Adjunctions between Fibrations & Enrichment

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1. Motivation: monoids and modules in monoidal categories
2. 2-categories of fibrations
3. Fibred adjunctions
4. Enrichment of modules in comodules

Universal measuring comonoid

Generalization of Sweedler's *universal measuring coalgebras*

$$\mathbf{Alg}_k(A, \mathrm{Hom}_k(C,B)) \cong \{C \otimes A \xrightarrow{\sigma} B \text{ measures}\} \cong \mathbf{Coalg}_k(C, P(A,B))$$

Suppose $\mathcal{V}$ is a symmetric monoidal closed category. The internal hom functor $[-,-] : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$ is lax monoidal, thus induces

$$\mathbf{Mon}[-,-] = H : \mathbf{Comon}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Mon}(\mathcal{V}) \to \mathbf{Mon}(\mathcal{V}).$$

Proposition

If $\mathcal{V}$ is also locally presentable, $\mathbf{Comon}(\mathcal{V}) \xrightarrow{H(-,B)^{\mathrm{op}}} \mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$ has a right adjoint $P(-,B)$ with a natural isomorphism

$$\mathbf{Mon}(\mathcal{V})(A, [C,B]) \cong \mathbf{Comon}(\mathcal{V})(C, P(A,B)).$$

$\mathbf{Comon}(\mathcal{V})$ inherits local presentability (cocomplete, co-well-powered with generator) and $H(-,B)^{\mathrm{op}}$ is cocontinuous.

Enrichment

The category of monoids $\mathbf{Mon}(\mathcal{V})$ is enriched in the category of comonoids $\mathbf{Comon}(\mathcal{V})$, with enriched hom-functor

$$P : \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Mon}(\mathcal{V}) \longrightarrow \mathbf{Comon}(\mathcal{V}).$$

Proof employs theory of *actions* of monoidal categories: functor $* : \mathcal{V} \times \mathcal{A} \to \mathcal{A}$ with coherent isos $(X \otimes Y) * A \cong X * (Y * A)$, $I * A \cong A$ (pseudomodule). Since functors H and

$$H^{\mathrm{op}} : \mathbf{Comon}(\mathcal{V}) \times \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \to \mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$$

are actions, it follows from

Theorem (*Janelidze, Kelly*)

If each $- * A$ has a right adjoint $F(A, -)$, we can enrich $\mathcal{A}$ in $\mathcal{V}$ with hom-objects $F(A, B)$.

Modules and Comodules

For fixed monoid A and comonoid C, have categories $\mathbf{Mod}_{\mathcal{V}}(A)$ and $\mathbf{Comod}_{\mathcal{V}}(C)$. The *global category of modules* **Mod** has

· objects modules M_A for an arbitrary monoid A;

· morphisms $M_A \to N_B$ consist of a monoid arrow $f : A \to B$ and $k : M \to N$ satisfying

$$\begin{array}{ccccc} A \otimes M & & \xrightarrow{\mu} & & M \\ {\scriptstyle 1\otimes k}\downarrow & & & & \downarrow{\scriptstyle k} \\ A \otimes N & \xrightarrow{f\otimes 1} & B \otimes N & \xrightarrow{\mu'} & N. \end{array}$$

Dually, global category of comodules **Comod**.

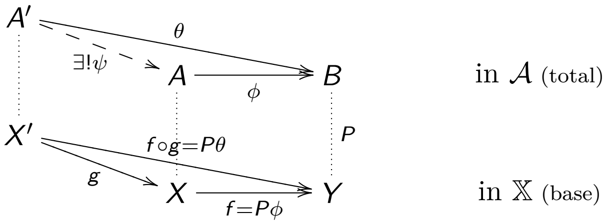
Mod/**Comod** are Grothendieck categories for functors mapping monoids/comonoids to the category of their modules/comodules.

Aim: enrichment of modules in comodules

$$\begin{array}{ccc} \mathbf{Mod} & \overset{?}{\dashrightarrow} & \mathbf{Comod} \\ {\scriptstyle \text{fibration}}\downarrow & & \downarrow{\scriptstyle \text{opfibration}} \\ \mathbf{Mon}(\mathcal{V}) & \overset{\text{enrichment}}{\dashrightarrow} & \mathbf{Comon}(\mathcal{V}) \end{array}$$

Fibrations and Opfibrations

A functor $P : \mathcal{A} \to \mathbb{X}$ is a *fibration* iff $\exists$ cartesian morphisms ϕ



for all B and f. The *fibre* $\mathcal{A}_X$ is $P^{-1}X$ with vertical arrows (mapped to 1_X). A choice $\phi \equiv \mathrm{Cart}(f, B) : f^*B \to B$ of cartesian liftings (unique up to vertical iso) is called *cleavage*, and induces a reindexing functor between the fibres

$$f^* : \mathcal{A}_Y \longrightarrow \mathcal{A}_X.$$

Dually, for an *opfibration* $U : \mathcal{C} \to \mathbb{X}$ with chosen cocartesian liftings, have functors $f_! : \mathcal{C}_X \longrightarrow \mathcal{C}_Y$.

- A *fibred 1-cell* $(S, F) : P \to Q$ between fibrations is a commutative square

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{S} & \mathcal{B} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathbb{X} & \xrightarrow[F]{} & \mathbb{Y} \end{array}$$

where S preserves cartesian arrows. In particular (over same base), *fibred functor* $(S, 1_{\mathbb{X}})$.

- A *fibred 2-cell* $(S, F) \Rightarrow (T, G)$ is a pair of natural transformations

$$\begin{array}{ccc} \mathcal{A} & \overset{S}{\underset{T}{\Downarrow \alpha}} & \mathcal{B} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathbb{X} & \overset{F}{\underset{G}{\Downarrow \beta}} & \mathbb{Y} \end{array}$$

with $Q\alpha = \beta_P$. In particular, *fibred natural transformation* $(\alpha, 1_{1_{\mathbb{X}}})$.

- Obtain 2-categories **Fib** and **Fib**$(\mathbb{X})$. Dually opfibrations, opfibred 1-cells (functors) and opfibred 2-cells (nat. transf.) form **OpFib** and **OpFib**$(\mathbb{X})$.

Any fibred 1-cell determines a collection of functors between the fibres:

$$S_X : \mathcal{A}_X \xrightarrow{S|_X} \mathcal{B}_{FX}.$$

Lemma

Suppose $(S, F) : P \to Q$ is a fibred 1-cell. The reindexing functors commute up to iso with the fibrewise S_X's, i.e.

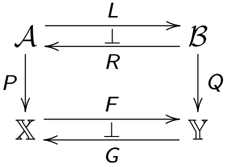
$$\begin{array}{ccc} \mathcal{A}_Y & \xrightarrow{S_Y} & \mathcal{B}_{FY} \\ {\scriptstyle f^*}\downarrow & \overset{\tau^f}{\cong} & \downarrow{\scriptstyle (Ff)^*} \\ \mathcal{A}_X & \xrightarrow[S_X]{} & \mathcal{B}_{FX}. \end{array}$$

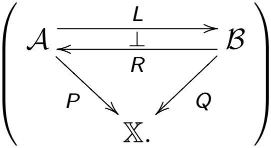
The image of a cartesian lifting under S is required to be cartesian:

$$\begin{array}{ccc} (Ff)^*(SA) & & \\ {\scriptstyle \cong}\downarrow & \searrow^{\mathrm{Cart}(Ff,SA)} & \\ S(f^*A) & \xrightarrow[S\mathrm{Cart}(f,A)]{} & SA \end{array}$$

Adjunctions in **Fib** and **Fib**$(\mathbb{X})$

A *general fibred adjunction* $(L, F) \dashv (R, G) : Q \to P$ (in particular, *fibred adjunction* $L \dashv R$) is displayed as





Proposition *"Existence of adjoint in* **Fib**$(\mathbb{X})$*"* (Borceux...)

The fibred functor $S : Q \to P$ has a fibred left adjoint L iff $L_X \dashv S_X$ for all X and the Beck-Chevalley condition holds for all τ^f, i.e.

$$L_X \circ f^* \cong f^* \circ L_Y.$$

"$\Leftarrow$" The fibrewise L_X's assemble into a fibred 'total' left adjoint L employing the invertible mates of τ^f.

Question: analogue for general fibred adjunctions?

Theorem *(Existence of adjoint in* **Fib***)*

For a fibred 1-cell (S, G), if there is an adjunction between the bases

$$\begin{array}{ccc}
\mathcal{A} & \xleftarrow{\quad S \quad} & \mathcal{B} \\
 & \overset{\perp}{\underset{L}{\dashrightarrow}} & \\
{\scriptstyle P}\big\downarrow & & \big\downarrow{\scriptstyle Q} \\
\mathbb{X} & \underset{G}{\overset{F}{\rightleftarrows}}\ \scriptstyle{\perp} & \mathbb{Y}
\end{array}$$

and the functor $\mathcal{B}_{FX} \xrightarrow{S_{FX}} \mathcal{A}_{GFX} \xrightarrow{\eta^*} \mathcal{A}_X$ has a left adjoint L_X, then S has a left adjoint L s.t. $(L, F) \dashv (S, G)$ in $\mathbf{Cat}^2$. If also the mates $L_X \circ f^* \Rightarrow (Ff)^* \circ L_Y$ are isos, obtain general fibred adjunction.

Establish a natural bijective correspondence

$$\mathcal{B}(L_X A, B) \cong \mathcal{A}(A, SB)$$

by employing the fibrewise adjoints L_X and base adjunction $F \dashv G$.

Universal Measuring Comodule

Back to global categories of modules and comodules, for $\mathcal{V}$ l.p. symmetric monoidal closed. Application (of dual result) to:

$$\begin{array}{ccc}
\mathbf{Comod} & \xrightarrow{\bar{H}(-,N_B)^{\mathrm{op}}} & \mathbf{Mod}^{\mathrm{op}} \\
\downarrow & & \downarrow \\
\mathbf{Comon}(\mathcal{V}) & \underset{P(-,B)}{\overset{H(-,B)^{\mathrm{op}}}{\rightleftarrows}}\ \bot & \mathbf{Mon}(\mathcal{V})^{\mathrm{op}}
\end{array}$$

· functor $\bar{H} : \mathbf{Comod}^{\mathrm{op}} \times \mathbf{Mod} \to \mathbf{Mod}$ also induced from internal hom functor $[-,-]$, moreover preserves cocartesian liftings, i.e. $(\bar{H}, H)$ opfibred 1-cell.

· base adjunction gives universal measuring comonoid.

· $\mathbf{Comod}_{\mathcal{V}} P(A,B) \xrightarrow{\bar{H}(-,N_B)^{\mathrm{op}}} \mathbf{Mod}^{\mathrm{op}}_{\mathcal{V}}[P(A,B),B] \xrightarrow{(\varepsilon_A)_!} \mathbf{Mod}^{\mathrm{op}}_{\mathcal{V}} A$ has a right adjoint $Q_A(-,N)$.

There is an adjunction between the global categories

$$\mathbf{Comod} \underset{Q(-,N_B)}{\overset{\bar{H}(-,N_B)^{\mathrm{op}}}{\rightleftarrows}} \mathbf{Mod}^{\mathrm{op}} \quad (\bot).$$

The $P(A,B)$-comodule $Q(M_A,N_B)$ is called *universal measuring comodule*.

· Functors ($\bar{H}$ and) $\bar{H}^{\mathrm{op}}$ are also actions, with parametrized adjoint $Q : \mathbf{Mod}^{\mathrm{op}} \times \mathbf{Mod} \to \mathbf{Comod}$.

Theorem

Mod *is enriched in* **Comod** *with enriched hom-functor Q, and*

$$\begin{array}{ccc} \mathbf{Comod} & \underset{Q(-,N)}{\overset{\bar{H}(-,N)^{\mathrm{op}}}{\rightleftarrows}} & \mathbf{Mod}^{\mathrm{op}} \\ \downarrow & \bot & \downarrow \\ \mathbf{Comon}(\mathcal{V}) & \underset{P(-,B)}{\overset{H(-,B)^{\mathrm{op}}}{\rightleftarrows}} & \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \end{array}$$

is a general opfibred adjunction.

Thank you for your attention!