathbf{Cat} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{V}\text{-}\mathbf{Cat} \arrow[d] \\
\mathcal{C} \arrow[r] & \mathbf{Set},
\end{tikzcd}$$

where the right edge is the fibration P which maps any $\mathcal{V}$-category to its set of objects, and the bottom edge is the bifibration which maps a $\mathcal{V}$-matrix with fixed domain 1 to its codomain. We will now construct a left adjoint to U, and the category of algebras for the induced monad will turn out to be $\mathcal{V}$-**Mod**. Define

$$\begin{tikzcd}
G : \mathcal{C} \times_{\mathbf{Set}} \mathcal{V}\text{-}\mathbf{Cat} \arrow[r] & \mathcal{V}\text{-}\mathbf{Mod} \\
(S_X, \mathcal{A}_X) \arrow[r, dotted] \arrow[d, "(\nu_f, F_f)"'] & (A \circ S)_{\mathcal{A}} \arrow[d] \\
(T_Y, \mathcal{B}_Y) \arrow[r, dotted] & (B \circ T)_{\mathcal{B}}
\end{tikzcd}$$

where the composite $\mathcal{V}$-matrix $1 \overset{S}{\twoheadrightarrow} X \overset{A}{\twoheadrightarrow} X$ obtains a left $\mathcal{A}$-action via multiplication of the monad A, and the image of the morphism between the two left $\mathcal{V}$-modules consists of the $\mathcal{V}$-functor $F_f : \mathcal{A}_X \to \mathcal{B}_Y$ and the family $\kappa_x : (A\circ S)(x) \to (B\circ T)(fx)$ of the composite arrows in $\mathcal{V}$

$$\begin{tikzcd}
\sum\limits_{x'\in X} A(x,x')\otimes S(x') \arrow[r, dashed] \arrow[d, "\sum F_{x,x'}\otimes\nu_{x'}"'] & \sum\limits_{y'\in Y} B(fx,y')\otimes T(y'). \\
\sum\limits_{x'\in X} B(fx,fx')\otimes S(fx') \arrow[ur, hook, "\iota"'] &
\end{tikzcd}$$

The above is possible only because ν and F have the same 'underlying function' f between the 'underlying sets' X and Y of the enriched modules and the enriched categories, since they determine an arrow in the specific pullback category. This morphism κ_F commutes with the $\mathcal{A}$-action and $\mathcal{B}$-action on $(A\circ S)$ and $(B\circ T)$ respectively, since $\mathcal{F}_f$ respects the composition laws of $\mathcal{A}$ and $\mathcal{B}$ which induce the actions. Now, there is a bijective correspondence between the hom-sets

$$\mathcal{V}\text{-}\mathbf{Mod}(G(S,\mathcal{A}_X),\Xi_{\mathcal{B}}) \cong (\mathcal{C}\times_{\mathbf{Set}} \mathcal{V}\text{-}\mathbf{Cat})((S,\mathcal{A}_X),U(\Xi,\mathcal{B}_Y))$$

for any left $\mathcal{B}_Y$-module Ξ, $\mathcal{V}$-matrix $S : 1 \twoheadrightarrow X$ and $\mathcal{V}$-category $\mathcal{A}_X$, as follows.

(i) Given a left $\mathcal{V}$-module morphism $\kappa_F : (A\circ S)_{\mathcal{A}} \to \Xi_{\mathcal{B}}$ with $\mathcal{V}$-functor F_f and arrows $\kappa_x : \sum_{x'} A(x,x')\otimes S(x') \to \Xi(fx)$ in $\mathcal{V}$, we can form a pair of morphisms $(\nu_f : S \to \Xi, F_f)$ in the pullback category, where ν in $\mathcal{C}$ is given by the function $f : X \to Y$ and the composite arrows in $\mathcal{V}$

$$\nu_x : S(x) \cong I\otimes S(x) \xrightarrow{\eta_x\otimes 1} A(x,x)\otimes S(x) \xrightarrow{\iota} \sum_{x'\in X} A(x,x')\otimes S(x') \xrightarrow{\kappa_x} \Xi(fx)$$

where η is the unit of the monad (A,X).

(ii) Given a pair of morphisms (σ_f, F_f) in the pullback, where

$$\sigma :\ (1 \overset{S}{\twoheadrightarrow} X\ ) \to\ (1 \overset{\Xi}{\twoheadrightarrow} Y\ )$$

with function $f : X \to Y$ and arrows $\sigma_x : S(x) \to \Xi(fx)$ in $\mathcal{V}$ is a morphism in $\mathcal{C}$, and $F_f : \mathcal{A}_X \to \mathcal{B}_Y$ is a $\mathcal{V}$-functor, we can form a left $\mathcal{V}$-module morphism $(A\circ S)_{\mathcal{A}} \to \Xi_{\mathcal{B}}$ with the same $\mathcal{V}$-functor F_f and family of arrows

$$\sum_{x\in x} A(x,x')\otimes S(x') \xrightarrow{\sum F_{x,x'}\otimes\sigma_{x'}} \sum_{x'\in X} B(fx,fx')\otimes\Xi(fx') \xrightarrow{\mu^{\Xi}_{fx}} \Xi(fx).$$

These two directions are inverse to each other, due to properties of the arrows involved, and also the bijection is natural, thus we established an adjunction

$$\begin{tikzcd}
\mathcal{C}\times_{\mathbf{Set}} \mathcal{V}\text{-}\mathbf{Cat} \arrow[r, bend left, "G"] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathbf{Mod} \arrow[l, bend left, "U"]
\end{tikzcd}$$

which gives rise to a monad $(GU, U\varepsilon_G, \eta)$ on $\mathcal{C}\times_{\mathbf{Set}} \mathcal{V}\text{-}\mathbf{Cat}$. The GU-algebras are precisely left $\mathcal{V}$-modules, since by definition they are objects in $\mathcal{V}\text{-}{}_{\mathcal{A}}\mathbf{Mod}$ for each different $\mathcal{V}$-category $\mathcal{A}$, and the diagram that a morphism between GU-algebras has to satisfy coincides with (7.44). Thus

$$\big(\mathcal{C} \times_{\mathbf{Set}} \mathcal{V}\text{-}\mathbf{Cat}\big)^{GU} \cong \mathcal{V}\text{-}\mathbf{Mod}.$$

Dually, we can show that the forgetful functor

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Comod} \arrow[r] & \mathcal{C} \times_{\mathbf{Set}} \mathcal{V}\text{-}\mathbf{Cocat} \\
\Phi_{\mathcal{C}} \arrow[r, dotted] \arrow[d, "\nu_G"'] & (\Phi\ ,\ \mathcal{C}_X) \arrow[d, "{(\nu,G_g)}"] \\
\Omega_{\mathcal{D}} \arrow[r, dotted] & (\Omega\ ,\ \mathcal{D}_Y)
\end{tikzcd}$$

has a right adjoint, such that the induced comonad on $\mathcal{C} \times_{\mathbf{Set}} \mathcal{V}$-**Cocat** is essentially the same as the global category of $\mathcal{V}$-comodules, hence $\mathcal{V}$-**Comod** is comonadic over the pullback category. ☐

The above proposition leads to a better understanding of the structure and properties of the global categories. For example, $\mathcal{V}$-**Mod** inherits completeness from the pullback category $\mathcal{C} \times_{\mathbf{Set}} \mathcal{V}$-**Cat** when $\mathcal{V}$ is complete, and the forgetful functor to $\mathcal{V}$-**Cat** strictly preserves all limits by construction. Hence by Corollary 5.3.11, the fibration N of Proposition 7.6.9 has all fibred limits, and Proposition 5.3.9 implies that the reindexing functors

$$(F_f)^* = (f^* \circ \text{-}) : \mathcal{V}\text{-}_{\mathcal{B}}\mathbf{Mod} \to \mathcal{V}\text{-}_{\mathcal{A}}\mathbf{Mod} \tag{7.49}$$

for a $\mathcal{V}$-functor $F_f : \mathcal{A}_X \to \mathcal{B}_Y$ preserve limits between the complete fibre categories.

As a further application, the functor $\bar{K}$ (7.48) between the global categories turns out to be an action, in essence because the functors K and ${}^c\mathrm{Hom}$ are actions.

PROPOSITION 7.7.2. *The functor $\bar{K}$ between the global categories of $\mathcal{V}$-modules and $\mathcal{V}$-comodules is an action, hence its opposite functor*

$$\bar{K}^{\mathrm{op}} : \mathcal{V}\text{-}\mathbf{Comod} \times \mathcal{V}\text{-}\mathbf{Mod}^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{Mod}^{\mathrm{op}}$$

*is an action of the symmetric monoidal category $\mathcal{V}$-**Comod** on the (ordinary) category $\mathcal{V}$-$\mathbf{Mod}^{\mathrm{op}}$.*

PROOF. As seen in Section 4.3, we need natural isomorphisms with components $\bar{K}(\Phi_{\mathcal{C}} \otimes \Omega_{\mathcal{D}}, \Psi_{\mathcal{A}}) \xrightarrow{\sim} \bar{K}(\Phi_{\mathcal{C}}, \bar{K}(\Omega_{\mathcal{D}}, \Psi_{\mathcal{A}}))$ and $\bar{K}(\mathbf{1}, \Psi_{\mathcal{A}}) \xrightarrow{\sim} \Psi_{\mathcal{A}}$ for $\mathcal{V}$-comodules $\Phi_{\mathcal{C}}$, $\Omega_{\mathcal{D}}$ and $\mathcal{V}$-modules $\Psi_{\mathcal{A}}$ in the global category $\mathcal{V}$-**Mod**. By definition of the functor $\bar{K}$, these are in fact of the form

$$\mathrm{Hom}(\Phi \otimes \Omega, \Psi)_{\mathrm{Hom}(\mathcal{C}\otimes\mathcal{D},\mathcal{A})_{Z X\times Y}} \cong \mathrm{Hom}(\Phi, \mathrm{Hom}(\Omega, \Psi))_{\mathrm{Hom}(\mathcal{C},\mathrm{Hom}(\mathcal{D},\mathcal{A}))_{Z^Y X}}$$
$$\mathrm{Hom}(\mathbf{1}, \Psi)_{\mathrm{Hom}(\mathcal{I},\mathcal{A})_{X1}} \cong \Psi_{\mathcal{A}_X}$$

where Hom is given by the product (7.7) in $\mathcal{V}$. Now, the functors

$$K : \mathcal{V}\text{-}\mathbf{Cocat}^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{V}\text{-}\mathbf{Cat}$$
$${}^c\mathrm{Hom}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$$

are actions, the former by Proposition 7.4.1 and the latters as the internal hom of $\mathcal{C}$. Thus we have isomorphisms

$$\mathrm{Hom}(\mathcal{C}\otimes\mathcal{D},\mathcal{A})\cong\mathrm{Hom}(\mathcal{C},\mathrm{Hom}(\mathcal{D},\mathcal{A})),\;\;\mathrm{Hom}(\mathcal{I},\mathcal{A})\cong\mathcal{A}\;\text{ in }\;\mathcal{V}\text{-}\mathbf{Cat}$$
$$\mathrm{Hom}(\Phi\otimes\Omega,\Psi)\cong\mathrm{Hom}(\Phi,\mathrm{Hom}(\Omega,\Psi)),\;\;\mathrm{Hom}(\mathbf{1},\Psi)\cong\Psi\;\text{ in }\;\mathcal{C}$$

for the two actions (notice they have the same mapping on objects). If we place these in pairs, they form natural isomorphisms in the pullback category $\mathcal{C}\times_{\mathbf{Set}}\mathcal{V}$-$\mathbf{Cat}$ for the chosen (co)modules over (co)categories. Since the forgetful functor from $\mathcal{V}$-$\mathbf{Mod}$ is monadic, it reflects all isomorphisms so these pairs lift to the required invertible arrows in $\mathcal{V}$-$\mathbf{Mod}$. Moreover, the diagrams (4.8) commute because they do for all objects of $\mathcal{C}$ and the arrows involved are in $\mathcal{V}$-$\mathbf{Mod}$. □

We aim to establish an enrichment of $\mathcal{V}$-$\mathbf{Mod}$ in $\mathcal{V}$-$\mathbf{Comod}$ by employing the theory of actions, and in particular Theorem 4.3.3. This process is in line with the ones which led to the enrichment of $\mathbf{Mon}(\mathcal{V})$ in $\mathbf{Comon}(\mathcal{V})$ in Section 6.1, of $\mathbf{Mod}$ in $\mathbf{Comod}$ in Section 6.3 and of $\mathcal{V}$-$\mathbf{Cat}$ in $\mathcal{V}$-$\mathbf{Cocat}$ in Section 7.4. Therefore, we need to show the existence of a parametrized adjoint of the action bifunctor $\bar{K}$, which will be the enriched hom functor of the ($\mathcal{V}$-$\mathbf{Comod}$)-enriched category with underlying category $\mathcal{V}$-$\mathbf{Mod}$. The theory of fibrations and opfibrations will be again of central importance, and so we begin with some lemmas which are helpful for the application of the main Theorem 5.3.7.

Lemma 7.7.3. *The diagram*

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Comod} \arrow[r, "\bar{K}(-{,}\Psi_{\mathcal{B}})^{\mathrm{op}}"] \arrow[d, "H"'] & \mathcal{V}\text{-}\mathbf{Mod}^{\mathrm{op}} \arrow[d, "N^{\mathrm{op}}"] \\
\mathcal{V}\text{-}\mathbf{Cocat} \arrow[r, "K(-{,}\mathcal{B}_Y)^{\mathrm{op}}"'] & \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}}
\end{tikzcd}$$

exhibits the pair of functors $(\bar{K}(-,\Psi_{\mathcal{B}})^{\mathrm{op}},K(-,\mathcal{B}_Y)^{\mathrm{op}})$ *as an opfibred 1-cell between the opfibrations* H *and* N^{op}.

Proof. The fact that this diagram commutes can be easily verified. For example, we already know that

$$K(\mathcal{C}_X,\mathcal{B}_Y)=\mathrm{Hom}_{(X,Y),(X,Y)}(\mathcal{C},\mathcal{B})_{Y^X}\;\;\text{and}$$
$$\bar{K}(\Phi_{\mathcal{C}},\Psi_{\mathcal{B}})=\mathrm{Hom}_{(1,1),(X,Y)}(\Phi,\Psi)_{K(\mathcal{C}_X,\mathcal{B}_Y)}$$

by definition of the two functors, which clearly implies that the $\mathcal{V}$-category action on some $\bar{K}(\Phi,\Psi)$ is precisely $K(\mathcal{C},\mathcal{B})$ for the $\mathcal{V}$-cocategory and $\mathcal{V}$-category which act on the initial $\mathcal{V}$-comodule and module.

We now have to show that the functor $\bar{K}(-,\Psi_{\mathcal{B}})^{\mathrm{op}}$ is cocartesian, *i.e.* maps a cocartesian lifting in $\mathcal{V}$-$\mathbf{Comod}$ to a cartesian lifting in $\mathcal{V}$-$\mathbf{Mod}$, since it is contravariant. By Proposition 7.6.10, we know that H is isomorphic to the opfibration which arose via the Grothendieck construction on the pseudofunctor $\mathscr{S}$, hence the canonical cocartesian lifting $\mathrm{Cocart}(F_f, \Phi_{\mathcal{C}}) : \Phi_{\mathcal{C}} \to (F_! \Phi)_{\mathcal{D}}$ is

$$\begin{tikzcd}
\Phi \arrow[r, "1_{f_*\Phi}"] \arrow[d, dotted] & f_*\Phi \arrow[d, dotted] & \text{in } \mathcal{V}\text{-}\mathbf{Comod} \\
\mathcal{C}_X \arrow[r, "F_f"'] & \mathcal{D}_Z & \text{in } \mathcal{V}\text{-}\mathbf{Cocat}
\end{tikzcd} \tag{7.50}$$

since $\mathscr{S}(F_f) = (f_* \circ -)$ is the reindexing functor. Notice that the pair notation of objects and arrows of the Grothendieck category is again dropped, because it is clear from the diagram where each element is mapped via the opfibration.

If we apply the functor $\bar{K}(-, \Psi_{\mathcal{B}})$, we get the arrow $\bar{K}((1_{f_*\Phi}, F_f), 1)$ with domain $\mathrm{Hom}((f_*\Phi)_{\mathcal{D}}, \Psi_{\mathcal{B}})$, whereas the canonical cartesian lifting of $\mathrm{Hom}(\Phi_{\mathcal{C}}, \Psi_{\mathcal{B}})$ along the $\mathcal{V}$-functor $K(F, 1)$ is

$$\begin{tikzcd}
(Y^f)^*\mathrm{Hom}(\Phi, \Psi) \arrow[r, "1_{(Y^f)^*\mathrm{Hom}(\Phi,\Psi)}"] \arrow[d, dotted] & \mathrm{Hom}(\Phi, \Psi) \arrow[d, dotted] & \text{in } \mathcal{V}\text{-}\mathbf{Mod} \\
\mathrm{Hom}(\mathcal{D}, \mathcal{B})_{Y^Z} \arrow[r, "K(F_f,1)"'] & \mathrm{Hom}(\mathcal{C}, \mathcal{B})_{Y^X} & \text{in } \mathcal{V}\text{-}\mathbf{Cat}.
\end{tikzcd}$$

This is the case because by Proposition 7.6.9, the reindexing functor of the isomorphic fibration coming from the pseudofunctor $\mathscr{H}$ is $\mathscr{H}(G_g) = (g^* \circ -)$.

For the image of (7.50) under $\bar{K}(-, \Psi)$ to be a cartesian arrow then, we have to show that the canonical arrow between the domains of the two arrows in $\mathcal{V}$-**Mod** is a vertical isomorphism. By definition of the operations involved, the domain of the canonical cartesian lifting is a family of objects in $\mathcal{V}$

$$(Y^f)^*\mathrm{Hom}(\Phi_{\mathcal{C}}, \Psi_{\mathcal{D}})(k) = I \otimes \prod_{x \in X} [\Phi(x), \Psi(kfx)]$$

for all $k \in Y^Z$, and the domain of the image of the cocartesian arrow in $\mathcal{V}$-**Comod**

$$\begin{aligned}
\mathrm{Hom}(f_*\Phi, \Psi)(k) &= \prod_{z \in Z} [(f_*\Phi)(z), \Psi(kz)] = \prod_{z} [\sum_{x \in f^{-1}z} I \otimes \Phi(x), \Psi(kz)] \\
&\cong \prod_{\substack{z \in Z \\ z = fx}} [I \otimes \Phi(x), \Psi(kz)] = \prod_{x} [I \otimes \Phi(x), \Psi(kfx)]
\end{aligned}$$

since the internal hom maps colimits to limits on the first variable. Thus the isomorphism is

$$\prod_x [I \otimes \Phi(x), \Psi(kfx)] \xrightarrow{\prod[l,1]} \prod_x [\Phi(x), \Psi(kfx)] \xrightarrow{l^{-1}} I \otimes \prod_x [\Phi(x), \Psi(kfx)]$$

for l the left unit constraint of $\mathcal{V}$, thus $\bar{K}(-, \Psi)^{\mathrm{op}}$ is a cocartesian functor. $\square$

Lemma 7.7.4. *Suppose $\Psi_{\mathcal{B}}$ is a $\mathcal{V}$-module and $\mathcal{A}_Z$, $\mathcal{B}_Y$ are $\mathcal{V}$-categories. If $\tilde{\varepsilon}$ is the counit of the adjunction*

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Cocat} \arrow[r, shift left, "K(-{,}\mathcal{B}_Y)^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \arrow[l, shift left, "T(-{,}\mathcal{B}_Y)"]
\end{tikzcd}$$

which defines the generalized Sweedler hom functor T, the composite functor

$$\mathcal{V}\text{-}\mathbf{Comod}_{T(\mathcal{A},\mathcal{B})} \xrightarrow{\bar{K}(-,\Psi)^{\mathrm{op}}} \mathcal{V}\text{-}\mathbf{Mod}^{\mathrm{op}}_{K(T(\mathcal{A},\mathcal{B}),\mathcal{B})} \xrightarrow{(\tilde{\varepsilon}_{\mathcal{A}})_!} \mathcal{V}\text{-}\mathbf{Mod}^{\mathrm{op}}_{\mathcal{A}} \tag{7.51}$$

has a right adjoint $\bar{T}_0(-,\Psi_{\mathcal{B}})$.

Proof. By Proposition 7.4.5, the functor T was defined as the parametrized adjoint of K^{op} and was retrieved by Proposition 7.5.9, where it was also shown that the underlying set of objects of the $\mathcal{V}$-cocategory $T(\mathcal{A}_X, \mathcal{B}_Y)$ is Y^X. The composite in question consists of functors between fibre categories, and we can view as

$$\begin{tikzcd}
\mathcal{V}\text{-}_{S(\mathcal{A},\mathcal{B})}\mathbf{Comod} \arrow[r, "\mathbf{Mod}(\mathrm{Hom}_{(1,1),(Y^Z,Y)})^{\mathrm{op}}"] \arrow[dr, dashed] & \mathcal{V}\text{-}_{K(S(\mathcal{A},\mathcal{B}),\mathcal{B})}\mathbf{Mod}^{\mathrm{op}} \arrow[d, "\mathscr{H}^{\mathrm{op}}(\tilde{\varepsilon}_{\varepsilon})"] \\
 & \mathcal{V}\text{-}_{\mathcal{A}}\mathbf{Mod}^{\mathrm{op}}
\end{tikzcd}$$

where $\varepsilon : Y^{Y^Z} \to Z$ is the counit of the exponential adjunction (7.42). The functor $\mathbf{Mod}(\mathrm{Hom})$ is continuous by the commutative diagram (7.47) for a fixed variable, and so is the reindexing functor $(\tilde{\varepsilon}_{\varepsilon})^*$ as in (7.49). Therefore the above composite of the opposite functors is cocontinuous. Since $\mathcal{V}\text{-}_{T(\mathcal{A},\mathcal{B})}\mathbf{Comod}$ is a locally presentable category by Proposition 7.6.3, it is cocomplete and it has a small dense subcategory. Thus, the cocontinuous composite (7.51) has a right adjoint. □

All conditions of Lemma 5.3.6 are now satisfied, hence the existence of a parametrized adjoint of $\bar{K}$ (more precisely, of its opposite functor) can be established as follows.

Proposition 7.7.5. *The functor $\bar{K}^{\mathrm{op}} : \mathcal{V}\text{-}\mathbf{Comod} \times \mathcal{V}\text{-}\mathbf{Mod}^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{Mod}^{\mathrm{op}}$ has a parametrized adjoint*

$$\bar{T} : \mathcal{V}\text{-}\mathbf{Mod}^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{Mod} \to \mathcal{V}\text{-}\mathbf{Comod} \tag{7.52}$$

which makes the following diagram of categories and functors serially commute:

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Comod} \arrow[r, shift left, "\bar{K}(-{,}\Psi_{\mathcal{B}})^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] \arrow[d, "H"'] & \mathcal{V}\text{-}\mathbf{Mod}^{\mathrm{op}} \arrow[l, shift left, "\bar{T}(-{,}\Psi_{\mathcal{B}})"] \arrow[d, "N^{\mathrm{op}}"] \\
\mathcal{V}\text{-}\mathbf{Cocat} \arrow[r, shift left, "K(-{,}\mathcal{B}_Y)^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathbf{Mod}^{\mathrm{op}}. \arrow[l, shift left, "T(-{,}\mathcal{B}_Y)"]
\end{tikzcd} \tag{7.53}$$

Proof. We have an opfibred 1-cell $(\bar{K}(-,\Psi_\mathcal{B})^{\mathrm{op}}, K(-,\mathcal{B}_Y)^{\mathrm{op}})$ between the opfibrations H and N^{op} by Lemma 7.7.3, and an adjunction (7.43) between the base categories of the opfibrations. Also, by Lemma 7.7.4, we have an adjunction

$$\begin{tikzcd}\mathcal{V}\text{-}_{T(\mathcal{A},\mathcal{B})}\mathbf{Comod} \arrow[r, shift left, "(\tilde{\varepsilon}_\mathcal{A})_!\circ\bar{K}^{\mathrm{op}}_{T(\mathcal{A},\mathcal{B})}(-,\Psi)"] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}_\mathcal{A}\mathbf{Mod}^{\mathrm{op}} \arrow[l, shift left, "\bar{T}_\mathcal{A}(-,\Psi)"]\end{tikzcd}$$

for any $\mathcal{V}$-category $\mathcal{A}$. By Theorem 5.3.7, these data suffice for the existence of a right adjoint

$$\bar{T}(-,\Psi_\mathcal{B}) : \mathcal{V}\text{-}\mathbf{Comod} \to \mathcal{V}\text{-}\mathbf{Mod}^{\mathrm{op}}$$

of the functor $\bar{K}^{\mathrm{op}}(-,\Psi_\mathcal{B})$ between the total categories of the opfibrations, with $\bar{T}_\mathcal{A}(-,\Psi)$ its mapping on objects. By construction of this adjoint, the opfibrations H and N^{op} constitute a map of adjunctions, thus (7.53) is an adjunction in $\mathbf{Cat}^2$. Moreover, since we have adjunctions $\bar{K}^{\mathrm{op}}(-,\Psi) \dashv \bar{T}(-,\Psi)$ for all left $\mathcal{B}_Y$-modules Ψ, there is a unique way to make $\bar{T}$ into a functor of two variables as in (7.52). This determines a parametrized adjoint of $\bar{K}^{\mathrm{op}}$ and the proof is complete. $\square$

Notice that by construction of $\bar{T}$, the $\mathcal{V}$-comodule $\bar{T}(\Omega_\mathcal{A},\Psi_\mathcal{B})$ is a $\mathcal{V}$-matrix with codomain the set Y^X, and a left $T(\mathcal{A}_X,\mathcal{B}_Y)$-action. This object evidently generalizes the universal measuring comodule of Proposition 6.3.1.

Using a similar series of arguments, we can also deduce that the global category of enriched comodules is a monoidal closed category, under assumptions which allow the category of $\mathcal{V}$-cocategories to be monoidal closed.

Proposition 7.7.6. *Suppose that $\mathcal{V}$ is a locally presentable symmetric monoidal closed category. The global category of left $\mathcal{V}$-comodules* $\mathcal{V}$-**Comod** *is a monoidal closed category too.*

Proof. We saw in the previous section how the global categories of modules and comodules are (symmetric) monoidal when $\mathcal{V}$ is. We are now going to use Lemma 5.3.6 once again, in order to obtain a right adjoint for the tensor product endofunctor $-\otimes\Phi_\mathcal{C}$ on $\mathcal{V}$-**Comod**.

By Proposition 7.4.4, the category $\mathcal{V}$-**Cocat** is also a symmetric monoidal closed category when $\mathcal{V}$ is, and its internal hom is denoted by ${}^g\textsc{Hom}$. Hence there is a square

$$\begin{tikzcd}\mathcal{V}\text{-}\mathbf{Comod} \arrow[r, "-\otimes\Phi_\mathcal{C}"] \arrow[d, "H"'] & \mathcal{V}\text{-}\mathbf{Comod} \arrow[d, "H"] \\ \mathcal{V}\text{-}\mathbf{Cocat} \arrow[r, "-\otimes\mathcal{C}_X"'] & \mathcal{V}\text{-}\mathbf{Cocat}\end{tikzcd} \tag{7.54}$$

which commutes by definition of the monoidal structure of $\mathcal{V}$-**Comod**, and also an adjunction between the base categories

$$\begin{tikzcd}\mathcal{V}\text{-}\mathbf{Cocat} \arrow[r, shift left, "(-\otimes\mathcal{C}_X)"] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathbf{Cocat} \arrow[l, shift left, "{}^g\textsc{Hom}(\mathcal{C}_X{,}-)"]\end{tikzcd}$$

as in (7.35). Moreover, the functor $(-\otimes\Phi_\mathcal{C})$ is cocartesian: it maps a cocartesian lifting to the top arrow of the triangle

$$\begin{tikzcd}
\Omega \arrow[r, "\mathrm{Cocart}(F{,}\Omega)"] \arrow[dd, dotted, no head] & F_!\Omega \arrow[dd, dotted, no head] & & \Omega\otimes\Phi \arrow[r, "\mathrm{Cocart}(F{,}\Omega)\otimes 1"] \arrow[dr, "\mathrm{Cocart}(F\otimes 1{,}\Omega\otimes\Phi)"'] \arrow[dd, dotted, no head] & F_!\Omega\otimes\Phi & \text{in } \mathcal{V}\text{-}\mathbf{Comod} \\
& & \mapsto & & (F\otimes 1)_!(\Omega\otimes\Phi) \arrow[u, dashed, "\exists!\gamma"'] \arrow[d, dotted, no head] & \\
\mathcal{D}_Y \arrow[r, "F_f"'] & \mathcal{E}_Z & & (\mathcal{D}\otimes\mathcal{C})_{Y\times X} \arrow[r, "(F\otimes 1)_{f\times 1}"'] & (\mathcal{E}\otimes\mathcal{C})_{Z\times X} & \text{in } \mathcal{V}\text{-}\mathbf{Cocat}
\end{tikzcd}$$

for any left $\mathcal{D}_Y$-comodule Ω. By Proposition 7.6.10, the reindexing functor $(F_f)_!$ for a $\mathcal{V}$-cofunctor with underlying function on objects f is given by post-composition with the induced $\mathcal{V}$-matrix f^*, *i.e.* $(F_f)_! = (f_*\circ\text{-})$. Now, the two $\mathcal{V}$-matrices

$$(f_*\circ\Omega)\otimes\Phi,\ (f\times 1)_*\circ(\Omega\otimes\Phi):\ 1 \mathrel{\longrightarrow\!\!\!\!\!\!\mapstochar\;\;\;} Z\times X$$

in $\mathcal{V}$-${}_{\mathcal{E}\otimes\mathcal{C}}$**Comod** are isomorphic: they are given by the families of objects in $\mathcal{V}$

$$\big((f^*\circ\Omega)\otimes\Phi\big)(z,x) = (f^*\circ\Omega)(z)\otimes\Phi(x) = \Big(\sum_{z=fy} I\otimes\Omega(y)\Big)\otimes\Phi(x)$$

$$(f\times 1)^*\circ(\Omega\otimes\Phi)(z,x) = \sum_{z=fy} I\otimes(\Omega\otimes\Phi)(y,x) = \sum_{z=fy} I\otimes\big(\Omega(y)\otimes\Phi(x)\big)$$

so the isomorphism between them is given by the fact that $\otimes$ commutes with sums. Furthermore the above triangle commutes, so the square (7.54) exhibits $(-\otimes\Phi_\mathcal{C}, -\otimes\mathcal{C}_X)$ as an opfibred 1-cell between H and H. Finally, if $\bar{\varepsilon}$ is the counit of the adjunction (7.35) which defines the internal hom ${}^g\textsc{Hom}$ for $\mathcal{V}$-cocategories, the composite functor between the fibres

$$\mathcal{V}\text{-}\mathbf{Comod}_{{}^g\textsc{Hom}(\mathcal{C},\mathcal{D})} \xrightarrow{-\otimes\Phi_\mathcal{C}} \mathcal{V}\text{-}\mathbf{Comod}_{{}^g\textsc{Hom}(\mathcal{C},\mathcal{D})\otimes\mathcal{C}} \xrightarrow{(\bar{\varepsilon}_\mathcal{D})_!} \mathcal{V}\text{-}\mathbf{Comod}_\mathcal{D}$$

has a right adjoint, call it ${}^g\overline{\textsc{Hom}}_\mathcal{D}(\Phi_\mathcal{C},-)$. This is because the category of left ${}^g\textsc{Hom}(\mathcal{C},\mathcal{D})$-comodules is locally presentable by Proposition 7.6.3, $(\bar{\varepsilon}_\mathcal{D})_!$ is cocontinuous because it is composition of $\mathcal{V}$-matrices, and $(-\otimes\Phi_\mathcal{C})$ is cocontinuous by the commutative diagram

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Comod} \arrow[r, "-\otimes\Phi_\mathcal{C}"] \arrow[d] & \mathcal{V}\text{-}\mathbf{Comod} \arrow[d] \\
\mathcal{C}\times_{\mathbf{Set}}\mathcal{V}\text{-}\mathbf{Cocat} \arrow[r, "(-\otimes\Phi)\times(-\otimes\mathcal{C}_X)"'] & \mathcal{C}\times_{\mathbf{Set}}\mathcal{V}\text{-}\mathbf{Cocat}.
\end{tikzcd}$$

Therefore we have an adjunction $(-\otimes\Phi_\mathcal{C})\dashv{}^g\overline{\textsc{Hom}}(\Phi_\mathcal{C},-)$ between the total categories for all $\mathcal{V}$-comodules $\Phi_\mathcal{C}$, exhibiting the induced bifunctor

$${}^g\overline{\textsc{Hom}}:\mathcal{V}\text{-}\mathbf{Comod}^{\mathrm{op}}\times\mathcal{V}\text{-}\mathbf{Comod}\to\mathcal{V}\text{-}\mathbf{Comod}$$

as the internal hom of $\mathcal{V}$-**Comod**. Also, ${}^g\overline{\textsc{Hom}}(\Phi_\mathcal{C},\Omega_\mathcal{D})$ is a ${}^g\textsc{Hom}(\mathcal{C},\mathcal{D})$-comodule. □

Consequently, we can now apply Corollaries 4.3.4 and 4.3.5 for the action $\bar{K}^{\mathrm{op}}$ of the symmetric monoidal closed category $\mathcal{V}$-**Comod** on the ordinary category $\mathcal{V}$-$\mathbf{Mod}^{\mathrm{op}}$ and obtain the pursued enrichment.

THEOREM 7.7.7. *Suppose that $\mathcal{V}$ is a locally presentable, symmetric monoidal closed category.*

1. *(1) The opposite of the global category of left $\mathcal{V}$-modules $\mathcal{V}$-$\mathbf{Mod}^{\mathrm{op}}$ is enriched in the global category of left $\mathcal{V}$-comodules $\mathcal{V}$-**Comod**, with hom-objects*

$$\mathcal{V}\text{-}\mathbf{Mod}^{\mathrm{op}}(\Psi_{\mathcal{A}}, \Xi_{\mathcal{B}}) = \bar{T}(\Xi, \Psi)_{T(\mathcal{B},\mathcal{A})}$$

   *where the ($\mathcal{V}$-**Comod**)-enriched category is denoted by the same name.*

2. *(2) The global category of left $\mathcal{V}$-modules $\mathcal{V}$-**Mod** is a cotensored ($\mathcal{V}$-**Comod**)-enriched category, with hom-objects*

$$\mathcal{V}\text{-}\mathbf{Mod}(\Psi_{\mathcal{A}}, \Xi_{\mathcal{B}}) = \bar{T}(\Psi, \Xi)_{T(\mathcal{A},\mathcal{B})}$$

   and cotensor product $\bar{K}(\Phi, \Xi)_{K(\mathcal{C},\mathcal{B})}$ for any $\mathcal{V}$-modules $\Psi_{\mathcal{A}}$, $\Xi_{\mathcal{B}}$ and $\mathcal{V}$-comodules $\Phi_{\mathcal{C}}$.

CHAPTER 8

An Abstract Framework

This last chapter is an attempt to exhibit some underlying motives of certain techniques used in the previous sections, and discuss possible generalizations of processes which resulted in the main theorems of the thesis. The previous chapter had as its clear goal to generalize the results of Chapter 6 in the next level of 'many-object' (co)monoids and (co)modules, namely $\mathcal{V}$-(co)categories and $\mathcal{V}$-(co)modules. The thorough investigation of this development reveals an intrinsic pattern of how the categories involved are expected to behave.

In the first section, the aim is to state and justify a definition of the notion of enriched fibration. More precisely, we would like to be able to characterize a (plain) fibration as being enriched in another, special kind of fibration, serving similar purposes as the monoidal base of usual enrichment of categories. There are two things that would incorporate the success of such a definition, in the frame of this thesis: firstly, the carefully examined cases of monoids/modules, enriched categories/enriched modules and dual structures should constitute examples of it, and secondly there should be a theorem which, under certain assumptions, would ensure the existence of an enriched fibration.

A first formal definition in this conceptual direction was given in [**GG76**], called 'a fibration relative to $\mathcal{A}$', where $\mathcal{A}$ was fibred over a monoidal category in an appropriate sense. As mentioned in the introduction, Shulman in [**Shu13**] develops a theory of 'enriched indexed categories', *i.e.* categories which are simultaneously indexed over a base category $\mathbb{S}$ with finite products, and also enriched in an $\mathbb{S}$-indexed monoidal category. The definition of an indexed $\mathcal{V}$-category was also given independently by Bunge in [**Bun13**]. The main issue is that even if we herein employ the same notion of a *monoidal fibration* (Definition 8.1.1), Bunge's and Shulman's approach only concerns enrichment in fibrations strictly over cartesian monoidal bases, which is not the chosen monoidal structure of, say, $\mathbf{Comon}(\mathcal{V})$ and $\mathcal{V}$-$\mathbf{Cocat}$. Moreover, the notion of an enriched indexed category refers only to a fibration enriched in another fibration over the same base, approximately depicted as

$$\begin{tikzcd}
\mathcal{A} \arrow[r, dashed, "\text{enriched}"] \arrow[dr, "\text{fibred}"'] & \mathcal{V} \arrow[d, "\text{fibred}"] \\
& \mathbb{S}.
\end{tikzcd}$$

In our examples, this is certainly not the case: we seek for enrichments between both the total and the base categories of the two fibrations involved.

In the second section of this chapter, the aim is to give an approximate description of a way in which the central results of this thesis fit into the theory of double categories. The motivation for this approach is that in the bicategory $\mathcal{V}$-**Mat**, fundamental for the development of the previous chapter, the functions f between the sets and especially the $\mathcal{V}$-matrices f_*, f^* induced by them were of importance for our constructions. This belongs to a variety of examples of bicategorical structures, where in fact two natural kinds of morphisms exist, typically some complicated ones (like $\mathcal{V}$-matrices between sets in our case) comprising the bicategory, and some more elementary ones which are discarded but in fact important. Therefore, having everything encompassed in a double category provides a conceptual advantage. Often, there is a lifting property which turns a vertical 1-cell into a horizontal 1-cell as in our situation, and this corresponds to the concept of a *fibrant double category*.

Due to the lack of machinery for dealing with double categories comparatively to bicategories or 2-categories, recently there has been some serious activity regarding the more systematic study and development of the theory of double categories. The exposition in this chapter is not meant to be a significant step in this direction, not being as rigorous or detailed as such an attempt deserves. Rather it introduces certain notions which might be of use to further research on the topic. Categories of monoids (or monads) in double categories have been methodically studied in [**FGK11**]. In the current treatment, they are combined with notions of comonoids, modules and comodules in double categories in order to exhibit a framework for the existence and properties of specific categories we dealt with in earlier chapters.

Various important facts about double categories such as detailed definitions for double functors and double natural transformations, monoidal structure, coherence for pseudo double categories and numerous examples can be found in the references provided in the introduction added to the ones mentioned later. The explicit definition of a *monoidal bicategory* can be found in [**Car95**], or in [**GPS95**] as a one-object tricategory.

8.1. Enriched fibrations

Chapters 6 and 7 were devoted to the establishment of the enrichment of certain, mostly well-studied categories like $\mathbf{Mon}(\mathcal{V})$, **Mod**, $\mathcal{V}$-**Cat** and $\mathcal{V}$-**Mod**, in their dual-flavored monoidal categories $\mathbf{Comon}(\mathcal{V})$, **Comod**, $\mathcal{V}$-**Cocat** and $\mathcal{V}$-**Comod**. Such enrichments were in fact combined, in a very natural way, with the theory of fibrations and opfibrations. The very adjunctions inducing enriched hom-functors often employed results regarding fibred functors, implying a strong relation between the two notions. Below we graphically summarize the results of the two previous chapters. The monoidal category $\mathcal{V}$ is required to be a locally presentable, symmetric monoidal closed category.

The category of monoids is enriched in the (symmetric monoidal closed) category of comonoids in $\mathcal{V}$, with enriched hom-functor the Sweedler hom

$$P:\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}\times\mathbf{Mon}(\mathcal{V})\to\mathbf{Comon}(\mathcal{V})$$

which is the parametrized adjoint of the opposite of the restricted internal hom

$$\begin{aligned} H:\ \mathbf{Comon}(\mathcal{V})^{\mathrm{op}}\times\mathbf{Mon}(\mathcal{V}) &\longrightarrow \mathbf{Mon}(\mathcal{V}) \\ (\ C\ ,\ A\ ) &\longmapsto [C,A] \end{aligned}$$

by Proposition 6.1.1 and Theorem 6.1.4. Moreover, the global category of modules is enriched in the (symmetric monoidal closed) global category of comodules in $\mathcal{V}$, with enriched hom-functor the universal measuring comodule functor

$$Q:\mathbf{Mod}^{\mathrm{op}}\times\mathbf{Mod}\to\mathbf{Comod}$$

which is the parametrized adjoint of the opposite of the further restricted

$$\begin{aligned} \bar{H}:\ \mathbf{Comod}^{\mathrm{op}}\times\mathbf{Mod} &\longrightarrow \mathbf{Mod} \\ (\ X_C\ ,\ M_A\ ) &\longmapsto [X,M]_{[C,A]} \end{aligned}$$

by Proposition 6.3.1 and Theorem 6.3.4. The diagram

$$\begin{tikzcd}
\mathbf{Mod}^{\mathrm{op}} \arrow[r, bend left, "Q(-{,}N_B)"] \arrow[r, phantom, "\top"] \arrow[d, "G^{\mathrm{op}}"'] & \mathbf{Comod} \arrow[l, bend left, "\bar{H}(-{,}N_B)^{\mathrm{op}}"] \arrow[d, "V"] \\
\mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \arrow[r, bend left, "P(-{,}B)"] \arrow[r, phantom, "\top"] & \mathbf{Comon}(\mathcal{V}) \arrow[l, bend left, "H(-{,}B)^{\mathrm{op}}"]
\end{tikzcd} \tag{8.1}$$

which describes the above situation is in fact an adjunction in the 2-category $\mathbf{Cat^2}$.

The category of $\mathcal{V}$-enriched categories is enriched in the (symmetric monoidal closed) category of $\mathcal{V}$-enriched cocategories, with enriched hom-functor the generalized Sweedler hom

$$T:\mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}}\times\mathcal{V}\text{-}\mathbf{Cat}\to\mathcal{V}\text{-}\mathbf{Cocat}$$

which is the parametrized adjoint of the opposite of the internal hom as $\mathcal{V}$-graphs

$$\begin{aligned} K:\ \mathcal{V}\text{-}\mathbf{Cocat}^{\mathrm{op}}\times\mathcal{V}\text{-}\mathbf{Cat} &\longrightarrow \mathcal{V}\text{-}\mathbf{Cat} \\ (\ \mathcal{C}_X\ ,\ \mathcal{B}_Y\ ) &\longmapsto \mathrm{Hom}(\mathcal{C},\mathcal{B})_{YX} \end{aligned}$$

defined by $\mathrm{Hom}(\mathcal{C},\mathcal{B})(k,s)=\prod_{x',x}[\mathcal{C}(x',x),\mathcal{B}(kx',sx)]$, by Proposition 7.5.9 and Theorem 7.4.6. Moreover, the global category of $\mathcal{V}$-enriched modules is enriched in the (symmetric monoidal closed) global category of $\mathcal{V}$-enriched comodules, with enriched hom-functor

$$\bar{T}:\mathcal{V}\text{-}\mathbf{Mod}^{\mathrm{op}}\times\mathcal{V}\text{-}\mathbf{Mod}\to\mathcal{V}\text{-}\mathbf{Comod}$$

which is the parametrized adjoint of the opposite of

$$\begin{aligned} \bar{K}:\ \mathcal{V}\text{-}\mathbf{Comod}^{\mathrm{op}}\times\mathcal{V}\text{-}\mathbf{Mod} &\longrightarrow \mathcal{V}\text{-}\mathbf{Mod} \\ (\ \Phi_{\mathcal{C}}\ ,\ \Psi_{\mathcal{B}}\ ) &\longmapsto \mathrm{Hom}(\Phi,\Psi)_{\mathrm{Hom}(\mathcal{C},\mathcal{B})} \end{aligned}$$

where $\mathrm{Hom}(\Phi,\Psi)(t)=\prod_x[\Phi(x),\Psi(tx)]$, by Proposition 7.7.5 and Theorem 7.7.7. The diagram

$$\begin{tikzcd}
\mathcal{V}\textbf{-Mod}^{\mathrm{op}} \arrow[r, shift left, bend left=5, "\bar{T}(-{,}\Psi_{\mathcal{B}})"] \arrow[r, phantom, "\top"] \arrow[d, "N^{\mathrm{op}}"'] & \mathcal{V}\textbf{-Comod} \arrow[l, shift left, bend left=5, "\bar{K}(-{,}\Psi_{\mathcal{B}})^{\mathrm{op}}"] \arrow[d, "H"] \\
\mathcal{V}\textbf{-Cat}^{\mathrm{op}} \arrow[r, shift left, bend left=5, "T(-{,}\mathcal{B}_Y)"] \arrow[r, phantom, "\top"] \arrow[d, "P^{\mathrm{op}}"'] & \mathcal{V}\textbf{-Cocat} \arrow[l, shift left, bend left=5, "K(-{,}\mathcal{B}_Y)^{\mathrm{op}}"] \arrow[d, "W"] \\
\textbf{Set}^{\mathrm{op}} \arrow[r, shift left, bend left=5, "Y^{(-)}"] \arrow[r, phantom, "\top"] & \textbf{Set} \arrow[l, shift left, bend left=5, "Y^{(-)\mathrm{op}}"]
\end{tikzcd} \tag{8.2}$$

depicts the above situation.

An appropriate enriched fibration notion would successfully encapsulate the rich structure of the above situations. Intuitively, we are looking for a definition which would ensure that the opfibration G^{op} is enriched in the opfibration V, and that the opfibrations N^{op} and P^{op} are enriched in the opfibrations H and W respectively.

Because of the nature of our examples, it is now evident that we are unable to employ the definitions and theory of [**Shu13**]. As mentioned earlier, the numerous examples therein restrict to fibrations (or indexed categories) over monoidal categories with tensor product the cartesian product. However, in the diagrams (8.1) and (8.2) the base categories (except **Set**) of the fibrations which we intend to use as base for enrichment are not viewed as cartesian monoidal categories. Moreover, and perhaps more importantly, the indexed enrichment (over the same base category) as stated in [**Shu13**, Definition 4.1] is conceived as 'fibrewise' enrichments between the fibres of the total categories, plus some preservation of the enriched structure via the reindexing functors. Apart from the absence of a monoidal structure on the fibre categories here, like $\mathbf{Comod}_{\mathcal{V}}(C)$, the fact that we require an enrichment between the (distinct) base categories of the fibrations makes a great difference.

Therefore, we are going to explore a new approach to this problem. The basic idea is to shift Theorem 4.3.3 from the context of categories to the context of fibrations. The reason for doing so is that this result provides an enrichment of an ordinary category in a monoidal category when certain conditions are satisfied, which can be rephrased if we replace categories by fibrations. This becomes clearer in the light of the following remarks (see also Remark 4.3.1(ii)).

- A monoidal category $(\mathcal{V},\otimes,I,a,l,r)$ is a *pseudomonoid* in the cartesian monoidal 2-category $(\mathbf{Cat},\times,\mathbf{1})$.
- An action $*$ of a monoidal category $\mathcal{V}$ on an ordinary category $\mathcal{A}$ is a *pseudoaction* of a pseudomonoid on an object of $(\mathbf{Cat},\times,\mathbf{1})$.
- A $\mathcal{V}$-representation $(\mathcal{A},*)$, *i.e.* an ordinary category on which $\mathcal{V}$ acts, is a *pseudomodule* for the pseudomonoid $\mathcal{V}$ in $(\mathbf{Cat},\times,\mathbf{1})$.

Theorem 4.3.3 and its following comments in fact give the one direction of the equivalence

$$\mathcal{V}\text{-Rep}_{\mathrm{cl}} \simeq \mathcal{V}\text{-}\mathbf{Cat}_{\otimes}$$

on the level of objects between *closed* $\mathcal{V}$-representations (*i.e.* equipped with a parametrized adjoint) and tensored $\mathcal{V}$-categories for $\mathcal{V}$ a monoidal closed category. This equivalence is in fact a special case of the more general [**GP97**, Theorem 3.7]. We would now like to produce an adjusted version of this, moving from $(\mathbf{Cat}, \times, \mathbf{1})$ to the monoidal 2-category $(\mathbf{Fib}, \times, 1_{\mathbf{1}})$, where $1_{\mathbf{1}}$ is the identity functor on the terminal category. Indeed, the 2-functor

$$\times : \mathbf{Cat} \times \mathbf{Cat} \to \mathbf{Cat}$$

which is the cartesian 2-monoidal structure on **Cat**, induces a monoidal structure on the 2-category $\mathbf{Cat}^{\mathbf{2}}$ which restricts to the sub-2-category **Fib**, since the cartesian product of two fibrations is still a fibration.

Initially, we would like to identify the pseudomonoids in this monoidal 2-category, which will be the analogue of monoidal categories. The concept of a pseudomonoid was formally defined in [**DS97**], and the more general pseudomonad viewpoint can be found in [**Mar97, Lac00**]. As an idea, a *tensor object* in [**JS93**] already captures the required structure. By applying this definition in the 2-category of fibrations, fibred 1-cells and fibred 2-cells, a *monoidal fibration* is a fibration $T : \mathcal{V} \to \mathbb{W}$ with arrows $M : T \times T \to T$, $\eta : 1 \to T$ equipped with natural isomorphisms

$$\begin{tikzcd}
T \times T \times T \arrow[r, "M\times 1"] \arrow[d, "1\times M"'] \arrow[dr, phantom, "\overset{a}{\cong}"] & T \times T \arrow[d, "M"] \\
T \times T \arrow[r, "M"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \times T \arrow[r, "\eta\times 1"] \arrow[dr, bend right=40] \arrow[dr, phantom, "\overset{l}{\cong}"] & T \times T \arrow[d, "M"] \arrow[dr, phantom, "\overset{r}{\cong}"] & T \times 1 \arrow[l, "1\times\eta"'] \arrow[dl, bend left=40] \\
 & T &
\end{tikzcd}$$

satisfying certain coherence conditions. More explicitly, there are fibred 1-cells $M = (M_{\mathcal{V}}, M_{\mathbb{W}})$, $\eta = (I_{\mathcal{V}}, I_{\mathbb{W}})$ displayed by the commutative squares

$$\begin{tikzcd}
\mathcal{V} \times \mathcal{V} \arrow[r, "M_{\mathcal{V}}"] \arrow[d, "T\times T"'] & \mathcal{V} \arrow[d, "T"] \\
\mathbb{W} \times \mathbb{W} \arrow[r, "M_{\mathbb{W}}"'] & \mathbb{W}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbf{1} \arrow[r, "I_{\mathcal{V}}"] \arrow[d, "1"'] & \mathcal{V} \arrow[d, "T"] \\
\mathbf{1} \arrow[r, "I_{\mathbb{W}}"'] & \mathbb{W}
\end{tikzcd}
\tag{8.3}$$

where the functors $M_{\mathcal{V}}$ and $I_{\mathcal{V}}$ are cartesian, and invertible fibred 2-cells $a = (a^{\mathcal{V}}, a^{\mathbb{W}})$, $r = (r^{\mathcal{V}}, r^{\mathbb{W}})$, $l = (l^{\mathcal{V}}, l^{\mathbb{W}})$ displayed as

$$\begin{tikzcd}[column sep=large]
\mathcal{V} \times \mathcal{V} \times \mathcal{V} \arrow[r, bend left, "M(M\times 1)"] \arrow[r, bend right, "M(1\times M)"'] \arrow[r, phantom, "\Downarrow a^{\mathcal{V}}"] \arrow[d, "T\times T\times T"'] & \mathcal{V} \arrow[d, "T"] \\
\mathbb{W} \times \mathbb{W} \times \mathbb{W} \arrow[r, bend left, "M(M\times 1)"] \arrow[r, bend right, "M(1\times M)"'] \arrow[r, phantom, "\Downarrow a^{\mathbb{W}}"] & \mathbb{W}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{V}\times 1 \arrow[r, bend left, "M(1\times I)"] \arrow[r, bend right, "\sim"'] \arrow[r, phantom, "\Downarrow r^{\mathcal{V}}"] \arrow[d, "T\times 1"'] & \mathcal{V} \arrow[d, "T"] \\
\mathbb{W}\times 1 \arrow[r, bend left, "M(1\times I)"] \arrow[r, bend right, "\sim"'] \arrow[r, phantom, "\Downarrow r^{\mathbb{W}}"] & \mathbb{W}
\end{tikzcd}
\qquad
\begin{tikzcd}
1\times\mathcal{V} \arrow[r, bend left, "M(I\times 1)"] \arrow[r, bend right, "\sim"'] \arrow[r, phantom, "\Downarrow l^{\mathcal{V}}"] \arrow[d, "1\times T"'] & \mathcal{V} \arrow[d, "T"] \\
1\times\mathbb{W} \arrow[r, bend left, "M(I\times 1)"] \arrow[r, bend right, "\sim"'] \arrow[r, phantom, "\Downarrow l^{\mathbb{W}}"] & \mathbb{W}.
\end{tikzcd}$$

Recall that the natural isomorphisms $a^{\mathcal{V}}, r^{\mathcal{V}}, l^{\mathcal{V}}$ lie above $a^{\mathbb{W}}, r^{\mathbb{W}}, l^{\mathbb{W}}$, by definitions in Section 5.1. The axioms that these data are required to satisfy turn out to give the usual axioms which make $(\mathcal{V}, M_{\mathcal{V}}, I_{\mathcal{V}})$ and $(\mathbb{W}, M_{\mathbb{W}}, I_{\mathbb{W}})$ into monoidal categories, with associativity, left and right unit constraints a, r, l respectively. This is due to the fact that the functors *dom*, *cod* : **Fib** → **Cat** are strict monoidal functors. In other words, the equality of pasted diagrams of 2-cells in **Fib** breaks down into equalities for the two natural transformations it consists of.

Moreover, the strict commutativity of the diagrams (8.3) imply that T preserves the tensor product and the unit object between $\mathcal{V}$ and $\mathbb{W}$ on the nose, *i.e.*

$$TA\otimes_{\mathbb{W}} TB = T(A\otimes_{\mathcal{V}} B), \quad I_{\mathbb{W}} = T(I_{\mathcal{V}})$$

if we denote $M = \otimes$. Along with the last conditions that $T(a^{\mathcal{V}}) = a^{\mathbb{W}}$, $T(l^{\mathcal{V}}) = l^{\mathbb{W}}$ and $T(r^{\mathcal{V}}) = r^{\mathbb{W}}$, these data define a strict monoidal structure on the functor T. Therefore we obtain the following definition, which coincides with [**Shu08**, 12.1].

Definition 8.1.1. A *monoidal fibration* is a fibration $T : \mathcal{V} \to \mathbb{W}$ such that

(i) $\mathcal{V}$ and $\mathbb{W}$ are monoidal categories,
(ii) T is a strict monoidal functor,
(iii) the tensor product $\otimes_{\mathcal{V}}$ of $\mathcal{V}$ preserves cartesian arrows.

In a dual way, we can define a *monoidal opfibration* to be an opfibration which is a strict monoidal functor, where the tensor product of the total category preserves cocartesian arrows. Also, if $\mathcal{V}$ and $\mathcal{W}$ are symmetric monoidal categories and T is a symmetric strict monoidal functor, call T a *symmetric monoidal fibration.*

We are now going to describe a pseudoaction of a pseudomonoid in **Fib**, and what it means for a fibration to be a pseudomodule for a monoidal fibration T. For a general 2-category or bicategory, the idea of a *pseudomodule* can be found in similar contexts in [**Mar97, Lac00**] (called (pseudo)algebra for a pseudomonad). Conceptually, as was the case for modules for monoids in a monoidal category, it arises as a pseudoalgebra for the pseudomonad $(M\otimes -)$ in our monoidal bicategory, where M is a fixed pseudomonoid.

In our case, a *pseudoaction* of a monoidal fibration $T : \mathcal{V} \to \mathbb{W}$ on an ordinary fibration $P : \mathcal{A} \to \mathbb{X}$ is a fibred 1-cell $\mu = (\mu^{\mathcal{A}}, \mu^{\mathbb{X}}) : T\times P \to P$ displayed by the commutative

$$\begin{tikzcd}
\mathcal{V}\times\mathcal{A} \arrow[r, "\mu^{\mathcal{A}}"] \arrow[d, "T\times P"'] & \mathcal{A} \arrow[d, "P"] \\
\mathbb{W}\times\mathbb{X} \arrow[r, "\mu^{\mathbb{X}}"'] & \mathbb{X}
\end{tikzcd} \tag{8.4}$$

where $\mu^{\mathcal{A}}$ is a cartesian functor, equipped with natural isomorphisms

$$\begin{tikzcd}
T\times T\times P \arrow[r, "M\times 1"] \arrow[d, "1\times\mu"'] \arrow[dr, phantom, "\overset{\chi}{\cong}"] & T\times P \arrow[d, "\mu"] \\
T\times P \arrow[r, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
1\times P \arrow[r, "\eta\times 1"] \arrow[dr, bend right, "\sim"'] \arrow[dr, phantom, "\overset{\nu}{\cong}"] & T\times P \arrow[d, "\mu"] \\
 & P
\end{tikzcd}$$

in **Fib**. These are invertible fibred natural transformations $\chi = (\chi^{\mathcal{A}}, \chi^{\mathbb{X}})$, $\nu = (\nu^{\mathcal{A}}, \nu^{\mathbb{X}})$ represented by

$$\begin{tikzcd}
 & \mathcal{V}\times\mathcal{A} \arrow[dr, "\mu"] & \\
\mathcal{V}\times\mathcal{V}\times\mathcal{A} \arrow[ur, "M\times 1"] \arrow[dr, "1\times\mu"'] \arrow[dddd, "T\times T\times P"'] & \Downarrow\chi^{\mathcal{A}} & \mathcal{A} \arrow[dddd, "P"] \\
 & \mathcal{V}\times\mathcal{A} \arrow[ur, "\mu"'] & \\
 & & \\
 & \mathbb{W}\times\mathbb{X} \arrow[dr, "\mu"] & \\
\mathbb{W}\times\mathbb{W}\times\mathbb{X} \arrow[ur, "M\times 1"] \arrow[dr, "1\times\mu"'] & \Downarrow\chi^{\mathbb{X}} & \mathbb{X} \\
 & \mathbb{W}\times\mathbb{X} \arrow[ur, "\mu"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{V}\times\mathcal{A} \arrow[dr, "\mu"] & \\
1\times\mathcal{A} \arrow[ur, "I\times 1"] \arrow[rr, bend right, "\sim"'] \arrow[dddd, "1\times P"'] & \Downarrow\nu^{\mathcal{A}} & \mathcal{A} \arrow[dddd, "P"] \\
 & & \\
 & & \\
 & \mathbb{W}\times\mathbb{X} \arrow[dr, "\mu"] & \\
1\times\mathbb{X} \arrow[ur, "I\times 1"] \arrow[rr, bend right, "\sim"'] & \Downarrow\nu^{\mathbb{X}} & \mathbb{X}
\end{tikzcd}$$

where $\chi^{\mathcal{A}}, \nu^{\mathcal{A}}$ are above $\chi^{\mathbb{X}}, \nu^{\mathbb{X}}$ with respect to the appropriate fibrations. These data are subject to certain axioms, which in fact again split up in two sets of commutative diagrams, for the components of the two natural isomorphisms that the fibred 2-cells χ and ν consist of. The resulting diagrams coincide with the ones for an action of a monoidal category (4.8).

DEFINITION 8.1.2. The fibration $P : \mathcal{A} \to \mathbb{X}$ is a *T-representation* (or a *T-module*) for a monoidal fibration $T : \mathcal{V} \to \mathbb{W}$, when it is equipped with a T-pseudoaction $\mu = (\mu^{\mathcal{A}}, \mu^{\mathbb{X}})$. This amounts to two actions

$$\mu^{\mathcal{A}} = * : \mathcal{V}\times\mathcal{A} \longrightarrow \mathcal{A}$$
$$\mu^{\mathbb{X}} = \diamond : \mathbb{W}\times\mathbb{X} \longrightarrow \mathbb{X}$$

of the monoidal categories $\mathcal{V}$, $\mathbb{W}$ on the categories $\mathcal{A}$ and $\mathbb{X}$ respectively, such that $\mu^{\mathcal{A}}$ preserves cartesian arrows and $P\chi^{\mathcal{A}}_{XYA} = \chi^{\mathbb{X}}_{(TX)(TY)(PA)}$, $P\nu^{\mathcal{A}}_{A} = \nu^{\mathbb{X}}_{PA}$ for all $X, Y \in \mathcal{V}$ and $A \in \mathcal{A}$.

The last two relations are easy to verify in specific examples. In greater detail, the commutative diagram (8.4) representing the pseudoaction implies that

$$P(X * A) = TX \diamond PA$$

for any $X \in \mathcal{V}$, $A \in \mathcal{A}$, hence the isomorphisms $\chi^{\mathcal{A}}_{XYA} : X * (Y * A) \cong (X \otimes_{\mathcal{V}} Y) * A$ lie above certain isomorphisms in $\mathbb{X}$

$$P\chi^{\mathcal{A}}_{XYA} : TX \diamond (TY \diamond PA) \xrightarrow{\sim} (TX \otimes_{\mathbb{W}} TY) \diamond PA$$

in $\mathbb{W}$, since T is strict monoidal. Similarly, $\nu^{\mathcal{A}}_{A} : I * A \cong A$ is mapped, under P, to

$$P\nu^{\mathcal{A}}_{A} : I_{\mathbb{X}} \diamond PA \xrightarrow{\sim} PA$$

since $P(I_{\mathcal{V}} * A) = T(I_{\mathcal{V}}) \diamond PA = I_{\mathbb{W}} \diamond PA$ by strict monoidality of T again. These isomorphisms then are required to coincide with the components of structure isomorphisms $\chi^{\mathbb{X}}$ and $\nu^{\mathbb{X}}$ of the $\mathbb{W}$-representation $\mathbb{X}$.

The last step in order to get a clear picture of how a modified correspondence between representations of a monoidal fibration and enriched fibrations would work, is to introduce a notion of a parametrized adjunction in **Fib**. For that, we first reformulate the 'adjunctions with a parameter' Theorem 3.0.2 in the context of $\mathbf{Cat^2}$. Even though the abstract definition of an adjunction applies to any 2-categorical, or bicategorical, setting as in Definition 2.3.5, for its appropriate parametrized version we need the 0-cells of our 2-category to be category-like themselves. Intuitively, in such cases, if we have a 1-cell with domain a product of two objects $t : A \times B \to C$, we are able to consider a 1-cell $t_a : B \to C$ by fixing an 'element' of one of the 0-cells, a in A.

Theorem 8.1.3 (Adjunctions with a parameter in $\mathbf{Cat^2}$). *Suppose we have a morphism (F, G) of two variables in $[\mathbf{2}, \mathbf{Cat}]$, given by a commutative square of categories and functors*

$$\begin{tikzcd}
\mathcal{A} \times \mathcal{B} \arrow[r, "F"] \arrow[d, "H \times J"'] & \mathcal{C} \arrow[d, "K"] \\
\mathbb{X} \times \mathbb{Y} \arrow[r, "G"'] & \mathbb{Z}.
\end{tikzcd} \tag{8.5}$$

Assume that, for every $B \in \mathcal{B}$ and $Y \in \mathbb{Y}$, there exist adjunctions $F(-, B) \dashv R(B, -)$ and $G(-, Y) \dashv S(Y, -)$, such that the 'partial' morphism $(F(-, B), G(-, JB))$ has a right adjoint $(R(B, -), S(JB, -))$ in $\mathbf{Cat^2}$. This is represented by the diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, shift left, "F(-{,}B)"] \arrow[r, phantom, "\perp"] \arrow[d, "H"'] & \mathcal{C} \arrow[l, shift left, "R(B{,}-)"] \arrow[d, "K"] \\
\mathbb{X} \arrow[r, shift left, "G(-{,}JB)"] \arrow[r, phantom, "\perp"] & \mathbb{Z} \arrow[l, shift left, "S(JB{,}-)"]
\end{tikzcd} \tag{8.6}$$

where both squares of left and right adjoints respectively commute, and (H, K) is a map of adjunctions. Then, there is a unique way to define a morphism of two variables

$$\begin{tikzcd}
\mathcal{B}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "R"] \arrow[d, "J^{\mathrm{op}} \times K"'] & \mathcal{A} \arrow[d, "H"] \\
\mathbb{Y}^{\mathrm{op}} \times \mathbb{Z} \arrow[r, "S"'] & \mathbb{X}
\end{tikzcd} \tag{8.7}$$

in $\mathbf{Cat}^2$, for which the natural isomorphisms

$$\mathcal{C}(F(A, B), C) \cong \mathcal{A}(A, R(B, C))$$
$$\mathbb{Z}(G(X, Y), Z) \cong \mathbb{X}(X, S(Y, Z))$$

are natural in all three variables.

Proof. The result is straightforward from the theory of parametrized adjunctions between categories. The fact that $(R(B,-),S(JB,-))$ is an arrow in $\mathbf{Cat^2}$ for all B's, ensures that the diagram (8.7) commutes on the second variable, and also on the first variable on objects, since $HR(B,C)=S(JB,KC)$. On arrows, commutativity follows from the unique way of defining $R(h,1)$ and $S(Jh,1)$ for any $h:B\to B'$ under these assumptions, given by (3.2). More explicitly, it is enough to consider the image of $R(h,1)$ under H and use the fact that the unit and counit of $F(-,B)\dashv R(B,-)$ are above the unit and counit of $G(-,JB)\dashv S(JB,-)$ with respect to the fibrations H and K. □

We call (S,R) the *parametrized adjoint* of (F,G) in $[\mathbf{2},\mathbf{Cat}]$. If we started with a morphism of two variables in $\mathbf{Fib}\subset\mathbf{Cat}^2$, *i.e.* a fibred 1-cell (F,G) depicted as (8.5), what would change in the above statement is that the diagram (8.6) would be required to be a general fibred adjunction as in Definition 5.3.1, *i.e.* the partial right adjoint $R(B,-)$ to be a cartesian functor itself. However, notice that by Lemma 5.3.2, right adjoints always preserve cartesian arrows in $\mathbf{Cat}^2$, therefore we do not need to request this as an extra condition. The pair (S,R) is then called the *fibred parametrized adjoint* of (F,G). On the other hand, in the context of **OpFib**, for the concept of an *opfibred parametrized adjoint* we request both F and $R(B,-)$ to be cocartesian.

We are now able to propose a definition of an enriched fibration, based on the evidence provided above. The theorem that follows justifies this statement, in the sense that it completes our initial goal: to generalize Theorem 4.3.3 from **Cat** to **Fib**, in order to establish an enrichment on the level of 0-cells of these 2-categories.

Definition 8.1.4 (Enriched Fibration). Suppose $T:\mathcal{V}\to\mathbb{W}$ is a monoidal fibration. We say that an (ordinary) fibration $P:\mathcal{A}\to\mathbb{X}$ is *enriched* in T when the following conditions are satisfied:

- the total category $\mathcal{A}$ is enriched in the total monoidal $\mathcal{V}$ and the base category $\mathbb{X}$ is enriched in the base monoidal $\mathbb{W}$, in such a way that

$$\begin{tikzcd}
\mathcal{A}^{\mathrm{op}}\times\mathcal{A} \arrow[r, "\mathcal{A}(-{,}-)"] \arrow[d, "P^{\mathrm{op}}\times P"'] & \mathcal{V} \arrow[d, "T"] \\
\mathbb{X}^{\mathrm{op}}\times\mathbb{X} \arrow[r, "\mathbb{X}(-{,}-)"'] & \mathbb{W}
\end{tikzcd} \tag{8.8}$$

  commutes;
- the composition law and the identities of the enrichments are compatible, in the sense that

$$\begin{aligned} TM^{\mathcal{A}}_{A,B,C} &= M^{\mathbb{X}}_{PA,PB,PC} \\ Tj^{\mathcal{A}}_{A} &= j^{\mathbb{X}}_{PA}; \end{aligned} \tag{8.9}$$

- the partial functor $\mathcal{A}(A,-)$ is cartesian.

It does not seem completely natural to ask for cartesianness of the enriched hom-functor between the total categories only on the second variable. However this condition is the only one with real effect, since the functor $\mathcal{A}(-,A):\mathcal{A}^{\mathrm{op}}\to\mathbb{X}$ goes from the total category of an opfibration to the total category of a fibration. We accordingly have the notion of an *enriched opfibration.*

The compatibility of the composition and identities of the two enrichments only says that if we take the image of the arrows

$$\begin{aligned} M^{\mathcal{A}}_{A,B,C} &: \mathcal{A}(B,C)\otimes_{\mathcal{V}}\mathcal{A}(A,B)\to\mathcal{A}(A,C)\\ j^{\mathcal{A}}_{A} &: I_{\mathcal{V}}\to\mathcal{A}(A,A)\end{aligned}$$

in $\mathcal{A}$ under the (monoidal) fibration T, we obtain the actual

$$\begin{aligned} M^{\mathbb{X}}_{PA,PB,PC} &: \mathbb{X}(PB,PC)\otimes_{\mathbb{W}}\mathbb{X}(PA,PB)\to\mathbb{X}(PA,PC)\\ j^{\mathbb{X}}_{PA} &: I_{\mathbb{W}}\to\mathbb{X}(PA,PA)\end{aligned}$$

where the domains and codomains already coincide by strict monoidality of T and the commutativity of (8.8).

Notice that in the above definition, there exists the usual abuse of notation, where the same name is given to the enriched categories and their underlying ordinary categories. If we wanted to be more rigorous, we should denote the categories with the additional enriched structure differently, for example $\mathbf{A}$ and $\mathbf{X}$. In that case the 'enriched hom-functor' (8.8), analogous to (4.5) for enrichment in **Cat**, would be written as

$$\begin{tikzcd} \mathcal{A}^{\mathrm{op}}\times\mathcal{A} \arrow[r, "\mathbf{A}(-,-)"] \arrow[d, "P^{\mathrm{op}}\times P"'] & \mathcal{V} \arrow[d, "T"] \\ \mathbb{X}^{\mathrm{op}}\times\mathbb{X} \arrow[r, "\mathbf{X}(-,-)"'] & \mathbb{W} \end{tikzcd}$$

and its partial 1-cell $(\mathbf{A}(A,-),\mathbf{X}(PA,-))$ is required to be a fibred 1-cell.

Remark 8.1.5. When an ordinary fibration $P:\mathcal{A}\to\mathbb{X}$ is enriched in a monoidal fibration $T:\mathcal{V}\to\mathbb{W}$, the latter has a strict monoidal structure hence by Proposition 4.1.1 we can make the $\mathcal{V}$-category $\mathcal{A}$ into a $\mathbb{W}$-enriched $\tilde{T}\mathcal{A}$, with the same set of objects ob$\mathcal{A}$ and hom-objects $T\mathcal{A}(A,B)=\mathbb{X}(PA,PB)$.

Then, the ordinary functor P can be viewed as a $\mathbb{W}$-enriched functor between the $\mathbb{W}$-categories $\tilde{T}\mathcal{A}$ and $\mathbb{X}$: on objects it is the function $\mathrm{ob}P:\mathrm{ob}\mathcal{A}\to\mathrm{ob}\mathbb{X}$ and on hom-objects it is the identity arrow $T\mathcal{A}(A,B)\xrightarrow{=}\mathbb{X}(PA,PB)$. The compatibility with the composition and the identities of the enriched categories, expressed by the commutativity of the diagrams (4.3), is ensured by the relations (8.9).

After a closer comparison between our Definition 8.1.4 of an enriched fibration, and Shulman's [**Shu13**, Definition 4.1] of an indexed $\mathcal{V}$-category, we conclude that even if there are conceptual similarities, our definition cannot even restrict in a straightforward way to the case of fibrations over the same base: the monoidal category $\mathbb{W}$ is not in principle enriched over itself, and certainly not via an identity functor. For a more accurate description of the similarities and differences of the two approaches to the subject, a detailed exposition of the ideas and theory in [**Shu13**] would be needed, but this would go beyond the scope of this thesis.

We now proceed to a result which asserts that to give a fibration and an action $(*, \diamond)$ of a monoidal fibration T with a fibred parametrized adjoint, is to give a T-enriched fibration.

Theorem 8.1.6. *Suppose that $T : \mathcal{V} \to \mathbb{W}$ is a monoidal fibration, which acts on an (ordinary) fibration $P : \mathcal{A} \to \mathbb{X}$ via the fibred 1-cell*

$$\begin{tikzcd}
\mathcal{V} \times \mathcal{A} \arrow[r, "*"] \arrow[d, "T\times P"'] & \mathcal{A} \arrow[d, "P"] \\
\mathbb{W} \times \mathbb{X} \arrow[r, "\diamond"'] & \mathbb{X}.
\end{tikzcd}$$

If this action has a parametrized adjoint $(R, S) : P^{\mathrm{op}} \times P \to T$ in **Fib***, then we can enrich the fibration P in the monoidal fibration T.*

Proof. By Definition 8.1.2, the T-action on P consists of two actions $*$ and $\diamond$ of the monoidal categories $\mathcal{V}$ and $\mathbb{W}$ on the ordinary categories $\mathcal{A}$ and $\mathbb{X}$ respectively. Moreover, by Theorem 8.1.3, we have two pairs of adjunctions

$$\begin{tikzcd}
\mathcal{A} \arrow[r, shift left, "-*A"] \arrow[r, phantom, "\perp"] & \mathcal{V} \arrow[l, shift left, "\bar{R}(A,-)"]
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\mathbb{X} \arrow[r, shift left, "-\diamond X"] \arrow[r, phantom, "\perp"] & \mathbb{W} \arrow[l, shift left, "R(X,-)"]
\end{tikzcd} \tag{8.10}$$

for all $A \in \mathcal{A}$ and $X \in \mathbb{X}$. By Theorem 4.3.3, there exists a $\mathcal{V}$-category with underlying category $\mathcal{A}$ and hom-objects $\bar{R}(A, B)$ and also a $\mathbb{W}$-category with underlying category $\mathbb{X}$ and hom-objects $R(X, Y)$. By the definition of fibred parametrized adjoints, we have that $(\bar{R}, R)$ is a 1-cell in $\mathbf{Cat}^2$ and moreover $(\bar{R}(A, -), R(PA, -))$ is a 1-cell in **Fib**.

Lastly, we need to show that the composition and identity laws of the enrichments are compatible as in (8.9). By computing the adjuncts of $M^{\mathcal{A}}_{A,B,C}$ and $j^{\mathcal{A}}_{A}$ under $(- * A) \dashv \bar{R}(A, -)$ which are given explicitly by the arrows (4.11) and (4.12) and taking their images under T, it can be seen that they bijectively correspond to the morphisms $M^{\mathbb{X}}_{PA,PB,PC}$ and $j^{\mathbb{X}}_{PA}$ under the adjunction $(- \diamond X) \dashv R(X, -)$. For this, we use that (P, T) is a map between the adjunctions (8.10), T is a strict monoidal functor and that the actions $*$ and $\diamond$ are compatible, in the sense of the definition of a T-representation. ☐

Clearly, there is a dual version of the above, characterizing the enrichment of an opfibration in a monoidal opfibration. In order for our examples to fit in this theory, we also need the notion of a fibration enriched in an opfibration and its dual.

Definition 8.1.7. Suppose that $T : \mathcal{V} \to \mathbb{W}$ is a symmetric monoidal opfibration. We say that a fibration $P : \mathcal{A} \to \mathbb{X}$ is enriched in T if the opfibration $P^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \to \mathbb{X}^{\mathrm{op}}$ is an enriched T-opfibration.

We can now apply Theorem 8.1.6 to obtain an enrichment of the fibration $G:\mathbf{Mod}\to\mathbf{Mon}(\mathcal{V})$ in the monoidal opfibration $V:\mathbf{Comod}\to\mathbf{Comon}(\mathcal{V})$. First of all, V is a strict monoidal functor (6.11) and $\otimes:\mathbf{Comod}\times\mathbf{Comod}\to\mathbf{Comod}$ preserves cocartesian arrows on the nose (see proof of Proposition 6.3.2), thus V is indeed a monoidal opfibration. Then, by Definition 8.1.2 we have an action of V on G^{op}, given by the actions H^{op} of $\mathbf{Comon}(\mathcal{V})$ on $\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$ and $\bar{H}^{\mathrm{op}}$ of $\mathbf{Comod}$ on $\mathbf{Mod}^{\mathrm{op}}$ as in (6.6) and (6.21). The compatibility conditions between these two actions hold and $\bar{H}^{\mathrm{op}}$ strictly preserves cocartesian liftings (see Section 6.2). Finally, there is evidence that the universal measuring comodule functor Q preserves cocartesian liftings on the first variable, which would make (Q,P) into an opfibred parametrized adjoint for the action $(\bar{H}^{\mathrm{op}},H^{\mathrm{op}})$. We can thus enrich G^{op} in V.

Proposition 8.1.8. *If $Q(-,N_B)$ is cocartesian, the fibration $G:\mathbf{Mod}\to\mathbf{Mon}(\mathcal{V})$ is enriched in the monoidal opfibration $V:\mathbf{Comod}\to\mathbf{Comon}(\mathcal{V})$.*

Of course, it would as well suffice to verify the conditions of Definition 8.1.4 for this particular case, in order to obtain the above result.

At this moment, similar complications arise for the proof that the generalized Sweedler hom functor $T(-,\mathcal{B}_Y)$ and the functor $\bar{T}(-,\Psi_{\mathcal{B}})$ between $\mathcal{V}$-modules and $\mathcal{V}$-comodules preserve cartesian liftings. As a result, we also cannot claim the enrichment of the fibrations N and P in the monoidal opfibrations H and W as in (8.2) unless this condition is satisfied (like the above proposition), even though the remaining conditions hold. We aim to verify these properties with future work.

8.2. Double categorical and bicategorical setting

We are now interested in generalizing the above development, starting with an arbitrary bicategory or even a double category in place of $\mathcal{V}$-$\mathbf{Mat}$. The fact that Chapter 7 is centered around the bicategory of $\mathcal{V}$-matrices and Chapter 6 addresses the one-object bicategory case are indicative of such an extension. So the driving question of this section is to determine what kind of structure a bicategory $\mathcal{K}$ should have, in order to recapture the main results of the previous two chapters.

There are two functors of bicategories which are fundamental for our purposes. Firstly, a homomorphism (pseudofunctor)

$$\otimes:\mathcal{K}\times\mathcal{K}\longrightarrow\mathcal{K}$$

which will be part of a monoidal structure on our bicategory, and also a lax functor

$$H:\mathcal{K}^{\mathrm{co}}\times\mathcal{K}\longrightarrow\mathcal{K}$$

which under circumstances, will lead to enrichment relations between total categories of certain fibrations and opfibrations. The above functors of bicategories provide (ordinary) functors

$$\begin{aligned}\otimes_{(A,B),(C,D)}&:\mathcal{K}(A,C)\times\mathcal{K}(B,D)\to\mathcal{K}(A\otimes B,C\otimes D) \qquad (8.11)\\ H_{(A,B),(C,D)}&:\mathcal{K}(A,C)^{\mathrm{op}}\times\mathcal{K}(B,D)\to\mathcal{K}(H(A,B),H(C,D))\end{aligned}$$

between the hom-categories. Moreover, as seen in Lemma 3.3.3, any lax functor of bicategories induces a functor between the categories of monoids of endoarrow hom-categories with horizontal composition. Here they produce

$$\begin{aligned}\mathbf{Mon}(\otimes_{(A,B)}) &: \mathbf{Mon}\mathcal{K}(A,A)\times\mathbf{Mon}\mathcal{K}(B,B)\to\mathbf{Mon}\mathcal{K}(A\otimes B,A\otimes B)\\ \mathbf{Mon}(H_{(A,B)}) &: \mathbf{Comon}\mathcal{K}(A,A)^{\mathrm{op}}\times\mathbf{Mon}\mathcal{K}(B,B)\to\mathbf{Mon}\mathcal{K}(H(A,B),H(A,B)).\end{aligned}$$

These functors are just restrictions of (8.11) on the appropriate categories, which in fact turn out to be fibres of total categories, crucial for the development. Since $\otimes$ is a pseudofunctor, *i.e.* also colax with respect to the horizontal composition, there is also an induced functor

$$\mathbf{Comon}(\otimes_{(A,B)}) : \mathbf{Comon}\mathcal{K}(A,A)\times\mathbf{Comon}\mathcal{K}(B,B)\to\mathbf{Comon}\mathcal{K}(A\otimes B,A\otimes B).$$

Under certain conditions, these functors 'between the fibres' induce total functors which give rise to specific structures of importance.

For $\mathcal{K}=\mathcal{V}\text{-}\mathbf{Mat}$ for example, these categories are $\mathbf{Mon}\mathcal{K}(A,A)=\mathcal{V}\text{-}\mathbf{Cat}_A$ and $\mathbf{Comon}\mathcal{K}(A,A)=\mathcal{V}\text{-}\mathbf{Cocat}_A$ for fixed sets of objects A. The bicategory of $\mathcal{V}$-matrices is in fact a monoidal bicategory with tensor product as in (7.4) which induces the monoidal structure of the total categories $\mathcal{V}$-**Cat** and $\mathcal{V}$-**Cocat**. Also, the lax functor $H=\mathrm{Hom}:(\mathcal{V}\text{-}\mathbf{Mat})^{\mathrm{co}}\times\mathcal{V}\text{-}\mathbf{Mat}\to\mathcal{V}\text{-}\mathbf{Mat}$ defined as in (7.5) gives rise to the functor K, whose adjoint induces the enrichment stated by Theorem 7.4.6.

Furthermore, by Proposition 2.2.10 the lax functors $\otimes$ and H induce

$$\begin{aligned}\mathcal{K}(A,C)^{\mathcal{K}(A,t)}\times\mathcal{K}(B,D)^{\mathcal{K}(B,s)} &\to \mathcal{K}(A\otimes B,C\otimes D)^{\mathcal{K}(A\otimes B,t\otimes s)}\\ \mathcal{K}(A,C)^{\mathcal{K}(A,u)^{\mathrm{op}}}\times\mathcal{K}(B,D)^{\mathcal{K}(B,s)} &\to \mathcal{K}(H(A,B),H(C,D))^{\mathcal{K}(H(A,B),H(u,s))}\end{aligned}$$

between the categories of left modules and comodules with fixed domains, for monads $t:C\to C$, $s:D\to D$ and comonad $u:C\to C$ in $\mathcal{K}$. These can also be written as

$$\begin{aligned}\mathbf{Mod}(\otimes_{(A,B),(C,D)}) &: {}^{A}_{t}\mathbf{Mod}\times{}^{B}_{s}\mathbf{Mod}\longrightarrow{}^{A\otimes B}_{t\otimes s}\mathbf{Mod}\\ \mathbf{Mod}(H_{(A,B),(C,D)}) &: {}^{A}_{u}\mathbf{Comod}^{\mathrm{op}}\times{}^{B}_{s}\mathbf{Mod}\longrightarrow{}^{H(A,B)}_{H(u,s)}\mathbf{Mod}\end{aligned}$$

by Definitions 2.2.3, 2.2.6. Again, since $\otimes$ is a homomorphism of bicategories, it also induces

$$\mathbf{Comod}(\otimes_{(A,B),(C,D)}) : {}^{A}_{u}\mathbf{Comod}\times{}^{B}_{v}\mathbf{Comod}\longrightarrow{}^{A\otimes B}_{u\otimes v}\mathbf{Comod}$$

between the categories of comodules. These functors between the fibres of the global categories are expected to give the monoidal structures to modules and comodules, and the enrichment of modules in comodules respectively. For the bicategory $\mathcal{V}$-**Mat**, the monoidal structures of $\mathcal{V}$-**Mod** and $\mathcal{V}$-**Comod** as well as Theorem 7.7.7 are obtained by employing instances of the above functors.

In order to identify suitable assumptions on the bicategory $\mathcal{K}$, we are going to employ the theory of double categories. This turns out to be an appropriate theoretical framework leading to enriched fibrations as discussed in last section, because it provides with a better understanding of the nature of the categories appearing in our examples. We largely follow the approach of [**Shu10**], where a method for constructing (symmetric) monoidal bicategories from (symmetric) monoidal double categories which satisfy a lifting condition is described. This process allows us to reduce a lengthy and demanding task of verifying the coherence conditions of monoidal structure on a bicategory into a much more concise and speedy procedure, essentially involving a pair of ordinary monoidal categories.

Definition 8.2.1. A *(pseudo) double category* $\mathbb{D}$ consists of a category of objects $\mathbb{D}_0$ and a category of arrows $\mathbb{D}_1$, with structure functors

$$\mathbf{1} : \mathbb{D}_0 \to \mathbb{D}_1, \quad \mathfrak{s}, \mathfrak{t} : \mathbb{D}_1 \rightrightarrows \mathbb{D}_0, \quad \odot : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$$

such that $\mathfrak{s}(1_A) = \mathfrak{t}(1_A) = A$, $\mathfrak{s}(M \odot N) = \mathfrak{s}(N)$, $\mathfrak{t}(M \odot N) = \mathfrak{t}(M)$ for all $A \in \mathrm{ob}\mathbb{D}_0$, $M, N \in \mathrm{ob}\mathbb{D}_1$, equipped with natural isomorphisms

$$\begin{aligned} \alpha &: (M \odot N) \odot P \xrightarrow{\sim} M \odot (N \odot P) \\ \lambda &: 1_{\mathfrak{s}(M)} \odot M \xrightarrow{\sim} M \\ \rho &: M \odot 1_{\mathfrak{t}(M)} \xrightarrow{\sim} M \end{aligned}$$

in $\mathbb{D}_1$ for all $M, N, E \in \mathrm{ob}\mathbb{D}_1$, such that $\mathfrak{t}(\alpha), \mathfrak{s}(\alpha), \mathfrak{t}(\lambda), \mathfrak{s}(\lambda), \mathfrak{t}(\rho), \mathfrak{s}(\rho)$ are all identities, and satisfying the usual coherence conditions (as for a bicategory).

The objects of $\mathbb{D}_0$ are called *0-cells* and the morphisms of $\mathbb{D}_0$ are called *1-morphisms* or *vertical 1-cells*, denoted as $f : A \to B$. The objects of $\mathbb{D}_1$ are the *(horizontal) 1-cells*, denoted as $M : A \xrightarrow{\bullet} B$ where $\mathfrak{s}(M) = A$ is the source and $\mathfrak{t}(M) = B$ the target of M. The morphisms of $\mathbb{D}_1$ are the *2-morphisms*, denoted as squares

$$\begin{tikzcd} A \arrow[r, "M", "\bullet" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "N"', "\bullet" marking] & D \end{tikzcd}$$

or ${}^f\alpha^g : M \Rightarrow N$, where $\mathfrak{s}(\alpha) = f$ and $\mathfrak{t}(\alpha) = g$. The composition of vertical 1-cells and the vertical composition of 2-morphisms are strictly associative since $\mathbb{D}_0$ and $\mathbb{D}_1$ are categories, whereas horizontal composition of horizontal 1-cells and 2-morphisms is associative up to isomorphism due to the isomorphisms $a_{M,N,P}$. These are respectively written as

$$\begin{tikzcd} A \arrow[r, "M", "\bullet" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "N", "\bullet" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta"] & D \arrow[d, "k"] \\ E \arrow[r, "P"', "\bullet" marking] & F \end{tikzcd} = \begin{tikzcd} A \arrow[r, "M", "\bullet" marking] \arrow[d, "hf"'] \arrow[dr, phantom, "\Downarrow\beta\alpha"] & B \arrow[d, "kg"] \\ E \arrow[r, "P"', "\bullet" marking] & F, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "M", "\bullet" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[r, "N", "\bullet" marking] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "h"] \\ D \arrow[r, "P"', "\bullet" marking] & E \arrow[r, "K"', "\bullet" marking] & F \end{tikzcd} = \begin{tikzcd} A \arrow[r, "N\odot M", "\bullet" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta\odot\alpha"] & C \arrow[d, "h"] \\ D \arrow[r, "K\odot P"', "\bullet" marking] & F. \end{tikzcd}$$

The vertical identity 1-cell $\mathrm{id}_A : A \to A$ for any object A and the identity 2-morphism 1_M for any 1-cell M make the vertical compositions also strictly unital. Also, the horizontal unit 1-cell $1_A : A \mathrel{-\!\!\bullet\!\!\!\rightarrow} A$ for every object A and the horizontal unit 2-morphism 1_f for any 1-morphism $f : A \to B$ make the horizontal compositions unital up to isomorphism. The identity 2-morphisms are denoted by

$$\begin{tikzcd}
A \arrow[r, "M", "\bullet"{marking}] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\Downarrow 1_M"] & B \arrow[d, "\mathrm{id}_B"] \\
A \arrow[r, "M"', "\bullet"{marking}] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "1_A", "\bullet"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow 1_f"] & A \arrow[d, "f"] \\
B \arrow[r, "1_B"', "\bullet"{marking}] & B
\end{tikzcd}$$

and in particular $1_{1_A} = 1_{\mathrm{id}_A}$. Functoriality of the horizontal composition $\odot$ results in the relation $1_N \odot 1_M = 1_{N \odot M}$ and the interchange law which the two different compositions obey:

$$(\beta'\beta) \odot (\alpha'\alpha) = (\beta' \odot \alpha')(\beta \odot \alpha).$$

The *opposite double category* $\mathbb{D}^{\mathrm{op}}$ is the double category with vertical category $\mathbb{D}_0^{\mathrm{op}}$ and horizontal category $\mathbb{D}_1^{\mathrm{op}}$. There also exist the *horizontally opposite* double category $\mathbb{D}^{\mathrm{hop}}$ and *vertically opposite* double category $\mathbb{D}^{\mathrm{vop}}$, where the horizontal and vertical categories respectively are the opposite ones.

A 2-morphism with identity source and target 1-morphisms, like a, l, r above, is called *globular*. Evidently, for every double category $\mathbb{D}$ there is a corresponding bicategory denoted by $\mathcal{H}(\mathbb{D})$ or just $\mathcal{D}$, called its *horizontal bicategory*. It consists of the objects, (horizontal) 1-cells and globular 2-morphisms. In a sense, this comes from discarding the vertical structure of the double category.

Many well-known bicategories arise as the horizontal bicategories of specific double categories. For example, consider the double category $\mathcal{V}$-$\mathbb{M}\mathbf{at}$: the category of objects is $\mathcal{V}$-$\mathbb{M}\mathbf{at}_0$=$\mathbf{Set}$, and the category of arrows $\mathcal{V}$-$\mathbb{M}\mathbf{at}_1$ consists of $\mathcal{V}$-matrices $S : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ as 1-cells, and 2-morphisms ${}^f\alpha^g : S \Rightarrow T$ given by families of arrows

$$\alpha_{y,x} : S(y,x) \to T(gy, fx)$$

in $\mathcal{V}$ for all $x \in X$ and $y \in Y$. The structure functor $\mathbf{1}$ gives the identity $\mathcal{V}$-matrix $1_X : X \relbar\joinrel\mapstochar\joinrel\rightarrow X$ for all sets X and the unit 2-morphism 1_f with components arrows

$$(1_f)_{x',x} : 1_X(x', x) \to 1_X(x', x) \equiv \begin{cases} I \xrightarrow{1_I} I, & \text{if } x = x' \\ 0 \to 0, & \text{if } x \neq x'. \end{cases}$$

The source and target functors give the evident sets and functions, and the functor

$$\odot : \mathcal{V}\text{-}\mathbb{M}\mathbf{at}_1 \times_{\mathcal{V}\text{-}\mathbb{M}\mathbf{at}_0} \mathcal{V}\text{-}\mathbb{M}\mathbf{at}_1 \to \mathcal{V}\text{-}\mathbb{M}\mathbf{at}_1$$

is given by the usual composition of $\mathcal{V}$-matrices as in (7.1) on objects, and on 2-morphisms ${}^f(\beta \odot \alpha)^g : T \circ S \Rightarrow T' \circ S'$ is given by the composite arrows

$$\begin{tikzcd}
\sum_y T(z,y) \otimes S(y,x) \arrow[r, "\sum \beta_{z,y} \otimes \alpha_{y,z}"] \arrow[dr, dashed] & \sum_y T'(hz, gy) \otimes S'(gy, fx) \arrow[d, hook, "\iota"] \\
& \sum_{y'} T'(hz, y') \otimes S'(y', fx)
\end{tikzcd}$$

in $\mathcal{V}$, for all $x \in X$ and $z \in Z$. Notice how this generalizes the operation (7.2) between $\mathcal{V}$-matrices of different domain and codomain. Compatibility conditions of source and target functors with composition can be easily checked, and the globular 2-isomorphisms are the ones described in Section 7.1. Of course, its horizontal bicategory $\mathcal{H}(\mathcal{V}\text{-}\mathbb{M}\mathbf{at})$ is precisely the bicategory $\mathcal{V}$-**Mat**.

Definition 8.2.2. For $\mathbb{D}$ and $\mathbb{E}$ (pseudo) double categories, a *pseudo double functor* $F : \mathbb{D} \to \mathbb{E}$ consists of functors $F_0 : \mathbb{D}_0 \to \mathbb{E}_0$ and $F_1 : \mathbb{D}_1 \to \mathbb{E}_1$ between the categories of objects and arrows, such that $\mathfrak{s} \circ F_1 = F_0 \circ \mathfrak{s}$ and $\mathfrak{t} \circ F_1 = F_0 \circ \mathfrak{t}$, and natural transformations $F_\odot$, F_U with components globular isomorphisms $F_1M \odot F_1N \xrightarrow{\sim} F_1(M \odot N)$ and $1_{F_0A} \xrightarrow{\sim} F_1(1_A)$ respectively, which satisfy the usual coherence axioms for a pseudofunctor.

We also have notions of *lax* and *colax double functors* between pseudo double categories, where the natural transformations $F_\odot$ and F_U have components globular 2-morphisms in one of the two possible directions respectively. The explicit definitions can be found in the appendix of [**GP99**] or [**GP04**]. In particular, naturality of $F_\odot$ in this context means the following: for any composable 2-morphisms ${}^f\alpha^g : M \Rightarrow M'$ and ${}^g\beta^h : N \Rightarrow N'$ in $\mathbb{D}$, the components of $F_\odot$ satisfy

$$\begin{tikzcd}
F_0A \arrow[r, "F_1M", "\bullet" marking] \arrow[d, "F_0f"'] \arrow[dr, phantom, "\Downarrow F_1\alpha"] & F_0B \arrow[r, "F_1N", "\bullet" marking] \arrow[d, "F_0g"] \arrow[dr, phantom, "\Downarrow F_1\beta"] & F_0C \arrow[d, "F_0h"] & & F_0A \arrow[r, "F_1M", "\bullet" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow F_\odot"] & F_0B \arrow[r, "F_1N", "\bullet" marking] & F_0C \arrow[d, equal] \\
F_0A' \arrow[r, "F_1M'"', "\bullet" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow F_\odot"] & F_0B' \arrow[r, "F_1N'"', "\bullet" marking] & F_0C' \arrow[d, equal] & = & F_0A \arrow[rr, "F_1(N\odot M)"', "\bullet" marking] \arrow[d, "F_0f"'] \arrow[drr, phantom, "\Downarrow F_1(\beta\odot\alpha)"] & & F_0C \arrow[d, "F_0h"] \\
F_0A' \arrow[rr, "F_1(N'\odot M')"', "\bullet" marking] & & F_0C' & & F_0A' \arrow[rr, "F_1(N'\odot M')"', "\bullet" marking] & & F_0C'.
\end{tikzcd} \tag{8.12}$$

Whenever we have a pseudo double functor $F : \mathbb{D} \to \mathbb{E}$, there is an induced pseudofunctor between the respective horizontal bicategories

$$\mathcal{H}F : \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{E})$$

which consists of the following data:

· for each 0-cell $A \in \mathbb{D}_0$ in the bicategory $\mathcal{H}(\mathbb{D})$, a 0-cell $F_0A \in \mathbb{E}_0$ in the bicategory $\mathcal{H}(\mathbb{E})$;

· for each two 0-cells $A, B \in \mathbb{D}_0$, a functor

$$\mathcal{H}F_{A,B} : \mathcal{H}(\mathbb{D})(A, B) \to \mathcal{H}(\mathbb{E})(F_0A, F_0B)$$

which maps a horizontal 1-cell $M : A \xrightarrow{\bullet} B$ to the 1-cell $F_1M : F_0A \xrightarrow{\bullet} F_0B$ and

$$\begin{tikzcd}
A \arrow[r, "M", "\bullet" marking] \arrow[d, "\mathrm{id}_{\mathrm{A}}"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "\mathrm{id}_{\mathrm{B}}"] & & F_0A \arrow[r, "F_1M", "\bullet" marking] \arrow[d, "\mathrm{id}_{(\mathrm{F_0A})}"'] \arrow[dr, phantom, "\Downarrow F_1\alpha"] & F_0B \arrow[d, "\mathrm{id}_{(\mathrm{F_0B})}"] \\
A \arrow[r, "N"', "\bullet" marking] & B & \mapsto & F_0A \arrow[r, "F_1N"', "\bullet" marking] & F_0B
\end{tikzcd}$$

using functoriality of F_0 and compatibility of F_0 and F_1 with sources and targets;

· for every triple of 0-cells A, B, C, a natural isomorphism with components invertible arrows

$$\delta^{N,M} : F_1 N \odot F_1 M \xrightarrow{\sim} F_1(N \odot M)$$

for $M : A \mathrel{-\!\!\!\bullet\!\!\!\rightarrow} B$ and $N : B \mathrel{-\!\!\!\bullet\!\!\!\rightarrow} C$, given by $F_\odot$;

· for every 0-cell A, a natural isomorphism with components invertible

$$\gamma^A : 1_{F_0 A} \xrightarrow{\sim} F_1(1_A)$$

given by F_U.

The coherence axioms are satisfied by definition of the pseudo double functor. Similarly we get (co)lax functors between bicategories from (co)lax double functors. This is indicative of the way that structure may be inherited from a pseudo double category to its horizontal bicategory. From now on, the adjective 'pseudo' will be dropped whenever it is clearly implied.

The formal definition of a monoidal double category can be found in [**Shu10**] and is omitted here. Notice that in [**GP04**] for example, the tensor product $\otimes$ as below is required to be a colax double functor rather than pseudo double. If we unpack the definition, we get the following simplified description.

Definition 8.2.3. A *monoidal double category* is a double category $\mathbb{D}$ equipped with (pseudo) double functors

$$\otimes : \mathbb{D} \times \mathbb{D} \to \mathbb{D} \quad \text{and} \quad \mathbf{I} : \mathbf{1} \to \mathbb{D},$$

such that $(\mathbb{D}_0, \otimes_0, I)$ and $(\mathbb{D}_1, \otimes_1, 1_I)$ are monoidal categories with $1_I : I \mathrel{-\!\!\!\bullet\!\!\!\rightarrow} I$ for $I = \mathbf{I}(*)$, the functors $\mathfrak{s}, \mathfrak{t}$ are strict monoidal and preserve associativity and unit constraints, and there exist globular isomorphisms

$$(M \otimes_1 N) \odot (M' \otimes_1 N') \cong (M \odot M') \otimes_1 (N \odot N')$$

$$1_{(A \otimes_0 B)} \cong 1_A \otimes_1 1_B$$

subject to coherence conditions.

For example, consider the double category $\mathcal{V}$-$\mathbb{M}\mathbf{at}$ where both categories of objects and arrows are monoidal categories. Indeed, $(\mathbf{Set}, \times, \{*\})$ is cartesian monoidal and $\mathcal{V}$-$\mathbb{M}\mathbf{at}_1$ has tensor product

$$\begin{tikzcd}
\otimes :\ \mathcal{V}\text{-}\mathbb{M}\mathbf{at}_1 \times \mathcal{V}\text{-}\mathbb{M}\mathbf{at}_1 \arrow[rr] & & \mathcal{V}\text{-}\mathbb{M}\mathbf{at}_1
\end{tikzcd} \tag{8.13}$$

$$\begin{tikzcd}
(X \arrow[r, "S", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & Y \arrow[d, "g"] & , & Z \arrow[r, "T", "|" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta"] & W) \arrow[d, "k"] \arrow[r, mapsto, dotted] & X \times Z \arrow[r, "S\otimes T", "|" marking] \arrow[d, "f\times h"'] \arrow[dr, phantom, "\Downarrow\alpha\otimes\beta"] & Y \times W \arrow[d, "g\times k"] \\
(X' \arrow[r, "S'"', "|" marking] & Y' & , & Z' \arrow[r, "T'"', "|" marking] & W') \arrow[r, mapsto, dotted] & X' \times Z' \arrow[r, "S'\otimes T'"', "|" marking] & Y' \times W'
\end{tikzcd}$$

given by the families $(S \otimes T)((y, w), (x, z)) := S(y, x) \otimes T(w, z)$ of objects in $\mathcal{V}$ and

$$(\alpha \otimes \beta)_{(y,w),(x,z)} := S(y, x) \otimes T(w, z) \xrightarrow{\alpha_{y,x} \otimes \beta_{w,z}} S'(gy, fx) \otimes T'(kw, hz)$$

of arrows in $\mathcal{V}$, and monoidal unit the $\mathcal{V}$-matrix $\mathcal{I} : \{*\} \mathrel{-\!\!\!+\!\!\!\rightarrow} \{*\}$ with $\mathcal{I}(*, *) = I_\mathcal{V}$. The conditions for $\mathfrak{s}$ and $\mathfrak{t}$ are satisfied, and the natural isomorphisms come down to combinations of associativity and unit constraints of $\mathcal{V}$ and the fact that the tensor product in $\mathcal{V}$ commutes with sums.

PROPOSITION 8.2.4. *The pseudo double category* $\mathcal{V}$-$\mathbb{M}\mathbf{at}$ *is monoidal.*

What is further required to obtain a monoidal structure on the horizontal bicategory of a monoidal double category is a way of turning vertical 1-morphisms into horizontal 1-cells. The links between vertical and horizontal 1-cells in a double category have been studied by various authors, and the terminology used below can be found in [**GP04, Shu08, DPP10**].

DEFINITION 8.2.5. Let $\mathbb{D}$ be a double category and $f : A \to B$ a vertical 1-morphism. A *companion* of f is a horizontal 1-cell $\hat{f} : A \mathrel{-\!\!\bullet\!\!\to} B$ together with 2-morphisms

$$\begin{tikzcd}
A \arrow[r, "\hat{f}", "\bullet" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow p_1"] & B \arrow[d, "\mathrm{id}_B"] \\
B \arrow[r, "1_B"', "\bullet" marking] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "1_A", "\bullet" marking] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\Downarrow p_2"] & A \arrow[d, "f"] \\
A \arrow[r, "\hat{f}"', "\bullet" marking] & B
\end{tikzcd}$$

such that $p_1p_2 = 1_f$ and $p_1 \odot p_2 \cong 1_{\hat{f}}$. Dually, a *conjoint* of f is a horizontal 1-cell $\check{f} : B \mathrel{-\!\!\bullet\!\!\to} A$ together with 2-morphisms

$$\begin{tikzcd}
B \arrow[r, "\check{f}", "\bullet" marking] \arrow[d, "\mathrm{id}_B"'] \arrow[dr, phantom, "\Downarrow q_1"] & A \arrow[d, "f"] \\
B \arrow[r, "1_B"', "\bullet" marking] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "1_A", "\bullet" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow q_2"] & A \arrow[d, "\mathrm{id}_A"] \\
B \arrow[r, "\check{f}"', "\bullet" marking] & A
\end{tikzcd}$$

such that $q_1q_2 = 1_f$ and $q_2 \odot q_1 \cong 1_{\check{f}}$.

The ideas which led to the above definitions go back to [**BS76**], where a *connection* on a double category corresponds to a strictly functorial choice of a companion for each vertical arrow. Now, a *fibrant double category* ([**Shu10**, Definition 3.4]) is a double category for which every vertical 1-morphism has a companion and a conjoint (called *framed bicategory* in [**Shu08**]). Many important properties for fibrant double categories can be obtained just from the definitions. For example, companions and conjoints of a specific 1-morphism are essentially unique (up to unique globular isomorphism), and $\hat{g} \odot \hat{f}$, $\check{g} \odot \check{f}$ are the companion and the conjoint of gf.

The significance of these notions is clear in the context of our primary example, the double category $\mathcal{V}$-$\mathbb{M}\mathbf{at}$. The companion of a function $f : X \to Y$ is the $\mathcal{V}$-matrix $f_* : X \mathrel{-\!\!+\!\!\to} Y$ and its conjoint is $\mathcal{V}$-matrix $f^* : Y \mathrel{-\!\!+\!\!\to} X$, as defined in (7.11). Properties of these $\mathcal{V}$-matrices, such as the adjunction $f^* \dashv f_*$ in the horizontal bicategory $\mathcal{V}$-**Mat** or Lemmas 7.1.3 and 7.1.4, are in fact true in the general setting of any fibrant double category. *I.e.* for any vertical 1-morphism f in $\mathbb{D}$, we have an adjunction $\hat{f} \dashv \check{f}$ in $\mathcal{H}(\mathbb{D})$.

Another important example of a fibrant double category is the one with horizontal bicategory $\mathcal{V}$-**BMod** (or $\mathcal{V}$-**Prof**) of enriched bimodules, as briefly described in

Section 4.2. In particular, the companion and conjoint for each $\mathcal{V}$-functor (which are the vertical 1-morphisms) are given by the 'representable' profunctors as in (4.7).

The main Theorem 5.1 in [**Shu10**] asserts that the horizontal bicategory of a fibrant monoidal double category inherits a monoidal structure. Explicitly, it consists of the induced pseudofunctor of bicategories $\mathcal{H}(\otimes) : \mathcal{H}(\mathbb{D}) \times \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{D})$ and the monoidal unit 1_I of $\mathbb{D}_1$. In particular, the double category of $\mathcal{V}$-matrices is a fibrant monoidal double category, hence the result follows for its horizontal bicategory $\mathcal{H}(\mathcal{V}\text{-}\mathbb{M}\mathbf{at})$.

PROPOSITION 8.2.6. *The bicategory* $\mathcal{V}$-**Mat** *of* $\mathcal{V}$*-matrices is a monoidal bicategory.*

The monoidal unit is the unit $\mathcal{V}$-matrix $\mathcal{I}$ and the induced tensor product pseudofunctor $\otimes : \mathcal{V}\text{-}\mathbf{Mat} \times \mathcal{V}\text{-}\mathbf{Mat} \to \mathcal{V}\text{-}\mathbf{Mat}$ maps two sets X, Y to their cartesian product $X \times Y$, and the functor

$$\otimes_{(X,Y),(Z,W)} : \mathcal{V}\text{-}\mathbf{Mat}(X,Z) \times \mathcal{V}\text{-}\mathbf{Mat}(Y,W) \to \mathcal{V}\text{-}\mathbf{Mat}(X \times Y, Z \times W),$$

is defined as in (8.13), for 2-morphisms with domain and codomain the identity vertical 1-morphisms.

We are now in position to examine how constructions and results of the previous chapter may fit in the general frame of any fibrant double category. As up to this point, our presentation aims to sketch the main ideas rather than rigorously establish a theory.

Suppose $\mathbb{D}$ is an arbitrary fibrant double category, with no monoidal structure to begin with. Define the category $\mathbb{D}_1^\bullet$ to be the (non-full) subcategory of $\mathbb{D}_1$ of all horizontal endo-1-cells and 2-morphisms with the same source and target. Explicitly, objects are all 1-cells of the form $M : A \to\!\!\!\!\bullet\; A$ and arrows are of the form

$$\begin{tikzcd} A \arrow[r, "M"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & A \arrow[d, "f"] \\ B \arrow[r, "N"'] & B \end{tikzcd}$$

denoted by $\alpha_f : M_A \to N_B$. In [**FGK11**], this category coincides with the vertical 1-category of the double category $\mathbb{E}\mathbf{nd}(\mathbb{D})$ of (horizontal) endomorphisms, horizontal endomorphism maps, vertical endomorphism maps and endomorphism squares in $\mathbb{D}$.

This definition is motivated by the fact that $\mathcal{V}\text{-}\mathbb{M}\mathbf{at}_1^\bullet = \mathcal{V}\text{-}\mathbf{Grph}$: objects are $\mathcal{V}$-graphs, *i.e.* endo-$\mathcal{V}$-matrices $G : X \to\!\!\!\!\!\shortmid\; X$ given by objects $\{G(x',x)\}$ in $\mathcal{V}$, and arrows $\alpha_f : G_X \to H_Y$ are $\mathcal{V}$-graph morphisms, *i.e.* a function $f : X \to Y$ and arrows $\alpha_{x',x} : G(x',x) \to H(fx',fx)$ in $\mathcal{V}$. In the view of [**FGK11**, Remark 2.5], this is analogous to the fact that the category $\mathbf{Grph}_\mathcal{E}$ of graphs and graph morphisms internal to a finitely complete $\mathcal{E}$ is identified with the category of endomorphisms and vertical endomorphism maps in the double category $\mathbb{S}\mathbf{pan}_\mathcal{E}$.

PROPOSITION 8.2.7. *Suppose* $\mathbb{D}$ *is a fibrant double category. The category* $\mathbb{D}_1^\bullet$ *is bifibred over* $\mathbb{D}_0$*.*

Proof. We can easily adjust a series of previous relevant proofs, in order to construct pseudofunctors whose Grothendieck construction gives rise to a fibration and an opfibration, isomorphic to the evident forgetful functor $\mathbb{D}_1^\bullet \to \mathbb{D}_0$ mapping G_X to X and α_f to f. Like Proposition 7.5.1, the respective pseudofunctors are

$$\begin{tikzcd} \mathscr{M}: \mathbb{D}_0^{\mathrm{op}} \arrow[r] & \mathbf{Cat}, & & \mathscr{F}: \mathbb{D}_0 \arrow[r] & \mathbf{Cat} \\ A \arrow[r, dotted, mapsto] \arrow[d, "f"'] & \mathcal{H}(\mathbb{D})(A,A) & & A \arrow[r, dotted, mapsto] \arrow[d, "f"'] & \mathcal{H}(\mathbb{D})(A,A) \arrow[d, "(\hat{f}\odot\text{-}\odot\check{f})"] \\ B \arrow[r, dotted, mapsto] & \mathcal{H}(\mathbb{D})(B,B) \arrow[u, "(\check{f}\odot\text{-}\odot\hat{f})"'] & & B \arrow[r, dotted, mapsto] & \mathcal{H}(\mathbb{D})(B,B). \end{tikzcd} \tag{8.14}$$

We can illustrate the isomorphism between, for example, the Grothendieck category $\mathfrak{G}\mathscr{M}$ and $\mathbb{D}_1^\bullet$, just by employing companions and conjoints. The objects are the same (horizontal endo-1-cells), and there is a bijective correspondence between the morphisms: given an arrow α_f in $\mathbb{D}_1^\bullet$, we obtain a composite 2-cell

$$\begin{tikzcd} A \arrow[r, "M", "\bullet" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & A \arrow[d, "f"] & & A \arrow[r, "1_A", "\bullet" marking] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\Downarrow p_2"] & A \arrow[r, "M", "\bullet" marking] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\alpha"] & A \arrow[r, "1_A", "\bullet" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow q_2"] & A \arrow[d, "\mathrm{id}_A"] \\ B \arrow[r, "N"', "\bullet" marking] & B & \mapsto & A \arrow[r, "\hat{f}"', "\bullet" marking] & B \arrow[r, "N"', "\bullet" marking] & B \arrow[r, "\check{f}"', "\bullet" marking] & B \end{tikzcd}$$

which is a morphism in $\mathfrak{G}\mathscr{M}$. This assignment is an isomorphism, with inverse mapping $\beta \mapsto (q_1 \odot 1_N \odot p_1)\beta$ for some $\beta : M \Rightarrow \check{f} \circ N \circ \hat{f}$ in $\mathcal{H}(\mathbb{D})(A, A)$.

Similarly $\mathfrak{G}\mathscr{F} \cong \mathbb{D}_1^\bullet$, but we can also deduce that $\mathbb{D}_1^\bullet$ is a bifibration by Remark 5.1.1, since we have an adjunction $(\check{f} \odot \text{-} \odot \hat{f}) \vdash (\hat{f} \odot \text{-} \odot \check{f})$ for all f. ☐

Even though the above result was independently established as a generalization of earlier proofs, the fibration part was also included in [**FGK11**, Proposition 3.3]. We now proceed to the definitions of structures in arbitrary double categories, which are fundamental for the formalization of our examples.

A *monoid* in a double category $\mathbb{D}$ is an endo-1-cell $M : A \xrightarrow{\bullet} A$, *i.e.* an object in $\mathbb{D}_1^\bullet$, equipped with globular 2-morphisms

$$\begin{tikzcd} A \arrow[r, "M", "\bullet" marking] \arrow[d, "\mathrm{id}_\mathrm{A}"'] & A \arrow[r, "M", "\bullet" marking] \arrow[d, phantom, "\Downarrow m"] & A \arrow[d, "\mathrm{id}_A"] & & A \arrow[r, "1_A", "\bullet" marking] \arrow[d, "\mathrm{id}_\mathrm{A}"'] \arrow[dr, phantom, "\Downarrow\eta"] & A \arrow[d, "\mathrm{id}_A"] \\ A \arrow[rr, "M"', "\bullet" marking] & & A, & & A \arrow[r, "M"', "\bullet" marking] & A \end{tikzcd}$$

satisfying the usual associativity and unit laws. In fact, this is the same as a monad in its horizontal bicategory $\mathcal{H}(\mathbb{D})$. A *monoid homomorphism* consists of an arrow $\alpha_f : M_A \to N_B$ in $\mathbb{D}_1^\bullet$ which respects multiplication and unit:

$$\begin{tikzcd} A \arrow[r, "M", "\bullet" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & A \arrow[r, "M", "\bullet" marking] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\alpha"] & A \arrow[d, "f"] & & A \arrow[r, "M", "\bullet" marking] \arrow[d, equal] & A \arrow[r, "M", "\bullet" marking] \arrow[d, phantom, "\Downarrow m"] & A \arrow[d, equal] \\ B \arrow[r, "N"', "\bullet" marking] \arrow[d, equal] & B \arrow[r, "N"', "\bullet" marking] \arrow[d, phantom, "\Downarrow m"] & B \arrow[d, equal] & = & A \arrow[rr, "M"', "\bullet" marking] \arrow[d, "f"'] & \arrow[d, phantom, "\Downarrow\alpha"] & A \arrow[d, "f"] \\ A \arrow[rr, "N"', "\bullet" marking] & & A & & B \arrow[rr, "N"', "\bullet" marking] & & B, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "1_A", "\bullet" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & A \arrow[d, equal] & & A \arrow[r, "1_A", "\bullet" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow 1_f"] & A \arrow[d, "f"] \\ A \arrow[r, "M"', "\bullet" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & A \arrow[d, "f"] & = & B \arrow[r, "1_B"', "\bullet" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & B \arrow[d, equal] \\ B \arrow[r, "N"', "\bullet" marking] & B & & B \arrow[r, "N"', "\bullet" marking] & B. \end{tikzcd}$$

We obtain a category $\mathbf{Mon}(\mathbb{D})$, which is a non-full subcategory of $\mathbb{D}_1$.

These definitions can be found in [**Shu08**] for fibrant double categories, and in [**FGK11**] as *monads* and *vertical monad maps* in a double category $\mathbb{D}$. In the terminology of the latter, $\mathbf{Mon}(\mathbb{D})$ is in fact the vertical category of $\mathbb{M}\mathbf{nd}(\mathbb{D})$, the double category of monads, horizontal and vertical monad maps and monad squares.

Remark. Considering monads in a double category rather than in a bicategory or 2-category presents certain advantages. For example, $\mathcal{V}$-$\mathbf{Cat}$ is precisely $\mathbf{Mon}(\mathcal{V}\text{-}\mathbb{M}\mathbf{at})$: objects are monads $A : X \relbar\joinrel\mapstochar\joinrel\rightarrow X$ in the horizontal bicategory $\mathcal{H}(\mathcal{V}$-$\mathbb{M}\mathbf{at})$, and morphisms are $\mathcal{V}$-graph morphisms (*i.e.* in $\mathcal{V}\text{-}\mathbb{M}\mathbf{at}_1^\bullet$) which respect the appropriate structure.

It was noted in Remark 7.3.1 that even if objects of $\mathcal{V}$-$\mathbf{Mat}$ are monads in the bicategory of $\mathcal{V}$-matrices, $\mathcal{V}$-functors do not correspond bijectively to monad (op)functors in $\mathcal{V}$-$\mathbf{Mat}$. So, in order to fully describe $\mathcal{V}$-$\mathbf{Cat}$ as in Lemma 7.3.3, we had to provide isomorphic characterizations for $\mathcal{V}$-functors. Now things are much clearer: we are able to recapture the whole category as the category of monoids in a double category, since a $\mathcal{V}$-functor properly matches the notion of a monoid morphism in $\mathcal{V}$-$\mathbb{M}\mathbf{at}$.

Dually, we can define a category $\mathbf{Comon}(\mathbb{D})$ for any double category. Objects are *comonoids* in $\mathbb{D}$, *i.e.* horizontal endo-1-cells $C : A \xrightarrow{\bullet} A$ equipped with globular 1-morphisms

$$\begin{tikzcd}
A \arrow[rr, "C", "\bullet" marking] \arrow[d, "\mathrm{id_A}"'] & & A \arrow[d, "\mathrm{id}_A"] \\
A \arrow[r, "C"', "\bullet" marking] \arrow[urr, phantom, "\Downarrow\Delta"] & A \arrow[r, "C"', "\bullet" marking] & A,
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "C", "\bullet" marking] \arrow[d, "\mathrm{id_A}"'] \arrow[dr, phantom, "\Downarrow\epsilon"] & A \arrow[d, "\mathrm{id}_A"] \\
A \arrow[r, "C"', "\bullet" marking] & A
\end{tikzcd}$$

satisfying the usual coassociativity and counit axioms for a comonad in the horizontal bicategory $\mathcal{H}(\mathbb{D})$. Morphisms are *comonoid homomorphisms*, *i.e.* $\alpha_f : C_A \to D_B$ in $\mathbb{D}_1^\bullet$ satisfying dual axioms to the monoid ones. Notice that $\mathbf{Mon}(\mathbb{D}^{\mathrm{op}}) = \mathbf{Comon}(\mathbb{D})^{\mathrm{op}}$.

For the double category $\mathbb{D} = \mathcal{V}$-$\mathbb{M}\mathbf{at}$, the above exactly describe the category of $\mathcal{V}$-cocategories as in the Definition 7.3.8, thus $\mathbf{Comon}(\mathcal{V}\text{-}\mathbb{M}\mathbf{at}) = \mathcal{V}\text{-}\mathbf{Cocat}$. This is again conceptually simpler and more straightforward than the isomorphic characterization of $\mathcal{V}$-$\mathbf{Cocat}$ as in Lemma 7.3.11.

Proposition 8.2.8. *Let $\mathbb{D}$ be a fibrant double category. The forgetful functors*

$$\mathbf{Mon}(\mathbb{D}) \to \mathbb{D}_0 \quad \textit{and} \quad \mathbf{Comon}(\mathbb{D}) \to \mathbb{D}_0$$

which map a horizontal endo-1-cell to its object and a 2-morphism to its vertical 1-morphism, are a fibration and an opfibration respectively.

Proof. We can again directly generalize Propositions 7.5.3 and 7.5.5 by restricting (8.14) to the categories $\mathbf{Mon}(\mathcal{H}(\mathbb{D})(A, A))$ and $\mathbf{Comon}(\mathcal{H}(\mathbb{D})(A, A))$ respectively.

Alternatively, we can exhibit the cartesian lifting of a monoid $N : B \to B$

$$\begin{tikzcd}
\check{f} \odot N \odot \hat{f} \arrow[r, "\mathrm{Cart}(f,N)"] \arrow[d, dotted] & N \arrow[d, dotted] & \text{in } \mathbf{Mon}(\mathbb{D}) \\
A \arrow[r, "f"] & B & \text{in } \mathbb{D}_0
\end{tikzcd}$$

along a 1-morphism f to be the 2-morphism

$$\begin{tikzcd}
A \arrow[r, "\hat{f}", "\bullet" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow p_1"] & B \arrow[r, "N", "\bullet" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow 1_N"] & B \arrow[r, "\check{f}", "\bullet" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow q_1"] & A \arrow[d, "f"] \\
B \arrow[r, "1_B"', "\bullet" marking] & B \arrow[r, "N"', "\bullet" marking] & B \arrow[r, "1_B"', "\bullet" marking] & B.
\end{tikzcd}$$

The universal property can be easily verified using the properties of companions and conjoints. Similarly, we can provide the cocartesian liftings

$$\mathrm{Cocart}(f, C) : C \Rightarrow \hat{f} \odot C \odot \check{f} \equiv p_2 \odot 1_C \odot q_2 \tag{8.15}$$

for the forgetful $\mathbf{Comon}(\mathbb{D}) \to \mathbb{D}_0$. ☐

In the proof of [**FGK11**, Proposition 3.3], the new multiplication and unit of $(\check{f} \odot N \odot \hat{f})$ for a monoid N is explicitly stated, and an analogous version for the comultiplication and counit of $(\hat{f} \odot C \odot \check{f})$ for a comonoid can be written. Essentially, they are the same as the ones of Lemmas 7.3.2 and 7.3.10 for the particular case of $\mathcal{V}$-categories and $\mathcal{V}$-cocategories.

We now proceed with the appropriate concepts of modules and comodules in double categories, and the (op)fibrations they form over $\mathbf{Mon}(\mathbb{D})$ and $\mathbf{Comon}(\mathbb{D})$.

A *(left) M-module* for a monoid $M : A \to A$ in a double category $\mathbb{D}$ is a horizontal 1-cell $\Psi : Z \to A$ with specified target A, equipped with a globular 2-morphism

$$\begin{tikzcd}
Z \arrow[r, "\Psi", "\bullet" marking] \arrow[d, equal] & A \arrow[r, "M", "\bullet" marking] \arrow[d, phantom, "\Downarrow \mu"] & A \arrow[d, equal] \\
Z \arrow[rr, "\Psi"', "\bullet" marking] & & A
\end{tikzcd}$$

called the *action*, which satisfies the usual compatibility axioms with the multiplication and unit of the monoid M. In fact this coincides with the concept of a left M-module for a monad M in the horizontal bicategory $\mathcal{H}(\mathbb{D})$.

A *(left) module homomorphism* between a left M-module Ψ and a left N-module Ξ consists of a monoid map α_f from M to N along with a 2-morphism

$$\begin{tikzcd}
Z \arrow[r, "\Psi", "\bullet" marking] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow \beta"] & A \arrow[d, "f"] \\
W \arrow[r, "\Xi"', "\bullet" marking] & B
\end{tikzcd}$$

with specified target f, which satisfies the equality

$$\begin{tikzcd}
Z \arrow[r, "\Psi", "\bullet"{marking}] \arrow[d, equal] & A \arrow[r, "M", "\bullet"{marking}] \arrow[d, phantom, "\Downarrow\mu"] & A \arrow[d, equal] & & Z \arrow[r, "\Psi", "\bullet"{marking}] \arrow[d, "k"'] & A \arrow[r, "M", "\bullet"{marking}] \arrow[d, "f"] & A \arrow[d, "f"] \\
Z \arrow[rr, "\Psi"', "\bullet"{marking}] \arrow[d, "k"'] & \arrow[d, phantom, "\Downarrow\beta"] & A \arrow[d, "f"] & = & W \arrow[r, "\Xi"', "\bullet"{marking}] \arrow[d, equal] & B \arrow[r, "N"', "\bullet"{marking}] \arrow[d, phantom, "\Downarrow\mu"] & B \arrow[d, equal] \\
W \arrow[rr, "\Xi"', "\bullet"{marking}] & & B & & W \arrow[rr, "\Xi"', "\bullet"{marking}] & & B.
\end{tikzcd}$$

with $\Downarrow\beta$ in the square Z, A, W, B and $\Downarrow\alpha$ in the square A, A, B, B of the right-hand side.

Denote the category of (left) modules and module homomorphisms as $\mathbf{Mod}(\mathbb{D})$.

There are certain subcategories of importance to us: we can consider all left modules with fixed source Z and arrows $^{k}\beta^{f}$ with $k = \mathrm{id}_Z$ which form a category $^{Z}\mathbf{Mod}(\mathbb{D})$; we can also consider the category ${}_M\mathbf{Mod}(\mathbb{D})$ of all left M-modules and module homomorphisms $^{k}\beta^{f}$ with $f = \mathrm{id}_A$; finally we have the category ${}^{Z}_{M}\mathbf{Mod}(\mathbb{D})$ of all M-modules with source Z and globular 2-morphisms. As expected, the latter is the category ${}^{Z}_{M}\mathbf{Mod}(\mathcal{H}(\mathbb{D})) = \mathcal{H}(\mathbb{D})(Z,A)^{\mathcal{H}(\mathbb{D})(Z,M)}$ as in Definition 2.2.3.

We can dualize the above definitions to obtain the category $\mathbf{Comod}(\mathbb{D})$ of *(left) comodules* and *comodule homomorphisms* for any double category $\mathbb{D}$. Explicitly, for a comonoid $C : A \mathrel{-\!\!\bullet\!\!\!\rightarrow} A$ in $\mathbb{D}$, a left C-comodule is a horizontal 1-cell $\Phi : W \mathrel{-\!\!\bullet\!\!\!\rightarrow} A$ with target A, equipped with a globular 2-morphism

$$\begin{tikzcd}
W \arrow[rr, "\Phi", "\bullet"{marking}] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow\delta"] & A \arrow[d, equal] \\
W \arrow[r, "\Phi"', "\bullet"{marking}] & A \arrow[r, "C"', "\bullet"{marking}] & A
\end{tikzcd}$$

called the *coaction*, compatible with the comultiplication and counit of the comonoid C. A comodule homomorphism between a C-comodule Φ and a D-comodule Ω consists of a comonoid map α_f between C and D and a 2-morphism $^{k}\beta^{f} : \Phi \Rightarrow \Omega$ which respects the coactions. Notice how for both module and comodule maps, the target agrees with the source (and target) of the (co)monoid map, *i.e.* $\mathfrak{t}(\beta) = \mathfrak{s}(\alpha)$.

Once again, we have the subcategories $^{W}\mathbf{Comod}(\mathbb{D})$ of left comodules with fixed source W, ${}_C\mathbf{Comod}(\mathbb{D})$ of left C-comodules for a fixed comonoid C, and the category of left C-comodules with fixed target W

$$ {}^{W}_{C}\mathbf{Comod}(\mathbb{D}) := {}^{W}_{C}\mathbf{Comod}(\mathcal{H}(\mathbb{D})) = \mathcal{H}(\mathbb{D})(W,A)^{\mathcal{H}(\mathbb{D})(W,C)}. $$

We could appropriately define categories of *right modules* and *comodules* in a double category $\mathbb{D}$, as well as *bimodules* and *bicomodules.* In fact, bimodules between monoids are the horizontal 1-cells for a double category $\mathbb{M}\mathbf{od}(\mathbb{D})$ studied in [**Shu08**], in the context of fibrant double categories. According to the notation followed in this thesis though, **Mod** corresponds only to one-sided modules and **BMod** to two-sided.

Motivated by Section 7.6, we now focus on $^{Z}\mathbf{Mod}(\mathbb{D})$ and $^{W}\mathbf{Comod}(\mathbb{D})$. Explicitly, for $\mathbb{D}$=$\mathcal{V}$-$\mathbb{M}\mathbf{at}$ the categories $^{1}\mathbf{Mod}(\mathcal{V}\text{-}\mathbb{M}\mathbf{at})$ and $^{1}\mathbf{Comod}(\mathcal{V}\text{-}\mathbb{M}\mathbf{at})$ are precisely the global categories $\mathcal{V}$-**Mod** and $\mathcal{V}$-**Comod**, where $1 = \{*\}$ is the singleton.

Whenever appropriate, we will briefly remark what the results for the more general categories of modules and comodules would look like.

Proposition 8.2.9. *Suppose* $\mathbb{D}$ *is a fibrant double category. The categories* ${}^Z\mathbf{Mod}(\mathbb{D})$ *and* ${}^W\mathbf{Comod}(\mathbb{D})$ *are fibred and opfibred respectively over* $\mathbf{Mon}(\mathbb{D})$ *and* $\mathbf{Comon}(\mathbb{D})$*, for any 0-cells* Z *and* W*.*

Proof. Analogously to Propositions 7.6.9 and 7.6.10, the indexed categories which give rise to the fibration and opfibration in this case are

$$\begin{tikzcd}
\mathscr{H}\colon \mathbf{Mon}(\mathbb{D})^{\mathrm{op}} \arrow[r] & \mathbf{Cat}, \\
M \arrow[r, mapsto, dotted] \arrow[d, "\alpha_f"'] & {}^Z_M\mathbf{Mod}(\mathbb{D}) \\
N \arrow[r, mapsto, dotted] & {}^Z_N\mathbf{Mod}(\mathbb{D}) \arrow[u, "(\check{f}\odot\text{-})"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{S}\colon \mathbf{Comon}(\mathbb{D}) \arrow[r] & \mathbf{Cat} \\
C \arrow[r, mapsto, dotted] \arrow[d, "\alpha_f"'] & {}^W_C\mathbf{Comod}(\mathbb{D}) \arrow[d, "(\hat{f}\odot\text{-})"] \\
D \arrow[r, mapsto, dotted] & {}^W_D\mathbf{Comod}(\mathbb{D}).
\end{tikzcd}$$

As an illustration, if $\Psi : Z \,{-\!\!\bullet\!\!\to}\, B$ is a left N-module, then $\check{f}\odot\Psi : Z \,{-\!\!\bullet\!\!\to}\, B \,{-\!\!\bullet\!\!\to}\, A$ obtains the structure of a left M-module, via the action

$$\begin{tikzcd}
Z \arrow[r, "\Psi", "\bullet"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow 1_\Psi"] & B \arrow[r, "\check{f}", "\bullet"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow 1_{\check{f}}"] & A \arrow[rr, "M", "\bullet"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \lambda^{\text{-}1}"] & & A \arrow[d, equal] \,. \\
Z \arrow[r, "\Psi"', "\bullet"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow 1_\Psi"] & B \arrow[r, "\check{f}"', "\bullet"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow q_1"] & A \arrow[r, "M"', "\bullet"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow \alpha"] & A \arrow[r, "1_A"', "\bullet"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow q_2"] & A \arrow[d, equal] \\
Z \arrow[r, "\Psi"', "\bullet"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow 1_\Psi"] & B \arrow[r, "1_B"', "\bullet"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \rho"] & B \arrow[r, "N"', "\bullet"{marking}] & B \arrow[r, "\check{f}"', "\bullet"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow 1_{\check{f}}"] & A \arrow[d, equal] \\
Z \arrow[r, "\Psi"', "\bullet"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow \mu"] & B \arrow[rr, "N"', "\bullet"{marking}] & & B \arrow[r, "\check{f}"', "\bullet"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow 1_{\check{f}}"] & A \arrow[d, equal] \\
Z \arrow[rrr, "\Psi"', "\bullet"{marking}] & & & B \arrow[r, "\check{f}"', "\bullet"{marking}] & A
\end{tikzcd}$$

This essentially generalizes Lemma 7.6.5, which is clearer if we suppress the natural isomorphisms λ, ρ of the pseudo double category. In a dual way, we can determine the induced D-coaction on a composite horizontal 1-cell $\hat{f}\odot\Phi : Z \,{-\!\!\bullet\!\!\to}\, A \,{-\!\!\bullet\!\!\to}\, B$ for a left C-module Φ, adjusting Lemma 7.6.6.

Alternatively, we can deduce that the forgetful functors ${}^Z\mathbf{Mod}(\mathbb{D}) \to \mathbf{Mon}(\mathbb{D})$ and ${}^W\mathbf{Comod}(\mathbb{D}) \to \mathbf{Comon}(\mathbb{D})$ are a fibration and opfibration respectively, by exhibiting the (co)cartesian arrows. For any left N-module Ψ and any monoid homomorphism $\alpha_f : M \to N$, the required cartesian lifting $\mathrm{Cart}(\Psi, \alpha_f) : \check{f}\odot\Psi \to \Psi$ in ${}^Z\mathbf{Mod}(\mathbb{D})$ is the left-module morphism

$$\begin{tikzcd}
Z \arrow[r, "\Psi", "\bullet"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow 1_\Psi"] & B \arrow[r, "\check{f}", "\bullet"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow q_1"] & A \arrow[d, "f"] \\
Z \arrow[r, "\Psi"', "\bullet"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \lambda"] & B \arrow[r, "1_B"', "\bullet"{marking}] & B \arrow[d, equal] \\
Z \arrow[rr, "\Psi"', "\bullet"{marking}] & & B.
\end{tikzcd}$$

The universal property is easily checked by the relations between q_1 and q_2, and similarly we can write the cocartesian liftings for the second forgetful functor. $\square$

We could also establish a fibration $\mathbf{Mod}(\mathbb{D}) \to \mathbf{Mon}(\mathbb{D})$ and an opfibration $\mathbf{Comod}(\mathbb{D}) \to \mathbf{Comon}(\mathbb{D})$ for the categories of left modules and comodules with arbitrary sources. The fibre categories would then be ${}_M\mathbf{Mod}(\mathbb{D})$ and ${}_C\mathbf{Comod}(\mathbb{D})$ respectively, and the reindexing functors the same as above.

Remark. Consider the categories ${}^X\mathbb{D}_1$ for any 0-cell X, of horizontal 1-cells with domain X and 2-morphisms with source id_X. We can generalize Proposition 7.7.1 and deduce that ${}^Z\mathbf{Mod}(\mathbb{D})$ is monadic over the pullback category ${}^Z\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbf{Mon}(\mathbb{D})$, and ${}^W\mathbf{Comod}(\mathbb{D})$ is comonadic over ${}^W\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbf{Comon}(\mathbb{D})$. This further clarifies the structure and properties of these categories. Similarly for (co)modules of arbitrary domain, if we replace ${}^X\mathbb{D}_1$ by plain $\mathbb{D}_1$.

We have so far totally recovered the fibrational view of Sections 7.5 and 7.6 in the abstract framework of fibrant double categories. As remarked earlier, the definitions of $\mathbf{Mon}(\mathcal{V}\text{-}\mathbb{M}\mathbf{at})$ and $\mathbf{Comon}(\mathcal{V}\text{-}\mathbb{M}\mathbf{at})$ wholly encapsulate the categories $\mathcal{V}$-**Cat** and $\mathcal{V}$-**Cocat**, and the same applies to the categories $\mathcal{V}$-**Mod** and $\mathcal{V}$-**Comod** which are identified with $\mathbf{Mod}(\mathcal{V}\text{-}\mathbb{M}\mathbf{at})$ and $\mathbf{Comod}(\mathcal{V}\text{-}\mathbb{M}\mathbf{at})$. We now turn to the issue of enrichment between those categories.

In order to generalize the main results of the previous chapter in the monoidal double categorical context, we require the existence of the following functors (compare also with the beginning of this section): a pseudo double functor

$$\otimes : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{D} \tag{8.16}$$

which constitutes the tensor product of the double category, and a lax double functor

$$H : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \longrightarrow \mathbb{D} \tag{8.17}$$

with the property that H_0 gives a monoidal closed structure on $(\mathbb{D}_0, \otimes_0, I)$ and H_1 a monoidal closed structure on $(\mathbb{D}_1, \otimes_1, 1_I)$.

We could assume that the extra structure given by this lax double functor H makes $\mathbb{D}$ into a *monoidal closed double category.* However, this seems to not be the case, even if there is an analogy with Definition 8.2.3 of a monoidal double category, where the pseudo double functor $\otimes = (\otimes_0, \otimes_1)$ induces monoidal structures to the vertical and horizontal categories $\mathbb{D}_0$ and $\mathbb{D}_1$.

In [**GP04**], a (weakly) monoidal closed pseudo double category $\mathbb{D}$ is a monoidal double category such that each pseudo double functor $(\text{-} \otimes D) : \mathbb{D} \to \mathbb{D}$ has a lax right adjoint, call it $\mathrm{Hom}^{\mathbb{D}}$. Notice that in fact, $(\text{-} \otimes D) = (\text{-} \otimes_0 D, \text{-} \otimes_1 1_D)$. This falls into the more general case of *pseudo/lax adjunction* between pseudo double categories as described in [**GP04**, 3.2], whereas double adjunctions are also studied in [**FGK12**] in detail. Explicitly, it consists of two ordinary adjunctions

$$\begin{tikzcd} \mathbb{D}_0 \arrow[r, shift left, "(-\otimes_0 D)"] \arrow[r, phantom, "\perp"] & \mathbb{D}_0 \arrow[l, shift left, "\mathrm{Hom}_0^{\mathbb{D}}(D,-)"] \end{tikzcd} , \qquad \begin{tikzcd} \mathbb{D}_1 \arrow[r, shift left, "-\otimes_1 1_D)"] \arrow[r, phantom, "\perp"] & \mathbb{D}_1 \arrow[l, shift left, "\mathrm{Hom}_1^{\mathbb{D}}(1_D,-)"] \end{tikzcd}$$

for any 0-cell D in $\mathbb{D}$, with units and counits $\eta_{0,1}, \varepsilon_{0,1}$ satisfying appropriate triangle identities, such that conditions expressing compatibility with the horizontal composition and identities are satisfied. It immediately follows that $\mathbb{D}_0$ is a monoidal closed category, but this cannot be deduced for $\mathbb{D}_1$ since 1_D is not an arbitrary horizontal 1-cell.

We call a monoidal pseudo double category equipped with a functor H as in (8.17) with such properties a *locally monoidal closed double category.* The above arguments justify that a monoidal closed structure on a double category does not imply a locally monoidal closed structure.

For example, consider the monoidal double category $\mathbb{V}$-$\mathbb{M}\mathbf{at}$. The tensor product is given by $\otimes_0 = \times$, the cartesian monoidal structure in **Set**, and $\otimes_1$ defined as in (8.13). Moreover, if $\mathcal{V}$ is monoidal closed and has products, there is a lax double functor $H = (H_0, H_1)$ defined as follows. On the vertical category, we have the exponentiation functor

$$H_0 : \mathbf{Set}^{\mathrm{op}} \times \mathbf{Set} \xrightarrow{(-)^{(-)}} \mathbf{Set}$$

which is the internal hom in **Set**. On the horizontal category

$$H_1 : \ \mathcal{V}\text{-}\mathbb{M}\mathbf{at}_1^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbb{M}\mathbf{at}_1 \longrightarrow \mathcal{V}\text{-}\mathbb{M}\mathbf{at}_1$$

$$\begin{tikzcd}
(X \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & Y \arrow[d, "g"] & {,} & Z \arrow[r, "T", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta"] & W) \arrow[d, "k"] \arrow[r, mapsto, dotted] & Z^X \arrow[r, "H_1(S,T)", "\shortmid"{marking}] \arrow[d, "h^f"'] \arrow[dr, phantom, "\Downarrow H_1(\alpha,\beta)"] & W^Y \arrow[d, "k^g"] \\
(X' \arrow[r, "S'"', "\shortmid"{marking}] & Y' & {,} & Z' \arrow[r, "T'"', "\shortmid"{marking}] & W') \arrow[r, mapsto, dotted] & Z'^{X'} \arrow[r, "H_1(S',T')"', "\shortmid"{marking}] & W'^{Y'}
\end{tikzcd}$$

is defined on objects as $H_1(S,T)(m,n) = \prod_{(y,x)}[S(y,x), T(m(y), n(x))]$ for all $m \in W^Y$, $n \in Z^X$, and on arrows as

$$\begin{gathered}
H_1(\alpha,\beta) : H_1(S,T)(m,n) \to H_1(S',T')(k^g(m), h^f(n)) \equiv \\
\prod_{\substack{y \in Y \\ x \in X}} [S(y,x), T(m(y), n(x))] \to \prod_{\substack{y' \in Y' \\ x' \in X'}} [S'(y',x'), T'(kmg(y'), hnf(x'))]
\end{gathered}$$

which corresponds under the adjunction (- $\otimes X) \dashv [X, -]$ in $\mathcal{V}$ for fixed y', x' to the composite

$$\begin{tikzcd}
\prod\limits_{\substack{y \in Y \\ x \in X}} [S(y,x), T(my,nx)] \otimes S'(y',x') \arrow[r, dashed] \arrow[d, "1 \otimes \alpha_{y',x'}"'] & T'(kmgy', hnfx') \\
\prod\limits_{\substack{y \in Y \\ x \in X}} [S(y,x), T(my,nx)] \otimes S(gy',fx') \arrow[d, "\pi_{gy',fx'} \otimes 1"'] & \\
{[S(gy',fx'), T(mgy',nfx')] \otimes S(gy',fx')} \arrow[r, "\mathrm{ev}"] & T(mgy',nfx'). \arrow[uu, "\beta_{mgy',nfx'}"']
\end{tikzcd}$$

The globular transformations

$$H_1(R,O) \odot H_1(S,T) \xrightarrow{\sim} H_1(R \odot S, O \odot T), \quad 1_{H_0(X,Y)} \xrightarrow{\sim} H_1(1_X, 1_Y)$$

which make $H = (H_0, H_1)$ into a lax double functor are as in (7.9), (7.10). The functor H_1 constitutes a monoidal closed structure for $(\mathcal{V}\text{-}\mathbb{M}\mathbf{at}_1, \otimes_1, 1_I)$, the proof being essential Proposition 7.2.3 in the more general case of arbitrary horizontal 1-cells and not only endoarrows like $\mathcal{V}$-graphs.

For an arbitrary locally monoidal closed double category $\mathbb{D}$, we now aim to investigate possible enrichment relations between the (op)fibrations of Propositions 8.2.8 and 8.2.9. The following properties of double functors resemble to properties of monoidal functors studied in Chapter 3.

Proposition 8.2.10. *Any lax double functor* $(F_0, F_1) : \mathbb{D} \to \mathbb{E}$ *induces an ordinary functor*

$$\mathbf{Mon}F : \mathbf{Mon}(\mathbb{D}) \to \mathbf{Mon}(\mathbb{E})$$

between the categories of monoids, which is F_1 *restricted to* $\mathbf{Mon}\mathbb{D}$. *Dually, any colax double functor induces a functor between the categories of comonoids.*

Remark. Since monoids in a double category are monads in its horizontal bicategory and a lax double functor induces a lax functor between the horizontal bicategories, the above statement on the level of objects coincides with Remark 2.2.2.

Proof. A monoid $M : A \xrightarrow{\bullet} A$ with $m : M \odot M \to M$ and $\eta : 1_M \to M$ is mapped to $F_1M : F_0A \xrightarrow{\bullet} F_0A$ with multiplication and unit

$$\begin{tikzcd}
F_0A \arrow[r, "F_1M", "\bullet" marking] \arrow[d, equal] & F_0A \arrow[r, "F_1M", "\bullet" marking] \arrow[d, phantom, "\Downarrow F_\odot"] & F_0A \arrow[d, equal] \\
F_0A \arrow[rr, "F_1(M\odot M)"', "\bullet" marking] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow F_1m"] & F_0A \arrow[d, equal] \\
F_0A \arrow[rr, "F_1M"', "\bullet" marking] & & F_0A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
F_0A \arrow[r, "F_1(1_A)", "\bullet" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow F_U"] & F_0A \arrow[d, equal] \\
F_0A \arrow[r, "1_{F_0A}"', "\bullet" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow F_1\eta"] & F_0A \arrow[d, equal] \\
F_0A \arrow[r, "F_1M"', "\bullet" marking] & F_0A
\end{tikzcd}$$

and the axioms follow from the axioms for $F_\odot$ and F_U. A monoid arrow $\alpha_f : M \to N$ is mapped to

$$\begin{tikzcd}
F_0A \arrow[r, "F_1M", "\bullet" marking] \arrow[d, "F_0f"'] \arrow[dr, phantom, "\Downarrow F_1\alpha"] & F_0A \arrow[d, "F_0f"] \\
F_0B \arrow[r, "F_1N"', "\bullet" marking] & F_0B
\end{tikzcd}$$

which respects multiplications and units by naturality of $F_\odot$ as in (8.12) and F_U. □

Proposition 8.2.11. *Any lax double functor* $F : \mathbb{D} \to \mathbb{E}$ *induces a functor*

$$^Z\mathbf{Mod}F : {}^Z\mathbf{Mod}(\mathbb{D}) \to {}^{F_0Z}\mathbf{Mod}(\mathbb{E})$$

between the categories of modules, which is a restriction of F_1. *Dually, any colax double functor* G *induces a functor*

$$^W\mathbf{Comod}G : {}^W\mathbf{Comod}(\mathbb{D}) \to {}^{G_0W}\mathbf{Comod}(\mathbb{E}).$$

Proof. On the level of objects, Proposition 2.2.10 gives functors

$$
\begin{aligned}
{}^{Z}_{M}\mathbf{Mod}F : {}^{Z}_{M}\mathbf{Mod}(\mathbb{D}) &\to {}^{F_0Z}_{F_1M}\mathbf{Mod}(\mathbb{E})\\
{}^{W}_{C}\mathbf{Comod}G : {}^{W}_{C}\mathbf{Comod}(\mathbb{D}) &\to {}^{G_0W}_{G_1C}\mathbf{Comod}(\mathbb{E})
\end{aligned}
$$

since (co)modules for a (co)monoid in a double category are (co)modules for a (co)monad in its horizontal bicategory. The F_1M-action on $F_1\Psi : F_0Z \xrightarrow{\bullet} F_0A$ for (Ψ, μ) a left M-module is just

$$\begin{tikzcd}
F_0Z \arrow[r, "F_1\Psi"] \arrow[d, equal] & F_0A \arrow[r, "F_1M"] \arrow[d, phantom, "\Downarrow F_\odot"] & F_0A \arrow[d, equal] \\
F_0Z \arrow[rr, "F_1(M\odot\Psi)"'] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow F_1\mu"] & F_0A \arrow[d, equal] \\
F_0Z \arrow[rr, "F_1\Psi"'] & & F_0A.
\end{tikzcd}$$

On arrows, the fact that the image ${}^{\mathrm{id}_{F_0Z}}(F_1\beta)^{F_0f} : F_1\Psi \Rightarrow F_1\Xi$ of a left module morphism β commutes with the induced actions on $F_1\Psi$, $F_1\Xi$ is easily verified, by naturality of $F_\odot$ and axioms for β. □

The functors ${}^{Z}\mathbf{Mod}F$ and ${}^{W}\mathbf{Comod}G$ are in fact special cases of the more general $\mathbf{Mod}F : \mathbf{Mod}(\mathbb{D}) \to \mathbf{Mod}(\mathbb{E})$ and $\mathbf{Comod}G : \mathbf{Comod}(\mathbb{D}) \to \mathbf{Comod}(\mathbb{E})$, between categories of (co)modules of arbitrary source, with a (co)action of any (co)monoid.

Motivated by our original examples, we wish to employ functors between categories of modules with strictly the same domain. The following lemma shows how under certain assumptions on $\mathbb{D}$ (but not in general), isomorphic 0-cells in $\mathbb{D}_0$ determine equivalent categories of modules with such domains.

Lemma 8.2.12. *Suppose $\mathbb{D}$ is a fibrant double category. If two objects Z and W are isomorphic in $\mathbb{D}_0$, there is an equivalence between the categories of (left) modules with fixed domain Z and W,* i.e. ${}^{Z}\mathbf{Mod}(\mathbb{D}) \simeq {}^{W}\mathbf{Mod}(\mathbb{D})$.

Proof. Recall that for any isomorphism f in $\mathbb{D}_0$, the adjunction $\hat{f} \dashv \check{f}$ in $\mathcal{H}(\mathbb{D})$ is an adjoint equivalence, and in particular the unit and counit $\check{\eta}$, $\check{\varepsilon}$ are isomorphisms ([**Shu10**, Lemma 3.21]).

Denote by $f : Z \xrightarrow{\sim} W$ the vertical isomorphism between the 0-cells. The functor $(\text{-} \odot \check{f}) : {}^{Z}\mathbb{D}_1 \to {}^{W}\mathbb{D}_1$ between categories of horizontal 1-cells with fixed domains and 2-morphisms with sources vertical identities, has an inverse up to isomorphism, namely the functor $(\text{-}\odot\hat{f})$. For example, there is a natural isomorphism

$$\begin{tikzcd}
Z \arrow[rrr, "\Psi"] \arrow[d, equal] & & \arrow[d, phantom, "\Downarrow \rho^{\text{-}1}"] & A \arrow[d, equal] \\
Z \arrow[rr, "1_Z"'] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow \check{\eta}"] & Z \arrow[r, "\Psi"'] \arrow[d, equal] \arrow[rd, phantom, "\Downarrow 1_\Psi"] & A \arrow[d, equal] \\
Z \arrow[r, "\hat{f}"'] & W \arrow[r, "\check{f}"'] & Z \arrow[r, "\Psi"'] & A
\end{tikzcd}$$

between Ψ and $\Psi \odot \check{f} \odot \hat{f}$, since in this case $\check{\eta}$ is invertible. This equivalence in fact lifts to the categories of horizontal 1-cells with the structure of a left M-module for an arbitrary monoid M in $\mathbb{D}$, *i.e.* ${}^{(\text{-})}\mathbf{Mod}(\mathbb{D})$. ☐

We can now apply the above results to the double functors $\otimes$ (8.16) and H (8.17) for our fibrant locally monoidal closed double category $\mathbb{D}$.

Firstly, in any monoidal double category, the tensor product of $(\mathbb{D}_1, \otimes_1, 1_I)$ restricts to the category $\mathbb{D}_1^\bullet$ of endo-1-cells, therefore $(\mathbb{D}_1^\bullet, \otimes_1, 1_I)$ is a monoidal category itself. Then, by Proposition 8.2.10, the pseudo double functor $\otimes$ induces (ordinary) functors

$$\mathbf{Mon}\otimes : \mathbf{Mon}(\mathbb{D}) \times \mathbf{Mon}(\mathbb{D}) \to \mathbf{Mon}(\mathbb{D})$$
$$\mathbf{Comon}\otimes : \mathbf{Comon}(\mathbb{D}) \times \mathbf{Comon}(\mathbb{D}) \to \mathbf{Comon}(\mathbb{D}),$$

given by $\otimes_1$ between the specific subcategories of $\mathbb{D}_1^\bullet$. The unit element is still $1_I : I \looparrowright I$ for I the unit of $\mathbb{D}_0$.

Proposition 8.2.13. *If $\mathbb{D}$ is a monoidal double category, then the categories $\mathbb{D}_1^\bullet$, $\mathbf{Mon}(\mathbb{D})$ and $\mathbf{Comon}(\mathbb{D})$ inherit a monoidal structure from $\mathbb{D}_1$.*

For the monoidal double category $\mathbb{D} = \mathcal{V}$-$\mathbb{M}\mathbf{at}$, this directly implies that the categories $\mathcal{V}$-**Grph**, $\mathcal{V}$-**Cat** and $\mathcal{V}$-**Cocat** obtain a monoidal structure essentially given by (8.13), which of course agrees with the previous chapter.

Furthermore, by Proposition 8.2.11 the tensor product also gives rise to functors

$${}^{(Z,Z')}\mathbf{Mod}\otimes : {}^{Z}\mathbf{Mod}(\mathbb{D}) \times {}^{Z'}\mathbf{Mod}(\mathbb{D}) \to {}^{Z\otimes_0 Z'}\mathbf{Mod}(\mathbb{D})$$
$${}^{(W,W')}\mathbf{Comod}\otimes : {}^{W}\mathbf{Comod}(\mathbb{D}) \times {}^{W'}\mathbf{Comod}(\mathbb{D}) \to {}^{W\otimes_0 W'}\mathbf{Comod}(\mathbb{D}).$$

For the general categories of (left) modules and comodules with arbitrary domain $\mathbf{Mod}(\mathbb{D})$ and $\mathbf{Comod}(\mathbb{D})$, these mappings turn out to induce monoidal structures with unit element 1_I. However, since we are here interested in categories with fixed domains and in particular ${}^{I}\mathbf{Mod}(\mathbb{D})$ and ${}^{I}\mathbf{Comod}(\mathbb{D})$ because of our motivating example, the following 'modified' monoidal structure is essential.

Lemma 8.2.14. *Suppose that $\mathbb{D}$ is a fibrant monoidal double category. The categories ${}^{I}\mathbf{Mod}(\mathbb{D})$ and ${}^{I}\mathbf{Comod}(\mathbb{D})$ inherit a 'tensor product' functor from $\mathbb{D}_1$.*

Proof. Since $\mathbb{D}_0$ is a monoidal category with $\otimes_0$, there exists a vertical isomorphism $r^0_I{=}l^0_I : I \otimes_0 I \xrightarrow{\sim} I$. Hence, by Lemma 8.2.12 we have an equivalence

$${}^{I\otimes_0 I}\mathbf{Mod}(\mathbb{D}) \simeq {}^{I}\mathbf{Mod}(\mathbb{D}) \tag{8.18}$$

between the categories of left modules with domain $I \otimes_0 I$ and of those with domain I. We can thus define a composite functor

$$\tilde{\otimes} : {}^{I}\mathbf{Mod}(\mathbb{D}) \times {}^{I}\mathbf{Mod}(\mathbb{D}) \xrightarrow{{}^{(I,I)}\mathbf{Mod}\otimes} {}^{I\otimes_0 I}\mathbf{Mod}(\mathbb{D}) \xrightarrow{\simeq} {}^{I}\mathbf{Mod}(\mathbb{D}) \tag{8.19}$$

where the first functor is $\otimes_1$ and the second is the equivalence $(\text{-} \odot \check{r}_I)$. It can be checked that this composite is equipped with natural coherent isomorphisms $(\Psi \tilde{\otimes} \Xi) \tilde{\otimes} \Theta \cong \Psi \tilde{\otimes} (\Xi \tilde{\otimes} \Theta)$, coming from the respective ones for $\otimes_1$. Similarly, we can work out a tensor product for ${}^{I}\mathbf{Comod}(\mathbb{D})$, making use of the equivalence ${}^{I\otimes_0 I}\mathbf{Comod}(\mathbb{D}) \simeq^{I} \mathbf{Comod}(\mathbb{D})$. ☐

Even though, intuitively, this functor should give rise to a monoidal structure on ${}^{I}\mathbf{Mod}(\mathbb{D})$, the natural choice of $1_I : I \mathrel{-\!\!\bullet\!\!\to} I$ does not serve as the monoidal unit for $\tilde{\otimes}$ as in (8.19). This is due to the fact that there is not an evident isomorphism between

$$I \overset{\check{r_I}}{\mathrel{-\!\!\bullet\!\!\to}} I \otimes_0 I \overset{\Psi\otimes_1 1_I}{\mathrel{-\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!\to}} A \otimes I \quad \text{and} \quad I \overset{\Psi}{\mathrel{-\!\!\bullet\!\!\to}} A$$

unless, for example, q_2 for the conjoint $\check{r_I}$ is invertible. However, when the equivalence (8.18) is an isomorphism, we can deduce that $({}^{I}\mathbf{Mod}(\mathbb{D}), \tilde{\otimes}, 1_I)$ is a monoidal category. This is again motivated by $\mathbb{D} = \mathcal{V}\text{-}\mathbb{M}\mathbf{at}$, where $I{=}\{*\}$ is the singleton set.

Now consider the lax double functor $H : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathbb{D}$ on a locally monoidal closed double category $\mathbb{D}$. First of all, it is easy to see that H_1 restricts to the subcategory $\mathbb{D}_1^{\bullet}$ of endo-1-cells. Also, the natural isomorphism

$$\mathbb{D}_1(M \otimes_1 N, P) \cong \mathbb{D}_1(M, H_1(N, P))$$

which defines the adjunction $(- \otimes_1 N) \dashv H_1(N, -)$ implies that $\mathbb{D}_1^{\bullet}$ is also a monoidal closed category. For example, for $\mathbb{D} = \mathcal{V}\text{-}\mathbb{M}\mathbf{at}$ this gives the monoidal closed structure on $\mathcal{V}$-$\mathbf{Grph}$. Then, by Proposition 8.2.10 there is an induced ordinary functor

$$\mathbf{Mon}H : \mathbf{Comon}(\mathbb{D})^{\mathrm{op}} \times \mathbf{Mon}(\mathbb{D}) \to \mathbf{Mon}(\mathbb{D}) \tag{8.20}$$

which is H_1 on the category $\mathbf{Mon}(\mathbb{D}^{\mathrm{op}} \times \mathbb{D}) \cong \mathbf{Mon}(\mathbb{D}^{\mathrm{op}}) \times \mathbf{Mon}(\mathbb{D})$. It is now easy to verify that for any monoid $M : A \mathrel{-\!\!\bullet\!\!\to} A$, the diagram

$$\begin{tikzcd}
\mathbf{Comon}(\mathbb{D})^{\mathrm{op}} \arrow[r, "H_1(-{,}M)"] \arrow[d] & \mathbf{Mon}(\mathbb{D}) \arrow[d] \\
\mathbb{D}_0^{\mathrm{op}} \arrow[r, "H_0(-{,}A)"'] & \mathbb{D}_0.
\end{tikzcd}$$

commutes. There is also an adjunction between the base categories

$$\begin{tikzcd}
\mathbb{D}_0 \arrow[r, shift left, "H_0^{\mathrm{op}}(-{,}A)"] \arrow[r, phantom, "\perp"] & \mathbb{D}_0^{\mathrm{op}} \arrow[l, shift left, "H_0(-{,}A)"]
\end{tikzcd}$$

for the monoidal closed category $\mathbb{D}_0$. If $\mathbb{D}$ is moreover fibrant, the legs of the diagram are fibrations by Proposition 8.2.8. Lastly, if $\mathbb{D}$ is *symmetric* monoidal (for the explicit definition, see [**Shu10**]), the internal homs H_0 and H_1 of the monoidal closed categories $\mathbb{D}_0$ and $\mathbb{D}_1$ are actions of the monoidal $\mathbb{D}_0^{\mathrm{op}}$, $\mathbb{D}_1^{\mathrm{op}}$ on the ordinary $\mathbb{D}_0$, $\mathbb{D}_1$ by Lemma 4.3.2. Subsequently H_0^{op} and

$$H_1^{\mathrm{op}} : \mathbb{D}_1 \times \mathbb{D}_1^{\mathrm{op}} \to \mathbb{D}_1^{\mathrm{op}}$$

are actions too. Then the opposite $\mathbf{Mon}H^{\mathrm{op}}$ of the induced functor between monoids as in (8.20) is an action of the monoidal category $\mathbf{Comon}(\mathbb{D})$ on the opposite category $\mathbf{Mon}(\mathbb{D})^{\mathrm{op}}$, since the forgetful $\mathbf{Mon}(\mathbb{D}) \to \mathbb{D}_1$ reflects isomorphisms.

We can now combine the above with Theorem 8.1.6 of the previous section to outline how we could obtain an enriched opfibration from the above data.

THEOREM 8.2.15. *Suppose* $\mathbb{D}$ *is a fibrant symmetric locally monoidal closed double category.*

(1) *If* $H_1^{\mathrm{op}} : \mathbf{Comon}(\mathbb{D}) \times \mathbf{Mon}(\mathbb{D})^{\mathrm{op}} \to \mathbf{Mon}(\mathbb{D})^{\mathrm{op}}$ *is cocartesian with a parametrized adjoint* P*, the categories* $\mathbf{Mon}(\mathbb{D})^{\mathrm{op}}$ *and* $\mathbf{Mon}(\mathbb{D})$ *are enriched in* $\mathbf{Comon}(\mathbb{D})$*.*

(2) *If furthermore*

$$\begin{tikzcd}
\mathbf{Comon}(\mathbb{D}) \arrow[r, shift left, "H_1^{\mathrm{op}}(-{,}M)"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathbf{Mon}(\mathbb{D})^{\mathrm{op}} \arrow[l, shift left, "P(-{,}M)"] \arrow[d] \\
\mathbb{D}_0 \arrow[r, shift left, "H_0^{\mathrm{op}}(-{,}A)"] \arrow[r, phantom, "\perp"] & \mathbb{D}_0^{\mathrm{op}} \arrow[l, shift left, "H_0(-{,}A)"]
\end{tikzcd}$$

is a general opfibred adjunction for any monoid $M : A \mathrel{-\!\!\bullet\!\!\!\rightarrow} A$ *in* $\mathbb{D}$*, then the fibration* $\mathbf{Mon}(\mathbb{D}) \to \mathbb{D}_0$ *is enriched in the monoidal opfibration* $\mathbf{Comon}(\mathbb{D}) \to \mathbb{D}_0$*.*

Notice that the forgetful $\mathbf{Comon}(\mathbb{D}) \to \mathbb{D}_0$ is a monoidal fibration for any fibrant monoidal double category $\mathbb{D}$: by definition, the tensor product of two comonoids has source and target the tensor product $\otimes_0$ of the 0-cells in $\mathbb{D}_0$, and it can also be verified that $\otimes_1$ preserves the cocartesian liftings (8.15) in $\mathbf{Comon}(\mathbb{D})$. Moreover, for the existence of such an adjoint P and the establishment of a parametrized adjunction in **OpFib** we can evidently employ Lemma 5.3.6 and Theorem 5.3.7 .

We now shift to the level of modules and comodules in a fibrant locally monoidal closed double category, still focusing on categories of horizontal 1-cells with fixed domain I, the monoidal unit of $\mathbb{D}_0$. By Proposition 8.2.11, the lax double functor H gives rise to a functor

$$^{(Z,W)}\mathbf{Mod}H : \ ^{Z}\mathbf{Comod}(\mathbb{D})^{\mathrm{op}} \times {}^{W}\mathbf{Mod}(\mathbb{D}) \to {}^{H_0(Z,W)}\mathbf{Mod}(\mathbb{D})$$

which is H_1 on ${}^{(Z,W)}\mathbf{Mod}(\mathbb{D}^{\mathrm{op}} \times \mathbb{D}) \cong {}^{Z}\mathbf{Mod}(\mathbb{D}^{\mathrm{op}}) \times {}^{W}\mathbf{Mod}(\mathbb{D})$. We now obtain a commutative diagram

$$\begin{tikzcd}
{}^{I}\mathbf{Comod}(\mathbb{D})^{\mathrm{op}} \arrow[r, "{}^{(I,I)}\mathbf{Mod}H(-{,}\Psi)"] \arrow[d] & {}^{H_0(I,I)}\mathbf{Mod}(\mathbb{D}) \arrow[r, "\simeq"] \arrow[d] & {}^{I}\mathbf{Mod}(\mathbb{D}) \arrow[dl] \\
\mathbf{Comon}(\mathbb{D})^{\mathrm{op}} \arrow[r, "\mathbf{Mon}H(-{,}M)"'] & \mathbf{Mon}(\mathbb{D}) &
\end{tikzcd}$$

for any left M-module Ψ, where by Lemma 8.2.12 the equivalence is the functor (- $\odot \breve{g}$), for $g : H_0(I, I) \cong I$ the isomorphism in the monoidal closed category $\mathbb{D}_0$.

The following roughly sketches how we can establish the enrichment of ${}^{I}\mathbf{Mod}(\mathbb{D})$ in ${}^{I}\mathbf{Comod}(\mathbb{D})$, as in our particular examples. Notice that the modified tensor product of Lemma 8.2.14 gives a monoidal structure on ${}^{I}\mathbf{Comod}(\mathbb{D})$ only when the equivalence between ${}^{I}\mathbf{Comod}(\mathbb{D})$ and ${}^{I\otimes_0 I}\mathbf{Comod}(\mathbb{D})$ is actually an isomorphism.

THEOREM 8.2.16. *Suppose that the assumptions of Theorem 8.2.15 hold, and also* ${}^{I}\mathbf{Comod}(\mathbb{D}) \cong {}^{I\otimes_0 I}\mathbf{Comod}(\mathbb{D})$. *If the functor* $\mathbf{Mod}H$ *has a parametrized adjoint* Q *such that for any left* M*-module* Ψ,

$$\begin{tikzcd}
{}^{I}\mathbf{Comod}(\mathbb{D}) \arrow[r, shift left, "H_1^{\mathrm{op}}(-{,}\Psi)\odot\check{g}"] \arrow[r, phantom, "\perp"] \arrow[d] & {}^{I}\mathbf{Mod}(\mathbb{D})^{\mathrm{op}} \arrow[l, shift left, "\bar{Q}(-\odot\hat{g}{,}\Psi)"] \arrow[d] \\
\mathbf{Comon}(\mathbb{D}) \arrow[r, shift left, "H_1^{\mathrm{op}}(-{,}M)"] \arrow[r, phantom, "\perp"] & \mathbf{Mon}(\mathbb{D})^{\mathrm{op}} \arrow[l, shift left, "Q(-{,}M)"]
\end{tikzcd}$$

is a general opfibred adjunction, then the fibration ${}^{I}\mathbf{Mod}(\mathbb{D}) \to \mathbf{Mon}(\mathbb{D})$ *is enriched in the monoidal opfibration* ${}^{I}\mathbf{Comod}(\mathbb{D}) \to \mathbf{Comon}(\mathbb{D})$.

We should stress that the above two theorems are just an attempt to place the most significant results and concepts of this thesis in a framework where they may arise in a natural way, rather than of actual importance on their own right as mathematical statements. What should be quite noticeable about this final section is that we are more interested in fitting this recurring duality and enrichment picture into a general theory via fibrations, than determining the more technical specifications required for the exact enrichments to appear, as was the focus in the previous two chapters. This explains why we have not addressed particular issues, such as existence of limits and colimits in the categories involved, monadicity, continuity of the key functors, cartesianness and fibrewise limits as well as local presentability, which were broadly studied previously.

Hence, the significance of this abstraction basically lies in the clarification of a setting for an enriched fibration picture between categories of a dual flavor, and moreover and perhaps most importantly, the possibility of further applications in the context of other double categories/bicategories. Regarding this last aspect we should point out the following, without proceeding into a more detailed description due to the conceptual limits of this thesis. In the context of a bicategory of $\mathcal{V}$*-symmetries*, following a similar process we would possibly be able to establish enrichments of categories of $\mathcal{V}$*-operads* in $\mathcal{V}$*-cooperads*, and $\mathcal{V}$-operad *modules* in $\mathcal{V}$-cooperad *comodules*. Evidently, this indicates the necessity of further work in this area.

Bibliography

[AJ13] Matthieu Anel and André Joyal. Sweedler theory of (co)algebras and the bar-cobar constructions. arXiv:1309.6952 [math.CT], 2013.

[AM10] Marcelo Aguiar and Swapneel Mahajan. *Monoidal functors, species and Hopf algebras*, volume 29 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2010. With forewords by Kenneth Brown and Stephen Chase and André Joyal.

[AP03] Jiří Adámek and Hans-E. Porst. On varieties and covarieties in a category. *Math. Structures Comput. Sci.*, 13(2):201–232, 2003. Coalgebraic methods in computer science (Genova, 2001).

[AR94] Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994.

[Bar74] Michael Barr. Coalgebras over a commutative ring. *J. Algebra*, 32(3):600–610, 1974.

[Bat91] Marjorie Batchelor. Measuring coalgebras, quantum group-like objects and noncommutative geometry. In *Differential geometric methods in theoretical physics (Rapallo, 1990)*, volume 375 of *Lecture Notes in Phys.*, pages 47–60. Springer, Berlin, 1991.

[Bat94] Marjorie Batchelor. Difference operators, measuring coalgebras, and quantum group-like objects. *Adv. Math.*, 105(2):190–218, 1994.

[Bat00] Marjorie Batchelor. Measuring comodules—their applications. *J. Geom. Phys.*, 36(3-4):251–269, 2000.

[BCSW83] Renato Betti, Aurelio Carboni, Ross Street, and Robert Walters. Variation through enrichment. *J. Pure Appl. Algebra*, 29(2):109–127, 1983.

[Bén67] Jean Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, pages 1–77. Springer, Berlin, 1967.

[Bén73] J. Bénabou. *Les distributeurs.* Rapport (Université catholique de Louvain (1970-).33 Séminaire de mathématique pure)). Institut de mathématique pure et appliquée, Université catholique de Louvain, 1973.

[Bén85] Jean Bénabou. Fibered categories and the foundations of naive category theory. *J. Symbolic Logic*, 50(1):10–37, 1985.

[BKP89] R. Blackwell, G. M. Kelly, and A. J. Power. Two-dimensional monad theory. *J. Pure Appl. Algebra*, 59(1):1–41, 1989.

[BL85] Richard E. Block and Pierre Leroux. Generalized dual coalgebras of algebras, with applications to cofree coalgebras. *J. Pure Appl. Algebra*, 36(1):15–21, 1985.

[BN96] John C. Baez and Martin Neuchl. Higher-dimensional algebra. I. Braided monoidal 2-categories. *Adv. Math.*, 121(2):196–244, 1996.

[Bor94a] Francis Borceux. *Handbook of categorical algebra. 1*, volume 50 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1994. Basic category theory.

[Bor94b] Francis Borceux. *Handbook of categorical algebra. 2*, volume 51 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1994. Categories and structures.

[BS76] Ronald Brown and Christopher B. Spencer. Double groupoids and crossed modules. *Cahiers Topologie Géom. Différentielle*, 17(4):343–362, 1976.

[Bun13] Marta Bunge. Tightly bounded completions. *Theory Appl. Categ.*, 28:No. 8, 213–240, 2013.

[Car95] Sean Carmody. *Cobordism categories.* PhD thesis, University of Cambridge, 1995.

[CGR12] Eugenia Cheng, Nick Gurski, and Emily Riehl. Multivariableadjunctions and mates. arXiv:1208.4520 [math.CT], 2012.

[DNR01] Sorin Dăscălescu, Constantin Năstăsescu, and Şerban Raianu. *Hopf algebras*, volume 235 of *Monographs and Textbooks in Pure and Applied Mathematics.* Marcel Dekker Inc., New York, 2001. An introduction.

[DPP10] Robert Dawson, Robert Paré, and Dorette Pronk. The span construction. *Theory Appl. Categ.*, 24:No. 13, 302–377, 2010.

[DS97] Brian Day and Ross Street. Monoidal bicategories and Hopf algebroids. *Adv. Math.*, 129(1):99–157, 1997.

[Dub68] Eduardo Dubuc. Adjoint triangles. In *Reports of the Midwest Category Seminar, II*, pages 69–91. Springer, Berlin, 1968.

[Ehr63] Charles Ehresmann. Catégories structurées. *Ann. Sci. École Norm. Sup. (3)*, 80:349–426, 1963.

[EK66] Samuel Eilenberg and G. Max Kelly. Closed categories. In *Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)*, pages 421–562. Springer, New York, 1966.

[FGK11] Thomas M. Fiore, Nicola Gambino, and Joachim Kock. Monads in double categories. *J. Pure Appl. Algebra*, 215(6):1174–1197, 2011.

[FGK12] Thomas M. Fiore, Nicola Gambino, and Joachim Kock. Double adjunctions and free monads. *Cah. Topol. Géom. Différ. Catég.*, 53(4):242–306, 2012.

[Fox93] Thomas F. Fox. The construction of cofree coalgebras. *J. Pure Appl. Algebra*, 84(2):191–198, 1993.

[GG76] M.F. Gouzou and R. Grunig. Fibration relatives. 1976.

[GJ14] Nicola Gambino and André Joyal. On operads, bimodules and analytic functors. arXiv:1405.7270 [math.CT], 2014.

[GP97] R. Gordon and A. J. Power. Enrichment through variation. *J. Pure Appl. Algebra*, 120(2):167–185, 1997.

[GP99] Marco Grandis and Robert Paré. Limits in double categories. *Cahiers Topologie Géom. Différentielle Catég.*, 40(3):162–220, 1999.

[GP04] Marco Grandis and Robert Paré. Adjoint for double categories. Addenda to: “Limits in double categories” [Cah. Topol. Géom. Différ. Catég. **40** (1999), no. 3, 162–220; mr1716779]. *Cah. Topol. Géom. Différ. Catég.*, 45(3):193–240, 2004.

[GPS95] R. Gordon, A. J. Power, and Ross Street. Coherence for tricategories. *Mem. Amer. Math. Soc.*, 117(558):vi+81, 1995.

[Gra66] John W. Gray. Fibred and cofibred categories. In *Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)*, pages 21–83. Springer, New York, 1966.

[Gra74] John W. Gray. *Formal category theory: adjointness for* 2*-categories.* Lecture Notes in Mathematics, Vol. 391. Springer-Verlag, Berlin, 1974.

[Gro61] Alexander Grothendieck. Catégories fibrées et descente, 1961. Seminaire de géometrie algébrique de l'Institut des Hautes Études Scientifiques (SGA 1), Paris.

[GS13] Richard Garner and Michael Shulman. Enriched categories as a free cocompletion. arXiv:1301.3191 [math.CT], 2013.

[GU71] P. Gabriel and F. Ulmer. *Lokal Präsentierbare Kategorien*, volume 221 of *Lecture Notes in Mathematics.* Springer-Verlag, 1971.

[Gur07] Nick Gurski. *An algebraic theory of tricategories.* PhD thesis, University of Chicago, 2007.

[Gur13] Nick Gurski. *Coherence in three-dimensional category theory*, volume 201 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 2013.

[Haz03] Michiel Hazewinkel. Cofree coalgebras and multivariable recursiveness. *J. Pure Appl. Algebra*, 183(1-3):61–103, 2003.

[Her93] Claudio Alberto Hermida. *Fibrations, logical predicates and indeterminates.* PhD thesis, University of Edinburgh, 1993.

[Her94] Claudio Hermida. On fibred adjunctions and completeness for fibred categories. In *Recent trends in data type specification (Caldes de Malavella, 1992)*, volume 785 of *Lecture Notes in Comput. Sci.*, pages 235–251. Springer, Berlin, 1994.

[Jac99] Bart Jacobs. *Categorical logic and type theory*, volume 141 of *Studies in Logic and the Foundations of Mathematics.* North-Holland Publishing Co., Amsterdam, 1999.

[JK02] G. Janelidze and G. M. Kelly. A note on actions of a monoidal category. *Theory Appl. Categ.*, 9:61–91, 2001/02. CT2000 Conference (Como).

[Joh02a] Peter T. Johnstone. *Sketches of an elephant: a topos theory compendium. Vol. 1*, volume 43 of *Oxford Logic Guides.* The Clarendon Press Oxford University Press, New York, 2002.

[Joh02b] Peter T. Johnstone. *Sketches of an elephant: a topos theory compendium. Vol. 2*, volume 44 of *Oxford Logic Guides.* The Clarendon Press Oxford University Press, Oxford, 2002.

[JS93] André Joyal and Ross Street. Braided tensor categories. *Adv. Math.*, 102(1):20–78, 1993.

[Kel64] G. M. Kelly. On MacLane's conditions for coherence of natural associativities, commutativities, etc. *J. Algebra*, 1:397–402, 1964.

[Kel72] G. M. Kelly. An abstract approach to coherence. In *Coherence in categories*, pages 106–147. Lecture Notes in Math., Vol. 281. Springer, Berlin, 1972.

[Kel74a] G. M. Kelly. Doctrinal adjunction. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pages 257–280. Lecture Notes in Math., Vol. 420. Springer, Berlin, 1974.

[Kel74b] G. M. Kelly. On clubs and doctrines. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pages 181–256. Lecture Notes in Math., Vol. 420. Springer, Berlin, 1974.

[Kel05] G. M. Kelly. Basic concepts of enriched category theory. *Repr. Theory Appl. Categ.*, (10):vi+137, 2005. Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714].

[Kel06] Bernhard Keller. A-infinity algebras, modules and functor categories. In *Trends in representation theory of algebras and related topics*, volume 406 of *Contemp. Math.*, pages 67–93. Amer. Math. Soc., Providence, RI, 2006.

[KK13] Anders Kock and Joachim Kock. Local fibred right adjoints are polynomial. *Math. Structures Comput. Sci.*, 23(1):131–141, 2013.

[KL01] G. M. Kelly and Stephen Lack. V-Cat is locally presentable or locally bounded if V is so. *Theory Appl. Categ.*, 8:555–575, 2001.

[KM07] Bernhard Keller and Oleksandr Manzyuk. Equalizers in the category of cocomplete cocategories. *J. Homotopy Relat. Struct.*, 2(1):85–97, 2007.

[KS74] G. Kelly and Ross Street. Review of the elements of 2-categories. In Gregory Kelly, editor, *Category Seminar*, volume 420 of *Lecture Notes in Mathematics*, pages 75–103. Springer Berlin / Heidelberg, 1974. 10.1007/BFb0063101.

[Lac00] Stephen Lack. A coherent approach to pseudomonads. *Adv. Math.*, 152(2):179–202, 2000.

[Lac10a] Stephen Lack. A 2-categories companion. In *Towards higher categories*, volume 152 of *IMA Vol. Math. Appl.*, pages 105–191. Springer, New York, 2010.

[Lac10b] Stephen Lack. Icons. *Appl. Categ. Structures*, 18(3):289–307, 2010.

[Lac10c] Stephen Lack. Note on the construction of free monoids. *Appl. Categ. Structures*, 18(1):17–29, 2010.

[Law73] F. William Lawvere. Metric spaces, generalized logic, and closed categories. *Rend. Sem. Mat. Fis. Milano*, 43:135–166 (1974), 1973.

[LF12] Ignacio Lopez Franco. 2-dimensional category theory, 2012. Part III graduate course lecture notes, University of Cambridge.

[Lin69] F. E. J. Linton. Coequalizers in categories of algebras. In *Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67)*, pages 75–90. Springer, Berlin, 1969.

[LP08] Stephen Lack and Simona Paoli. 2-nerves for bicategories. *K-Theory*, 38(2):153–175, 2008.

[Lyu03] Volodymyr Lyubashenko. Category of A_∞-categories. *Homology Homotopy Appl.*, 5(1):1–48, 2003.

[Mar97] F. Marmolejo. Doctrines whose structure forms a fully faithful adjoint string. *Theory Appl. Categ.*, 3:No. 2, 24–44, 1997.

[McC00a] Paddy McCrudden. Balanced coalgebroids. *Theory Appl. Categ.*, 7:No. 6, 71–147, 2000.

[McC00b] Paddy McCrudden. Categories of representations of coalgebroids. *Adv. Math.*, 154(2):299–332, 2000.

[ML63] Saunders Mac Lane. Natural associativity and commutativity. *Rice Univ. Studies*, 49(4):28–46, 1963.

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[MLP85] Saunders Mac Lane and Robert Paré. Coherence for bicategories and indexed categories. *J. Pure Appl. Algebra*, 37(1):59–80, 1985.

[MP89] Michael Makkai and Robert Paré. *Accessible categories: the foundations of categorical model theory*, volume 104 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1989.

[Por06] Hans-E. Porst. On corings and comodules. *Arch. Math. (Brno)*, 42(4):419–425, 2006.

[Por08a] Hans-E. Porst. Dual adjunctions between algebras and coalgebras. *Arab. J. Sci. Eng. Sect. C Theme Issues*, 33(2):407–411, 2008.

[Por08b] Hans-E. Porst. Fundamental constructions for coalgebras, corings, and comodules. *Appl. Categ. Structures*, 16(1-2):223–238, 2008.

[Por08c] Hans-E. Porst. On categories of monoids, comonoids, and bimonoids. *Quaest. Math.*, 31(2):127–139, 2008.

[Pow89] A. J. Power. A general coherence result. *J. Pure Appl. Algebra*, 57(2):165–173, 1989.

[Pow90] A. J. Power. A 2-categorical pasting theorem. *J. Algebra*, 129(2):439–445, 1990.

[Shu08] Michael Shulman. Framed bicategories and monoidal fibrations. *Theory Appl. Categ.*, 20:No. 18, 650–738, 2008.

[Shu10] Michael Shulman. Constructing symmetric monoidal bicategories. arXiv:1004.0993 [math.CT], 2010.

[Shu13] Michael Shulman. Enriched indexed categories. *Theory Appl. Categ.*, 28:616–696, 2013.

[Str72] Ross Street. The formal theory of monads. *J. Pure Appl. Algebra*, 2(2):149–168, 1972.

[Str80] Ross Street. Fibrations in bicategories. *Cahiers Topologie Géom. Différentielle*, 21(2):111–160, 1980.

[Str96] Ross Street. Categorical structures. In *Handbook of algebra, Vol. 1*, volume 1 of *Handb. Algebr.*, pages 529–577. Elsevier/North-Holland, Amsterdam, 1996.

[Str07] Ross Street. *Quantum groups*, volume 19 of *Australian Mathematical Society Lecture Series*. Cambridge University Press, Cambridge, 2007. A path to current algebra.

[Swe69] Moss E. Sweedler. *Hopf algebras*. Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York, 1969.

[Vas12] Christina Vasilakopoulou. Enrichment of categories of algebras and modules. pre-print arXiv:1205.6450 [math.CT], 2012.

[Web04] Mark Weber. Generic morphisms, parametric representations and weakly cartesian monads. *Theory Appl. Categ.*, 13:191–234, 2004.

[Win90] Glynn Winskel. A compositional proof system on a category of labelled transition systems. *Inform. and Comput.*, 87(1-2):2–57, 1990.

[Wis75] Manfred B. Wischnewsky. On linear representations of affine groups. I. *Pacific J. Math.*, 61(2):551–572, 1975.

[Wol74] Harvey Wolff. V-cat and V-graph. *J. Pure Appl. Algebra*, 4:123–135, 1974.