



# ON ENRICHED FIBRATIONS

*Christina VASILAKOPOULOU*

**Résumé.** Nous introduisons la notion de fibration enrichie, à savoir une fibration dont la catégorie totale et la catégorie de base sont enrichies d'une manière appropriée dans celles d'une fibration. De plus, nous proposons un moyen d'obtenir une telle structure, à partir des actions de catégories monoïdales avec des adjoints paramétrés. L'objectif est de capturer certaines classes d'exemples, comme la fibration des modules sur des algèbres enrichie dans l'opfibration de comodules sur des coalgèbres.

**Abstract.** We introduce the notion of an enriched fibration, i.e. a fibration whose total category and base category are enriched in those of a monoidal fibration in an appropriate way. Furthermore, we provide a way to obtain such a structure, starting from actions of monoidal categories with parameterized adjoints. The motivating goal is to capture certain example cases, like the fibration of modules over algebras enriched in the opfibration of comodules over coalgebras.

**Keywords.** monoidal category, fibration, enriched category, parameterized adjunction.

**Mathematics Subject Classification (2010).** 18D10,18D20,18D30.

## 1. Introduction

Enriched category theory [Kel05], as well as the theory of fibrations [Gro61], have both been of central importance to developments in many contexts. Both are classical theories for formal category theory; however, they do not seem to 'go together' in some evident way.

The goal of the present work is to introduce a notion of an *enriched fibration*. This should combine elements of both concepts in an appropriate and natural way; the enriched structure of a category cannot really be internalized in order to provide a definite answer. In any case, ‘being enriched in’ and ‘being internal to’ are two major but separate generalizations of ordinary category theory, whereas ‘being fibred over’ is often considered as a third one.

More explicitly, we would like to characterize a fibration as being enriched in some special kind of fibration, serving similar purposes as the monoidal base of usual enriched categories; this has already been identified as a *monoidal fibration* [Shu08]. For the desired enriched fibration definition, there are two main factors that determine its relevance. First of all, it should be able to adequately capture certain cases that first arose in [Vas14] and furthermore studied in [HLFV17a, HLFV17b, Vas17], and in fact motivated these explorations. Further details of these examples and how they ultimately fit in the described framework can be found in Section 4. The original driving example case is the enrichment of algebras in coalgebras via Sweedler’s *measuring coalgebra* construction [Swe69], together with the enrichment of a global category of modules in comodules; the latter categories are respectively fibred and opfibred over algebras and coalgebras. This also extends to their many-object generalizations, namely (enriched) categories and cocategories and their (enriched) modules and comodules. These cases can be roughly depicted as

$$\begin{array}{ccc}
 \text{Mod} & \xrightarrow{\text{enriched}} & \text{Comod} \\
 \text{fibred} \downarrow & & \downarrow \text{opfibred} \\
 \text{Alg}(\mathcal{V}) & \xrightarrow{\text{enriched}} & \text{Coalg}(\mathcal{V})
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{V}\text{-Mod} & \xrightarrow{\text{enriched}} & \mathcal{V}\text{-Comod} \\
 \text{fibred} \downarrow & & \downarrow \text{opfibred} \\
 \mathcal{V}\text{-Cat} & \xrightarrow{\text{enriched}} & \mathcal{V}\text{-Cocat}.
 \end{array}$$

Secondly, the introduced enriched fibration concept should theoretically constitute an as-close-as-possible fibred analogue of the usual enrichment of categories. In order to initiate such an effort, we provide a theorem which under certain assumptions ensures the existence of such a structure. This theorem lifts a standard result, which combines the theory of *actions* of monoidal categories and parameterized adjunctions to produce an enrichment [GP97, JK02], to the fibred context.

Notably, a strongly related notion called *enriched indexed category* has been studied, from a slightly different point of view, originally in [Bun13] and also independently in [Shu13]. However, the main definitions and constructions diverge from the ones presented here. We postpone a short discussion on these differences until the very end of the paper, Section 4.3.

Finally, it should be indicated that this paper deliberately includes only what is necessary to first of all sufficiently describe the examples at hand. It elaborates on and extends a sketched narrative from [Vas14, §8.1], and provides the first steps in such a research direction. Future work may build on the current development, towards a theory of enriched fibrations and related structures.

### Acknowledgements

The author would like to thank Martin Hyland for posing the driving questions that led to this work, as well as providing valuable insight; also Marta Bunge for carefully reviewing this manuscript and offering multiple useful comments; finally Mitchell Buckley, for many discussions that affected various parts of this work.

## 2. Background

In this section, we recall some basic definitions and known results which serve as background material in what follows, and we also fix terminology.

### 2.1 Monoidal categories, actions and enrichment

We assume familiarity with the basics of monoidal categories, see [JS93, ML98]. A monoidal category is denoted by  $(\mathcal{V}, \otimes, I)$  with associator and left and right unit constraints  $a, \ell, r$ . A lax monoidal structure on a functor  $F: \mathcal{V} \rightarrow \mathcal{W}$  between monoidal categories is denoted by  $(\phi, \phi_0)$ , with components  $\phi_{AB}: FX \otimes FY \rightarrow F(X \otimes Y)$  and  $\phi_0: I \rightarrow FI$  satisfying usual axioms. If these are isomorphisms/identities, this is a strong/strict monoidal structure.

A (left) monoidal closed category is one where the functor  $(- \otimes X)$  has a right adjoint  $[X, -]$ , for all objects  $X$ . This induces the internal hom functor

$[-, -]: \mathcal{V}^{\text{op}} \times \mathcal{V} \rightarrow \mathcal{V}$ , as a result of the classic *parameterized adjunctions* theorem [ML98, §IV.7.3]:

**Theorem 2.1.** *Suppose that, for a functor of two variables  $F: \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ , there exists an adjunction*

$$\mathcal{A} \begin{array}{c} \xrightarrow{F(-, B)} \\ \perp \\ \xleftarrow{G(B, -)} \end{array} \mathcal{C}$$

for each  $B \in \mathcal{B}$ , with an isomorphism  $\mathcal{C}(F(A, B), C) \cong \mathcal{A}(A, G(B, C))$  natural in  $A$  and  $C$ . Then, there is a unique way to make  $G$  into a functor of two variables  $\mathcal{B}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{A}$  for which the isomorphism is natural also in  $B$ .

The functor  $G$  is called the (right) *parameterized adjoint* of  $F$ , and we denote this as  $F \dashv_p G$ . In particular,  $\otimes \dashv_p [-, -]$  in any monoidal (left) closed category. We could also decide to fix the other parameter, and have that  $F(A, -) \dashv H(A, -)$  for  $H: \mathcal{A}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{B}$ . For a 2-categorical proof and generalizations, see [CGR14].

We now recall some basics of the theory of actions of monoidal categories, [JK02].

**Definition 2.2.** A (left) action of a monoidal category  $\mathcal{V}$  on a category  $\mathcal{D}$  is given by a functor  $*$ :  $\mathcal{V} \times \mathcal{D} \rightarrow \mathcal{D}$  along with two natural isomorphisms  $\chi, \nu$  with components

$$\chi_{XYD}: (X \otimes Y) * D \xrightarrow{\sim} X * (Y * D), \quad \nu_D: I * D \xrightarrow{\sim} D \quad (1)$$

satisfying the commutativity of

$$\begin{array}{ccc} ((X \otimes Y) \otimes Z) * D & \xrightarrow{\chi} & (X \otimes Y) * (Z * D) \xrightarrow{\chi} X * (Y * (Z * D)) \\ \downarrow a * 1 & & \uparrow 1 * \chi \\ (X \otimes (Y \otimes Z)) * D & \xrightarrow{\chi} & X * ((Y \otimes Z) * D) \\ \\ (I \otimes X) * D & \xrightarrow{\chi} & I * (X * D) \quad (X \otimes I) * D \xrightarrow{\chi} X * (I * D) \\ \swarrow l * 1 \quad \searrow \nu & & \swarrow r * 1 \quad \searrow 1 * \nu \\ & X * D & X * D \end{array} \quad (2)$$

The category  $\mathcal{D}$  is called a  $\mathcal{V}$ -representation, or a  $\mathcal{V}$ -actegory [McC00].

For example, every monoidal category has a canonical action on itself via its tensor product,  $\otimes = * : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ , and  $\chi = a, \nu = \ell$ ; it is called the *regular*  $\mathcal{V}$ -representation. Moreover, for any monoidal closed category, its internal hom constitutes an action of the monoidal  $\mathcal{V}^{\text{op}}$  (with the same tensor product  $\otimes^{\text{op}}$ ) on  $\mathcal{V}$ , via the standard natural isomorphisms

$$\chi_{XYZ} : [X \otimes Y, D] \xrightarrow{\sim} [X, [Y, Z]], \quad \nu_D : [I, D] \xrightarrow{\sim} D$$

which satisfy (2) using the transpose diagrams under the tensor-hom adjunction.

Familiarity with enrichment theory is also assumed, see [Kel05]. We denote the 2-category of  $\mathcal{V}$ -enriched categories, along with enriched functors and enriched natural transformations,  $\mathcal{V}\text{-Cat}$ ; we call  $\mathcal{V}$  the monoidal base of the enrichment. If  $\mathcal{A}$  is a  $\mathcal{V}$ -enriched category with hom-objects  $\mathcal{A}(A, B) \in \mathcal{V}$ , we will write  $j_A : I \rightarrow \mathcal{A}(A, A)$  for its identities and  $M_{ABC} : \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \rightarrow \mathcal{A}(A, C)$  for the composition. Its *underlying category*  $\mathcal{A}_0$  has the same objects, while morphisms  $f : A \rightarrow B$  in  $\mathcal{A}_0$  are just ‘elements’  $f : I \rightarrow \mathcal{A}(A, B)$  in  $\mathcal{V}$ , i.e.  $\mathcal{A}_0(A, B) = \mathcal{V}(I, \mathcal{A}(A, B))$  as sets. In fact, we can define a functor

$$\mathcal{A}(-, -) : \mathcal{A}_0^{\text{op}} \times \mathcal{A}_0 \rightarrow \mathcal{V} \tag{3}$$

called the *enriched hom-functor*, which maps  $(A, B)$  to  $\mathcal{A}(A, B)$ , and a pair of arrows  $(A' \xrightarrow{f} A, B \xrightarrow{g} B')$  in  $\mathcal{A}_0^{\text{op}} \times \mathcal{A}_0$  to the top arrow

$$\begin{array}{ccc} \mathcal{A}(A, B) & \overset{\mathcal{A}(f,g)}{\dashrightarrow} & \mathcal{A}(A', B') \\ \begin{array}{c} \downarrow r^{-1} \\ \mathcal{A}(A, B) \otimes I \\ \downarrow 1 \otimes f \end{array} & & \begin{array}{c} \uparrow M \\ \mathcal{A}(B, B') \otimes \mathcal{A}(A', B) \\ \uparrow g \otimes 1 \end{array} \\ \mathcal{A}(A, B) \otimes \mathcal{A}(A', A) & \xrightarrow{M} \mathcal{A}(A', B) \xrightarrow{l^{-1}} & I \otimes \mathcal{A}(A', B) \end{array}$$

Speaking loosely, we say that an ordinary category  $\mathcal{C}$  is enriched in a monoidal category  $\mathcal{V}$  when we have a  $\mathcal{V}$ -enriched category  $\mathcal{A}$  (often denoted  $\underline{\mathcal{C}}$ ) and an isomorphism  $\mathcal{A}_0 \cong \mathcal{C}$ . Consequently, *to be enriched in*  $\mathcal{V}$  is not a property, but additional structure. Of course, a given ordinary category may

be enriched in more than one monoidal category; this is evident in view of Proposition 2.3. But also, a category  $\mathcal{C}$  may be enriched in  $\mathcal{V}$  in more than one way.

**Proposition 2.3** (Change of Base). *Suppose  $F : \mathcal{V} \rightarrow \mathcal{W}$  is a lax monoidal functor between two monoidal categories. There is an induced 2-functor*

$$\tilde{F} : \mathcal{V}\text{-Cat} \longrightarrow \mathcal{W}\text{-Cat}$$

between the 2-categories of  $\mathcal{V}$  and  $\mathcal{W}$ -enriched categories, which maps any  $\mathcal{V}$ -category  $\mathcal{A}$  to a  $\mathcal{W}$ -category with the same objects as  $\mathcal{A}$  and hom-objects  $F\mathcal{A}(A, B)$ .

*Sketch of proof.* On the level of objects, the composition and identities are given by

$$\begin{array}{ccc} F\mathcal{A}(B, C) \otimes F\mathcal{A}(A, B) & \dashrightarrow & F\mathcal{A}(A, C) \\ \downarrow \phi_{\mathcal{A}(B,C), \mathcal{A}(A,B)} & \nearrow FM_{ABC} & \\ F(\mathcal{A}(B, C) \otimes \mathcal{A}(A, B)) & & \end{array} \quad \begin{array}{ccc} I_{\mathcal{W}} & \dashrightarrow & F\mathcal{A}(A, A) \\ \downarrow \phi_0 & \nearrow Fj_A & \\ FI_{\mathcal{V}} & & \end{array}$$

□

A crucial result for what follows is that given a category  $\mathcal{D}$  with an action from a monoidal category  $\mathcal{V}$  with a parameterized adjoint, we obtain a  $\mathcal{V}$ -enriched category.

**Theorem 2.4.** *Suppose that  $\mathcal{V}$  is a monoidal category which acts on a category  $\mathcal{D}$  via a functor  $* : \mathcal{V} \times \mathcal{D} \rightarrow \mathcal{D}$ , such that  $- * D$  has a right adjoint  $F(D, -)$  for every  $D \in \mathcal{D}$ . Then we can enrich  $\mathcal{D}$  in  $\mathcal{V}$ , with hom-objects  $\underline{\mathcal{D}}(A, B) = F(A, B)$ .*

The proof and further details can be found in [JK02] or [Vas14, § 4.3]. Briefly, due to the adjunction  $- * D \dashv F(D, -)$ , we have natural isomorphisms

$$\mathcal{D}(X * D, E) \cong \mathcal{V}(X, F(D, E)) \tag{4}$$

which give rise to a functor  $F : \mathcal{D}^{\text{op}} \times \mathcal{D} \rightarrow \mathcal{V}$  by Theorem 2.1. This serves as the enriched hom-functor of the induced enrichment of  $\mathcal{D}$  in  $\mathcal{V}$ : we can

define a composition law  $F(B, C) \otimes F(A, B) \rightarrow F(A, C)$  as the adjunct of the composite

$$\begin{array}{ccc}
 (F(B, C) \otimes F(A, B)) * A & \xrightarrow{\chi} & F(B, C) * (F(A, B) * A) \\
 & \searrow \text{dashed} & \downarrow 1 * \varepsilon_B \\
 & & F(B, C) * B \\
 & & \downarrow \varepsilon_C \\
 & & C
 \end{array} \tag{5}$$

and identities  $I \rightarrow F(A, A)$  as the adjuncts of

$$I * A \xrightarrow{\nu} A \tag{6}$$

where  $\chi$  and  $\nu$  give the action structure (1) and  $\varepsilon$  is the counit of the adjunction. The associativity and identity axioms for the enrichment can be established using the action axioms. Finally,  $\underline{\mathcal{D}}_0 \cong \mathcal{D}$  since they have the same objects, and

$$\underline{\mathcal{D}}_0(A, B) = \mathcal{V}(I, F(A, B)) \stackrel{(4)}{\cong} \mathcal{D}(I * A, B) \stackrel{\nu}{\cong} \mathcal{D}(A, B).$$

In fact, Theorem 2.4 gives part of one direction of an equivalence

$$\mathcal{V}\text{-Rep}_{\text{cl}} \simeq \mathcal{V}\text{-Cat}_{\otimes}$$

between *closed*  $\mathcal{V}$ -representations (those with action equipped with a parameterized adjoint) and *tensoring*  $\mathcal{V}$ -categories (those with specific weighted limits), for  $\mathcal{V}$  a monoidal closed category. This equivalence, discussed in [JK02, §6], is a special case of the much more general [GP97, Theorem 3.7] for bicategory-enriched categories.

**Remark 2.5.** When  $\mathcal{V}$  is monoidal closed, the regular action  $\otimes: \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$  has a parameterized adjoint  $[-, -]: \mathcal{V}^{\text{op}} \times \mathcal{V} \rightarrow \mathcal{V}$ . We thus re-discover the well-known enrichment of a monoidal closed category in itself via the internal hom, as a direct application of Theorem 2.4.

**2.2 Pseudomonoids and pseudomodules**

Recall that a *monoidal 2-category*  $(\mathcal{K}, \otimes, I)$  is a 2-category equipped with a pseudofunctor  $\otimes: \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$  and a unit  $I: \mathbf{1} \rightarrow \mathcal{K}$  which are associative and unital up to coherence equivalence, see [GPS95].

**Definition 2.6.** [DS97, §3] A pseudomonoid  $A$  in  $\mathcal{K}$  is an object equipped with multiplication  $m: A \otimes A \rightarrow A$  and unit  $j: I \rightarrow A$  along with invertible 2-cells satisfying coherence conditions.

$$\begin{array}{ccc}
 A \otimes A \otimes A & \xrightarrow{1 \otimes m} & A \otimes A \\
 m \otimes 1 \downarrow & \cong \downarrow & \downarrow m \\
 A \otimes A & \xrightarrow{m} & A
 \end{array}
 \qquad
 \begin{array}{ccc}
 A \otimes I & \xrightarrow{1 \otimes j} & A \otimes A & \xleftarrow{j \otimes 1} & I \otimes A \\
 \sim \searrow & \cong \downarrow & \downarrow m & \cong \downarrow & \swarrow \sim \\
 & & A & & 
 \end{array}
 \tag{7}$$

The notion of a pseudomodule for a pseudomonoid in a monoidal 2-category (or bicategory) can be found in similar contexts [Mar97, Lac00]; conceptually, as it is the case for modules for monoids in a monoidal category, it arises as a pseudoalgebra for the pseudomonad  $(A \otimes -)$  induced by a pseudomonoid  $A$  in  $\mathcal{K}$ .

**Definition 2.7.** A (left)  $A$ -pseudomodule is an object  $M$  in  $(\mathcal{K}, \otimes, I)$  equipped with  $\mu: A \otimes M \rightarrow M$  (the pseudoaction) and invertible 2-cells

$$\begin{array}{ccc}
 A \otimes A \otimes M & \xrightarrow{1 \otimes \mu} & A \otimes M \\
 m \otimes 1 \downarrow & \cong \downarrow & \downarrow \mu \\
 A \otimes M & \xrightarrow{\mu} & A
 \end{array}
 \qquad
 \begin{array}{ccc}
 I \otimes M & \xrightarrow{j \otimes 1} & A \otimes M \\
 \sim \searrow & \cong \downarrow & \downarrow \mu \\
 & & A
 \end{array}
 \tag{8}$$

satisfying coherence conditions.

**Example 2.8.** As a fundamental example of a (cartesian) monoidal 2-category, consider  $\mathbf{Cat}$  equipped with the 2-functor  $\times: \mathbf{Cat} \times \mathbf{Cat} \rightarrow \mathbf{Cat}$  and  $\mathcal{I}$  the unit category. It is a standard fact that a pseudomonoid therein is precisely a monoidal category  $(\mathcal{V}, \otimes, I, a, \ell, r)$ . Moreover, an action of a monoidal category  $\mathcal{V}$  on an ordinary category  $\mathcal{A}$  as defined in Definition 2.2 is precisely a  $\mathcal{V}$ -pseudoaction inside  $(\mathbf{Cat}, \times, \mathcal{I})$ , exhibiting  $\mathcal{A}$  as a  $\mathcal{V}$ -pseudomodule.



### 2.3 Fibrations and adjunctions

We now briefly recall some basic concepts and constructions from the theory of fibred categories. Relevant references for what follows are [Bor94, Her94].

Consider a functor  $P : \mathcal{A} \rightarrow \mathbb{X}$ . A morphism  $\phi : A \rightarrow B$  in  $\mathcal{A}$  over a morphism  $f = P(\phi) : X \rightarrow Y$  in  $\mathbb{X}$  is called *cartesian* if and only if, for all  $g : X' \rightarrow X$  in  $\mathbb{X}$  and  $\theta : A' \rightarrow B$  in  $\mathcal{A}$  with  $P\theta = f \circ g$ , there exists a unique arrow  $\psi : A' \rightarrow A$  such that  $P\psi = g$  and  $\theta = \phi \circ \psi$ :

$$\begin{array}{ccc}
 A' & \xrightarrow{\theta} & B \\
 \exists! \psi \dashrightarrow & & \downarrow \\
 A & \xrightarrow{\phi} & B \\
 \downarrow & & \downarrow \\
 X' & \xrightarrow{f \circ g = P\theta} & Y \\
 g \searrow & & \downarrow \\
 X & \xrightarrow{f = P\phi} & Y
 \end{array}
 \begin{array}{l}
 \text{in } \mathcal{A} \\
 \\
 \text{in } \mathbb{X}
 \end{array}$$

For  $X \in \text{ob}\mathbb{X}$ , the *fibre of  $P$  over  $X$*  written  $\mathcal{A}_X$ , is the subcategory of  $\mathcal{A}$  which consists of objects  $A$  such that  $P(A) = X$  and morphisms  $\phi$  with  $P(\phi) = 1_X$ , called *vertical* morphisms. The functor  $P : \mathcal{A} \rightarrow \mathbb{X}$  is called a *fibration* if and only if, for all  $f : X \rightarrow Y$  in  $\mathbb{X}$  and  $B \in \mathcal{A}_Y$ , there is a cartesian morphism  $\phi$  with codomain  $B$  above  $f$ ; it is called a *cartesian lifting* of  $B$  along  $f$ . The category  $\mathbb{X}$  is then called the *base* of the fibration, and  $\mathcal{A}$  its *total* category.

Dually, the functor  $U : \mathcal{C} \rightarrow \mathbb{X}$  is an *opfibration* if  $U^{\text{op}}$  is a fibration, *i.e.* for every  $C \in \mathcal{C}_X$  and  $g : X \rightarrow Y$  in  $\mathbb{X}$ , there is a cocartesian morphism with domain  $C$  above  $g$ , the *cocartesian lifting* of  $C$  along  $g$ .

If  $P : \mathcal{A} \rightarrow \mathbb{X}$  is a fibration, assuming the axiom of choice we may select a cartesian arrow over each  $f : X \rightarrow Y$  in  $\mathbb{X}$  and  $B \in \mathcal{A}_Y$ , denoted by  $\text{Cart}(f, B) : f^*(B) \rightarrow B$ . Such a choice of cartesian liftings is called a *cleavage* for  $P$ , which is then called a *cloven* fibration; any fibration is henceforth assumed to be cloven. Dually, if  $U$  is an opfibration, for any  $C \in \mathcal{C}_X$  and  $g : X \rightarrow Y$  in  $\mathbb{X}$  we can choose a cocartesian lifting of  $C$  along  $g$ ,  $\text{Cocart}(g, C) : C \rightarrow g_!(C)$ . The choice of (co)cartesian liftings in an (op)fibration induces a so-called *reindexing functor* between the fibre categories

$$f^* : \mathcal{A}_Y \rightarrow \mathcal{A}_X \quad \text{and} \quad g_! : \mathcal{C}_X \rightarrow \mathcal{C}_Y$$

respectively, for each morphism  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  in the base category, mapping each object to the (co)domain of its lifting.

An *oplax morphism of fibrations* (or oplax fibred 1-cell)  $(S, F)$  between  $P : \mathcal{A} \rightarrow \mathbb{X}$  and  $Q : \mathcal{B} \rightarrow \mathbb{Y}$  is given by a commutative square of categories and functors

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{S} & \mathcal{B} \\
 P \downarrow & & \downarrow Q \\
 \mathbb{X} & \xrightarrow{F} & \mathbb{Y}
 \end{array} \tag{9}$$

as in [Shu08, Def. 3.5]. If moreover  $S$  preserves cartesian arrows, meaning that if  $\phi$  is  $P$ -cartesian then  $S\phi$  is  $Q$ -cartesian, the pair  $(S, F)$  is called a *fibred 1-cell* or *strong morphism of fibrations*. Dually, we have the notion of an *lax morphism of opfibrations*  $(K, F)$ , and *opfibred 1-cell* when  $K$  is co-cartesian. Notice that any oplax fibred 1-cell  $(S, F)$  determines a collection of functors between the fibres  $S_X : \mathcal{A}_X \rightarrow \mathcal{B}_{FX}$  as the restriction of  $S$  to the corresponding subcategories.

A *fibred 2-cell* between oplax fibred 1-cells  $(S, F)$  and  $(T, G)$  is a pair of natural transformations  $(\alpha : S \Rightarrow T, \beta : F \Rightarrow G)$  with  $\alpha$  above  $\beta$ , i.e.  $Q(\alpha_A) = \beta_{PA}$  for all  $A \in \mathcal{A}$ , displayed

$$\begin{array}{ccc}
 \mathcal{A} & \begin{array}{c} \xrightarrow{S} \\ \Downarrow \alpha \\ \xrightarrow{T} \end{array} & \mathcal{B} \\
 P \downarrow & & \downarrow Q \\
 \mathbb{X} & \begin{array}{c} \xrightarrow{F} \\ \Downarrow \beta \\ \xrightarrow{G} \end{array} & \mathbb{Y}.
 \end{array} \tag{10}$$

Notice that if the 1-cells are strong, the definition of a 2-cell between them remains the same. Dually, we have the notion of an *opfibred 2-cell* between (lax) opfibred 1-cells.

We obtain 2-categories  $\mathbf{Fib}_{\text{opl}}$  and  $\mathbf{Fib}$  of fibrations over arbitrary base categories, (oplax) fibred 1-cells and fibred 2-cells. Evidently, these are both subcategories of  $\mathbf{Cat}^2$ .  $\mathbf{Fib}_{\text{opl}}$  is a full sub-2-category of those objects which are fibrations, and  $\mathbf{Fib}$  is the non-full sub-2-category whose morphism are commutative squares where the top functor is cartesian. Dually,

$\mathbf{OpFib} \subset \mathbf{OpFib}_{\text{lax}} \subset_{\text{full}} \mathbf{Cat}^2$ . These 2-categories are monoidal, inheriting the tensor product from  $\mathbf{Cat}^2$ : the cartesian product of two fibrations is still a fibration. The unit is  $1_{\mathcal{I}}: \mathcal{I} \rightarrow \mathcal{I}$ , the identity on the terminal category.

Notice that the terminology for oplax morphisms of fibrations and lax morphisms of opfibrations is justified by a relaxed version of the fundamental equivalence between fibrations and pseudofunctors (Grothendieck construction). For more details, see [Shu08, Prop. 3.6].

We now turn to notions of adjunctions between fibrations, internally to any of the above 2-categories of (op)fibrations.

**Definition 2.9.** *Given fibrations  $P : \mathcal{A} \rightarrow \mathbb{X}$  and  $Q : \mathcal{B} \rightarrow \mathbb{Y}$ , a general (oplax) fibred adjunction  $(L, F) \dashv (R, G)$  is given by a pair of (oplax) fibred 1-cells  $(L, F) : P \rightarrow Q$  and  $(R, G) : Q \rightarrow P$  together with fibred 2-cells  $(\zeta, \eta) : (1_{\mathcal{A}}, 1_{\mathbb{X}}) \Rightarrow (RL, GF)$  and  $(\xi, \varepsilon) : (LR, FG) \Rightarrow (1_{\mathcal{B}}, 1_{\mathbb{Y}})$  such that  $L \dashv R$  via  $\zeta, \xi$  and  $F \dashv G$  via  $\eta, \varepsilon$ . This is displayed as*

$$\begin{array}{ccc}
 \mathcal{A} & \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{1} \\ \xrightarrow{R} \end{array} & \mathcal{B} \\
 P \downarrow & & \downarrow Q \\
 \mathbb{X} & \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{1} \\ \xrightarrow{G} \end{array} & \mathbb{Y}
 \end{array}$$

Notice that by definition,  $\zeta$  is above  $\eta$  and  $\xi$  is above  $\varepsilon$ , hence  $(P, Q)$  is an ordinary map between adjunctions. Dually, we have the notion of *general (lax) opfibred adjunction* in  $\mathbf{OpFib}_{(\text{lax})}$ .

The following result establishes certain (co)cartesian properties of adjoints.

**Lemma 2.10.** [Win90, 4.5] *Right adjoints in the 2-category  $\mathbf{Cat}^2$  preserve cartesian morphisms; dually left adjoints preserve cocartesian morphisms.*

Finally, in [HLFV17b, §3.2], conditions under which a fibred 1-cell has an adjoint are investigated in detail, and that proves very useful in determining enrichment relations in conjunction with Theorem 2.4. Here we recall a main result providing a general lax opfibred adjunction, with regards to the applications of Section 4.

**Theorem 2.11.** *Suppose  $(K, F) : U \rightarrow V$  is an opfibred 1-cell and  $F \dashv G$  is an adjunction with counit  $\varepsilon$  between the bases of the opfibrations, as in*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{K} & \mathcal{D} \\ U \downarrow & & \downarrow V \\ \mathbb{X} & \xrightleftharpoons[F]{\perp} & \mathbb{Y} \\ & \xleftarrow{G} & \end{array}$$

If, for each  $Y \in \mathbb{Y}$ , the composite functor  $\mathcal{C}_{GY} \xrightarrow{K_{GY}} \mathcal{D}_{FGY} \xrightarrow{(\varepsilon_Y)_!} \mathcal{D}_Y$  between the fibres has a right adjoint for each  $Y \in \mathbb{Y}$ , then  $K$  has a right adjoint  $R$  between the total categories and  $(K, F) \dashv (R, G)$  is an general oplax adjunction.

### 3. Enriched fibrations

This section’s goal is to introduce a notion of an enriched fibration. It will do so in a way that an adjusted version of Theorem 2.4, instead of providing an enrichment of an ordinary category in a monoidal category, will give an enrichment of an ordinary fibration in a *monoidal* one. The key idea is to shift all necessary structure (Example 2.8) from the context of categories to fibrations, moving from  $(\mathbf{Cat}, \times, \mathcal{I})$  to the monoidal 2-category  $(\mathbf{Fib}, \times, 1_{\mathcal{I}})$ .

First of all, a pseudomonoid (Definition 2.6) in the 2-category of fibrations, which will serve as the base of the enrichment, is a fibration  $T : \mathcal{V} \rightarrow \mathbb{W}$  equipped with a multiplication  $m : T \times T \rightarrow T$  and unit  $j : 1_{\mathcal{I}} \rightarrow T$ , along with 2-isomorphisms  $a, \ell, r$  as in (7). More explicitly, the multiplication and unit are fibred 1-cells  $m = (\otimes_{\mathcal{V}}, \otimes_{\mathbb{W}})$  and  $j = (I_{\mathcal{V}}, I_{\mathbb{W}})$  (9), displayed as

$$\begin{array}{ccc} \mathcal{V} \times \mathcal{V} & \xrightarrow{\otimes_{\mathcal{V}}} & \mathcal{V} \\ T \times T \downarrow & & \downarrow T \\ \mathbb{W} \times \mathbb{W} & \xrightarrow{\otimes_{\mathbb{W}}} & \mathbb{W} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{I} & \xrightarrow{I_{\mathcal{V}}} & \mathcal{V} \\ 1 \downarrow & & \downarrow T \\ \mathcal{I} & \xrightarrow{I_{\mathbb{W}}} & \mathbb{W} \end{array} \quad (11)$$

where  $\otimes_{\mathcal{V}}$  and  $I_{\mathcal{V}}$  are cartesian, and invertible fibred 2-cells  $a = (a^{\mathcal{V}}, a^{\mathbb{W}})$ ,

$r = (r^{\mathcal{V}}, r^{\mathbb{W}})$ ,  $\ell = (\ell^{\mathcal{V}}, \ell^{\mathbb{W}})$  (10), displayed as

$$\begin{array}{ccc}
 \mathcal{V} \times \mathcal{V} \times \mathcal{V} & \begin{array}{c} \xrightarrow{\otimes(\otimes \times 1)} \\ \Downarrow a^{\mathcal{V}} \\ \xrightarrow{\otimes(1 \times \otimes)} \end{array} & \mathcal{V} \\
 \downarrow T \times T \times T & & \downarrow T \\
 \mathbb{W} \times \mathbb{W} \times \mathbb{W} & \begin{array}{c} \xrightarrow{\otimes(\otimes \times 1)} \\ \Downarrow a^{\mathbb{W}} \\ \xrightarrow{\otimes(1 \times \otimes)} \end{array} & \mathbb{W}
 \end{array}$$
  

$$\begin{array}{ccc}
 \mathcal{V} \times 1 & \begin{array}{c} \xrightarrow{\otimes(1 \times I)} \\ \Downarrow r^{\mathcal{V}} \\ \sim \end{array} & \mathcal{V} \\
 \downarrow T \times 1 & & \downarrow T \\
 \mathbb{W} \times 1 & \begin{array}{c} \xrightarrow{\otimes(1 \times I)} \\ \Downarrow r^{\mathbb{W}} \\ \sim \end{array} & \mathbb{W}
 \end{array}
 \qquad
 \begin{array}{ccc}
 1 \times \mathcal{V} & \begin{array}{c} \xrightarrow{\otimes(I \times 1)} \\ \Downarrow \ell^{\mathcal{V}} \\ \sim \end{array} & \mathcal{V} \\
 \downarrow 1 \times T & & \downarrow T \\
 1 \times \mathbb{W} & \begin{array}{c} \xrightarrow{\otimes(I \times 1)} \\ \Downarrow \ell^{\mathbb{W}} \\ \sim \end{array} & \mathbb{W}
 \end{array}$$

where by definition  $a^{\mathcal{V}}, r^{\mathcal{V}}, \ell^{\mathcal{V}}$  lie above  $a^{\mathbb{W}}, r^{\mathbb{W}}, \ell^{\mathbb{W}}$ . The coherence axioms they satisfy turn out to give the usual axioms which make  $(\mathcal{V}, \otimes_{\mathcal{V}}, I_{\mathcal{V}})$  and  $(\mathbb{W}, \otimes_{\mathbb{W}}, I_{\mathbb{W}})$  into monoidal categories with the respective associativity, left and right unit constraints.

**Remark 3.1.** The latter can be deduced also by the fact that the domain and codomain 2-functors  $dom, cod : \mathbf{Fib} \subset \mathbf{Cat}^2 \rightarrow \mathbf{Cat}$  are in fact strict monoidal, i.e. preserve the cartesian structure on the nose. In other words, the equality of pasted diagrams of 2-cells in  $\mathbf{Fib}$  breaks down into equalities  $\mathbf{Cat}$  for the two (ordinary) natural transformations they consist of.

Moreover, the strict commutativity of the diagrams (11) implies that  $T$  strictly preserves the tensor product and the unit object between  $\mathcal{V}$  and  $\mathbb{W}$ , i.e.

$$TA \otimes_{\mathbb{W}} TB = T(A \otimes_{\mathcal{V}} B), \quad I_{\mathbb{W}} = T(I_{\mathcal{V}}).$$

Along with the conditions that  $T(a^{\mathcal{V}}) = a^{\mathbb{W}}, T(\ell^{\mathcal{V}}) = \ell^{\mathbb{W}}$  and  $T(r^{\mathcal{V}}) = r^{\mathbb{W}}$ , these data define a strict monoidal structure on  $T$ ; we obtain the following definition, which coincides with [Shu08, 12.1].

**Definition 3.2.** A monoidal fibration is a fibration  $T : \mathcal{V} \rightarrow \mathbb{W}$  such that

- (i)  $\mathcal{V}$  and  $\mathbb{W}$  are monoidal categories,

- (ii)  $T$  is a strict monoidal functor,
- (iii) the tensor product  $\otimes_{\mathcal{V}}$  of  $\mathcal{V}$  preserves cartesian arrows.

If  $\mathcal{V}$  and  $\mathcal{W}$  are symmetric monoidal categories and  $T$  is a symmetric strict monoidal functor, we call  $T$  a *symmetric monoidal fibration*. In a dual way, we can define a (*symmetric*) *monoidal opfibration* to be an opfibration which is a (symmetric) strict monoidal functor, where the tensor product of the total category preserves cocartesian arrows. Notice that a monoidal opfibration is still a pseudomonoid (and not a pseudocomonoid), this time in **OpFib**. Finally, a *monoidal bifibration* is one where the tensor product of the total category preserves both cartesian and cocartesian liftings.

We now describe a pseudomodule for a pseudomonoid in  $(\mathbf{Fib}, \times, 1_{\mathcal{I}})$ ; in analogy to Theorem 2.4, this will be the object which will eventually have the enriched structure. According to Definition 2.7, a *pseudoaction* of a monoidal fibration  $T : \mathcal{V} \rightarrow \mathbb{W}$  on an ordinary fibration  $P : \mathcal{A} \rightarrow \mathbb{X}$  is a fibred 1-cell  $\mu = (\mu^{\mathcal{A}}, \mu^{\mathbb{X}}) : T \times P \rightarrow P$

$$\begin{array}{ccc} \mathcal{V} \times \mathcal{A} & \xrightarrow{\mu^{\mathcal{A}}} & \mathcal{A} \\ T \times P \downarrow & & \downarrow P \\ \mathbb{W} \times \mathbb{X} & \xrightarrow{\mu^{\mathbb{X}}} & \mathbb{X} \end{array} \quad (12)$$

where  $\mu^{\mathcal{A}}$  is cartesian, along with 2-isomorphisms  $\chi, \nu$  as in (8) in **Fib**. Explicitly, these are invertible fibred 2-cells  $\chi = (\chi^{\mathcal{A}}, \chi^{\mathbb{X}})$ ,  $\nu = (\nu^{\mathcal{A}}, \nu^{\mathbb{X}})$  represented by

$$\begin{array}{ccc} \mathcal{V} \times \mathcal{V} \times \mathcal{A} & \xrightarrow{M \times 1} \mathcal{V} \times \mathcal{A} \xrightarrow{\mu} & \mathcal{A} \\ & \searrow 1 \times \mu \downarrow \chi^{\mathcal{A}} & \downarrow P \\ \mathcal{V} \times \mathcal{A} & \xrightarrow{\mu} & \mathcal{A} \\ T \times T \times P \downarrow & & \\ \mathbb{W} \times \mathbb{W} \times \mathbb{X} & \xrightarrow{M \times 1} \mathbb{W} \times \mathbb{X} \xrightarrow{\mu} & \mathbb{X} \\ & \searrow 1 \times \mu \downarrow \chi^{\mathbb{X}} & \\ \mathbb{W} \times \mathbb{X} & \xrightarrow{\mu} & \mathbb{X} \end{array} \quad \begin{array}{ccc} 1 \times \mathcal{A} & \xrightarrow{I \times 1} \mathcal{V} \times \mathcal{A} \xrightarrow{\mu} & \mathcal{A} \\ & \searrow \downarrow \nu^{\mathcal{A}} & \downarrow P \\ 1 \times \mathcal{A} & \xrightarrow{\mu} & \mathcal{A} \\ 1 \times P \downarrow & & \\ 1 \times \mathbb{X} & \xrightarrow{I \times 1} \mathbb{W} \times \mathbb{X} \xrightarrow{\mu} & \mathbb{X} \\ & \searrow \downarrow \nu^{\mathbb{X}} & \\ 1 \times \mathbb{X} & \xrightarrow{\mu} & \mathbb{X} \end{array}$$

where  $\chi^{\mathcal{A}}, \nu^{\mathcal{A}}$  are above  $\chi^{\mathbb{X}}, \nu^{\mathbb{X}}$  with respect to the appropriate fibrations. These data are subject to certain axioms, which in fact again split up in

two sets of commutative diagrams for the the two natural isomorphisms that  $\chi$  and  $\nu$  consist of; these coincide with the action of a monoidal category axioms (Definition 2.2).

**Definition 3.3.** *A  $T$ -representation for a monoidal fibration  $T: \mathcal{V} \rightarrow \mathbb{W}$  is a fibration  $P: \mathcal{A} \rightarrow \mathbb{X}$  equipped with a  $T$ -pseudoaction  $\mu = (\mu^A, \mu^{\mathbb{X}})$ . This amounts to two actions*

$$\begin{aligned}\mu^A = * : \mathcal{V} \times \mathcal{A} &\longrightarrow \mathcal{A} \\ \mu^{\mathbb{X}} = \diamond : \mathbb{W} \times \mathbb{X} &\longrightarrow \mathbb{X}\end{aligned}$$

of the monoidal categories  $\mathcal{V}$ ,  $\mathbb{W}$  on the categories  $\mathcal{A}$  and  $\mathbb{X}$  respectively, satisfying the commutativity of (12) where  $\mu^A$  preserves cartesian arrows, such that for all  $X, Y \in \mathcal{V}$  and  $A \in \mathcal{A}$  the following conditions hold:

$$P\chi_{XYA}^A = \chi_{(TX)(TY)(PA)}^{\mathbb{X}}, \quad P\nu_A^A = \nu_{PA}^{\mathbb{X}}. \quad (13)$$

The compatibility conditions of the above definition are natural, since by (12)

$$P(X * A) = TX \diamond PA$$

for any  $X \in \mathcal{V}$ ,  $A \in \mathcal{A}$ , hence the isomorphisms  $\chi_{XYA}^A : X * (Y * A) \cong (X \otimes_{\mathcal{V}} Y) * A$  in  $\mathcal{A}$  lie above certain isomorphisms

$$P\chi_{XYA}^A : TX \diamond (TY \diamond PA) \xrightarrow{\sim} (TX \otimes_{\mathbb{W}} TY) \diamond PA \quad (14)$$

in  $\mathbb{X}$ , due to the strict monoidality of  $T$ . Similarly,  $\nu_A^A : I * A \cong A$  is mapped to

$$P\nu_A^A : I_{\mathbb{X}} \diamond PA \xrightarrow{\sim} PA \quad (15)$$

since  $P(I_{\mathcal{V}} * A) = T(I_{\mathcal{V}}) \diamond PA = I_{\mathbb{W}} \diamond PA$ . Thus (13) demand that (14) and (15) coincide with the components of  $\chi^{\mathbb{X}}$  and  $\nu^{\mathbb{X}}$ , from the  $\mathbb{W}$ -representation  $\mathbb{X}$ .

The last step in modifying Theorem 2.4 to obtain a correspondence between representations of a monoidal fibration and the desired enriched fibrations, is to introduce a notion of a parameterized adjunction in **Fib**. For that, we first re-formulate the ‘adjunctions with a parameter’ Theorem 2.1 in the context of **Cat**<sup>2</sup>.

**Theorem 3.4.** *Suppose we have a morphism  $(F, G)$  of two variables in  $\mathbf{Cat}^2$ , given by a commutative square of categories and functors*

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{B} & \xrightarrow{F} & \mathcal{C} \\ H \times J \downarrow & & \downarrow K \\ \mathbb{X} \times \mathbb{Y} & \xrightarrow{G} & \mathbb{Z}. \end{array} \quad (16)$$

*Assume that, for every  $B \in \mathcal{B}$  and  $Y \in \mathbb{Y}$ , there exist adjunctions  $F(-, B) \dashv R(B, -)$  and  $G(-, Y) \dashv S(Y, -)$ , such that  $(F(-, B), G(-, JB))$  has a right adjoint  $(R(B, -), S(JB, -))$  in  $\mathbf{Cat}^2$ . This is represented by*

$$\begin{array}{ccc} \mathcal{A} & \begin{array}{c} \xrightarrow{F(-, B)} \\ \perp \\ \xleftarrow{R(B, -)} \end{array} & \mathcal{C} \\ H \downarrow & & \downarrow K \\ \mathbb{X} & \begin{array}{c} \xrightarrow{G(-, JB)} \\ \perp \\ \xleftarrow{S(JB, -)} \end{array} & \mathbb{Z} \end{array} \quad (17)$$

*where  $(H, K)$  is a map of adjunctions (both squares commute and  $\varepsilon_K = K\varepsilon$ ,  $H\eta = \eta_H$ ). Then, there is a unique way to define a morphism of two variables*

$$\begin{array}{ccc} \mathcal{B}^{\text{op}} \times \mathcal{C} & \xrightarrow{R} & \mathcal{A} \\ J^{\text{op}} \times K \downarrow & & \downarrow H \\ \mathbb{Y}^{\text{op}} \times \mathbb{Z} & \xrightarrow{S} & \mathbb{X} \end{array} \quad (18)$$

*in  $\mathbf{Cat}^2$ , for which  $\mathcal{C}(F(A, B), C) \cong \mathcal{A}(A, R(B, C))$ ,  $\mathbb{Z}(G(X, Y), Z) \cong \mathbb{X}(X, S(Y, Z))$  are natural in all three variables.*

*Proof.* The result clearly follows from ordinary parameterized adjunctions. The fact that  $(R(B, -), S(JB, -))$  is an arrow in  $\mathbf{Cat}^2$  for all  $B$ 's ensures that the diagram (18) commutes on the second variable, and also on the first variable on objects, since  $HR(B, C) = S(JB, KC)$ . On arrows, commutativity follows from the unique way of defining  $R(h, 1)$  and  $S(Jh, 1)$  for any  $h : B \rightarrow B'$  under these assumptions.  $\square$

We call  $(R, S)$  the *parameterized adjoint* of  $(F, G)$  in  $\mathbf{Cat}^2$ , written  $(F, G) \dashv_{\mathcal{P}} (R, S)$ .



**Remark 3.5.** Although the notion of an adjunction can be internalized in any bicategory, its parameterized version seems to be much more involved. In any monoidal bicategory with duals, we could ask for 1-cells  $A \cong A \otimes I \xrightarrow{1 \times b} A \otimes B \xrightarrow{t} C$  to have adjoints  $g_b: C \rightarrow A$ , for every  $b: I \rightarrow B$ . For the cartesian 2-monoidal case at least, with ‘category-like’ objects like in **Fib**, the 2-categorical approach of [CGR14, Thm. 2.4] clarifies things.

Restricting to fibrations, consider a morphism of two variables in  $\mathbf{Fib} \subset \mathbf{Cat}^2$  i.e. a fibred 1-cell  $(F, G)$  as in (16) with  $F$  cartesian, with the property that (17) is a general fibred adjunction as in Definition 2.9, i.e. the partial right adjoint  $R(B, -)$  is also cartesian. Dually, in  $\mathbf{OpFib}$  we request both  $F$  and  $R(B, -)$  to be cocartesian. Notice that in both cases, the parameterized adjoint of two variables  $(R, S)$  can neither be a fibred nor an opfibred 1-cell ‘wholly’, since at (18) the vertical  $J^{\text{op}} \times K$  is a product of a fibration with an opfibration, hence neither of the two.

If we lift the (co)cartesian requirements, we end up with the (op)lax version of these adjunctions. Since those cases are the most relevant to our examples, we abuse notation as to call (op)fibred parameterized adjunctions the (op)lax ones. Based on the remark that follows, this abuse is in fact only fractional.

**Remark 3.6.** There exists an interesting asymmetry regarding the (co)cartesianness requirement of the left/right partial adjoints, due to Lemma 2.10. Since right adjoints always preserve cartesian arrows in  $\mathbf{Cat}^2$  and dually left adjoints always preserve cocartesian ones, we can deduce that any fibred 1-cell  $(F, G)$  has a (right) fibred parameterized adjoint as long as it has a  $\mathbf{Cat}^2$ -parameterized adjoint. Dually, an opfibred 1-cell has a (left) opfibred parameterized adjoint as long as it has it in  $\mathbf{Cat}^2$ .

**Definition 3.7.** Suppose  $H, K$  are fibrations. A fibred parameterized adjunction is a parameterized adjunction  $(F, G) \dashv (R, S)$  in  $\mathbf{Cat}^2$ , between two 1-cells

$$\begin{array}{ccc}
 \mathcal{A} \times \mathcal{B} & \xrightarrow{F} & \mathcal{C} \\
 \downarrow H \times J & & \downarrow K \\
 \mathbb{X} \times \mathbb{Y} & \xrightarrow{G} & \mathbb{Z}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{B}^{\text{op}} \times \mathcal{C} & \xrightarrow{R} & \mathcal{A} \\
 \downarrow J^{\text{op}} \times K & & \downarrow H \\
 \mathbb{Y}^{\text{op}} \times \mathbb{Z} & \xrightarrow{S} & \mathbb{X}
 \end{array}$$

where  $R(B, -)$  is by default cartesian. Dually, an opfibred parameterized adjunction is as above, where  $F(-, B)$  is by default cocartesian.

The proposed definition of an enriched fibration is justified by the subsequent Theorem 3.11 which fulfills our initial goal, i.e. to generalize Theorem 2.4 to the context of (op)fibrations. In Remark 3.9 we give an equivalent formulation in terms of enriched functors. The enriched hom-functor is defined as in (3), writing  $\mathcal{A}$  for both the enriched and the underlying category.

**Definition 3.8 (Enriched Fibration).** *Suppose  $T : \mathcal{V} \rightarrow \mathbb{W}$  is a monoidal fibration. A fibration  $P : \mathcal{A} \rightarrow \mathbb{X}$  is enriched in  $T$  when the following conditions are satisfied:*

- the total category  $\mathcal{A}$  is enriched in the total monoidal category  $\mathcal{V}$  and the base category  $\mathbb{X}$  is enriched in the base monoidal category  $\mathbb{W}$ , in such a way that the following commutes:

$$\begin{array}{ccc} \mathcal{A}^{\text{op}} \times \mathcal{A} & \xrightarrow{\mathcal{A}(-,-)} & \mathcal{V} \\ P^{\text{op}} \times P \downarrow & & \downarrow T \\ \mathbb{X}^{\text{op}} \times \mathbb{X} & \xrightarrow{\mathbb{X}(-,-)} & \mathbb{W} \end{array} \quad (19)$$

- the composition law and the identities of the enrichments are compatible, in the sense that

$$\begin{aligned} TM_{A,B,C}^{\mathcal{A}} &= M_{PA,PB,PC}^{\mathbb{X}} \\ Tj_A^{\mathcal{A}} &= j_{PA}^{\mathbb{X}} \end{aligned} \quad (20)$$

The compatibilities (20) only state that the composition and identities

$$M_{A,B,C}^{\mathcal{A}} : \mathcal{A}(B, C) \otimes_{\mathcal{V}} \mathcal{A}(A, B) \rightarrow \mathcal{A}(A, C), \quad j_A^{\mathcal{A}} : I_{\mathcal{V}} \rightarrow \mathcal{A}(A, A)$$

of the  $\mathcal{V}$ -enriched  $\mathcal{A}$  are mapped, under  $T$ , exactly to those of the  $\mathbb{W}$ -enriched  $\mathbb{X}$ :

$$\begin{aligned} M_{PA,PB,PC}^{\mathbb{X}} &: \mathbb{X}(PB, PC) \otimes_{\mathbb{W}} \mathbb{X}(PA, PB) \rightarrow \mathbb{X}(PA, PC) \\ j_{PA}^{\mathbb{X}} &: I_{\mathbb{W}} \rightarrow \mathbb{X}(PA, PA) \end{aligned}$$

where the domains and codomains already coincide by strict monoidality of  $T$  and the commutativity of (19).

For the above definition, it could be argued that some sort of cartesian condition for the enriched hom-functor  $\mathcal{A}(-, -)$  should be asked; notice however that for  $P$  a fibration, the product  $P^{\text{op}} \times P$  has neither a fibration nor an opfibration structure. If we required that the partial functor  $\mathcal{A}(A, -): \mathcal{A} \rightarrow \mathcal{V}$  is cartesian, all results below would still be valid with minor adjustments. Since the examples (Section 4) so far do not seem to satisfy this extra condition, for the moment we adhere to this more general definition.

**Remark 3.9** (*Enriched fibrations as enriched functors*). For a  $T$ -enriched fibration  $P$  as above, the strict monoidal structure of  $T$  induces a 2-functor  $\tilde{T}: \mathcal{V}\text{-Cat} \rightarrow \mathbb{W}\text{-Cat}$  by Proposition 2.3. Hence we can make the  $\mathcal{V}$ -category  $\mathcal{A}$  into a  $\mathbb{W}$ -category  $\tilde{T}\mathcal{A}$ , with the same set of objects  $\text{ob}\mathcal{A}$  and hom-objects  $\tilde{T}\mathcal{A}(A, B) = \mathbb{X}(PA, PB)$ . Then  $P: \mathcal{A} \rightarrow \mathbb{X}$  can be verified to have the structure of a  $\mathbb{W}$ -enriched functor between the  $\mathbb{W}$ -categories  $\tilde{T}\mathcal{A}$  and  $\mathbb{X}$ , with hom-objects mapping  $\tilde{T}\mathcal{A}(A, B) \xrightarrow{=} \mathbb{X}(PA, PB)$ . The compatibility with the composition and the identities is ensured by (20).

From this perspective, the definition of a  $(T: \mathcal{V} \rightarrow \mathbb{W})$ -enriched fibration between a  $\mathcal{V}$ -category  $\mathcal{A}$  and a  $\mathbb{W}$ -category  $\mathbb{X}$  could be reformulated as a strictly fully faithful  $\mathbb{W}$ -enriched functor  $P: \tilde{T}\mathcal{A} \rightarrow \mathbb{X}$ , whose underlying ordinary functor  $P_0: \mathcal{A}_0 \rightarrow \mathbb{X}_0$  is a fibration (the commutativity of (19) follows).

Dually, we have the notion of an *enriched opfibration*, as well as the following combined version.

**Definition 3.10.** *Suppose that  $T: \mathcal{V} \rightarrow \mathbb{W}$  is a symmetric monoidal opfibration. We say that a fibration  $P: \mathcal{A} \rightarrow \mathbb{X}$  is enriched in  $T$  if the opfibration  $P^{\text{op}}: \mathcal{A}^{\text{op}} \rightarrow \mathbb{X}^{\text{op}}$  is an enriched  $T$ -opfibration.*

Finally, we prove that to give a fibration with an action  $(*, \diamond)$  of a monoidal fibration  $T$  (Definition 3.3) with a fibred parameterized adjoint (Definition 3.7), is to give a  $T$ -enriched fibration (Definition 3.8).

**Theorem 3.11.** *Suppose that  $T : \mathcal{V} \rightarrow \mathbb{W}$  is a monoidal fibration, which acts on an (ordinary) fibration  $P : \mathcal{A} \rightarrow \mathbb{X}$  via the fibred 1-cell*

$$\begin{array}{ccc} \mathcal{V} \times \mathcal{A} & \xrightarrow{*} & \mathcal{A} \\ T \times P \downarrow & & \downarrow P \\ \mathbb{W} \times \mathbb{X} & \xrightarrow{\diamond} & \mathbb{X}. \end{array}$$

*If this action has a parameterized adjoint  $(R, S)$  as in*

$$\begin{array}{ccc} \mathcal{A}^{\text{op}} \times \mathcal{A} & \xrightarrow{R} & \mathcal{V} \\ P^{\text{op}} \times P \downarrow & & \downarrow T \\ \mathbb{X}^{\text{op}} \times \mathbb{X} & \xrightarrow{S} & \mathbb{W} \end{array}$$

*we can enrich the fibration  $P$  in the monoidal fibration  $T$ .*

*Proof.* Recall by Definition 3.3 that the  $T$ -action in particular consists of two actions  $*$  and  $\diamond$  of the monoidal categories  $\mathcal{V}$  and  $\mathbb{W}$  on the categories  $\mathcal{A}$  and  $\mathbb{X}$  respectively. Since  $(*, \diamond) \dashv_p (R, S)$ , by Theorem 3.4 we have two ordinary adjunctions

$$\mathcal{V} \begin{array}{c} \xrightarrow{-*A} \\ \perp \\ \xleftarrow{R(A,-)} \end{array} \mathcal{A} \quad \text{and} \quad \mathbb{W} \begin{array}{c} \xrightarrow{-\diamond X} \\ \perp \\ \xleftarrow{S(X,-)} \end{array} \mathbb{X}$$

for all  $A \in \mathcal{A}$  and  $X \in \mathbb{X}$ . By Theorem 2.4, there exists a  $\mathcal{V}$ -category  $\underline{\mathcal{A}}$  with underlying category  $\mathcal{A}$  and hom-objects  $\underline{\mathcal{A}}(A, B) = R(A, B)$  and also a  $\mathbb{W}$ -category  $\underline{\mathbb{X}}$  with underlying category  $\mathbb{X}$  and hom-objects  $\underline{\mathbb{X}}(X, Y) = S(X, Y)$ . Also, the enriched hom-functors satisfy the required commutativity  $TS(-, -) = R(P-, P-)$  by (18).

Finally, we need to show that the composition and identity laws of the enrichments are compatible as in (20), i.e.  $TM_{A,B,C}^A = M_{PA,PB,PC}^{\mathbb{X}}$  and  $Tj_A^A = j_{PA}^{\mathbb{X}}$ . For that, it is enough to confirm that their adjuncts under  $(-\diamond X) \dashv S(X, -)$  coincide. The latter ones are explicitly given by (5)

and (6), i.e.

$$\begin{array}{ccc}
 (S(PB, PC) \otimes S(PA, PB)) \diamond PA & \dashrightarrow & PC \\
 \chi^{\mathbb{X}} \downarrow & & \uparrow \varepsilon_{PC} \\
 S(PB, PC) \diamond (S(PA, PB) \diamond PA) & \xrightarrow{1 \diamond \varepsilon_{PB}} & S(PB, PC) \diamond PB \\
 I \diamond PA \xrightarrow{\nu^{\mathbb{X}}} PA & & 
 \end{array}$$

For the former ones, since  $(P, T)$  is a map of adjunctions

$$\begin{array}{ccc}
 \mathcal{V} & \begin{array}{c} \xrightarrow{- * A} \\ \perp \\ \xleftarrow{R(A, -)} \end{array} & \mathcal{A} \\
 T \downarrow & & \downarrow P \\
 \mathcal{W} & \begin{array}{c} \xrightarrow{- \diamond PA} \\ \perp \\ \xleftarrow{S(PA, -)} \end{array} & \mathbb{X},
 \end{array}$$

taking the images of  $M_{A,B,C}^A$  and  $j_A^A$  under  $T$  and translating under the adjunction  $(- \diamond X) \dashv S(X, -)$  is the same as first translating under  $(- * A) \dashv R(A, -)$  and then applying  $P$ . That produces

$$\begin{array}{ccc}
 P(R(B, C) \otimes R(A, B)) * A & \xrightarrow{P\chi^A} & P(R(B, C) * R(A, B) * A) \\
 & \searrow \text{dashed} & \downarrow P(1 * \varepsilon_B) \\
 & & P(R(B, C) * B) \\
 & & \downarrow P(\varepsilon_C) \\
 & & PC
 \end{array}$$

$$P(I * A) \xrightarrow{P(\nu^A)} PA$$

Since  $P\chi^A = \chi^{\mathbb{X}}$  and  $P\nu^A = \nu^{\mathbb{X}}$  from (13), and also  $P\varepsilon = \varepsilon_P$  as a map of adjunctions, the above composites coincide and the proof is complete.  $\square$

An important first example that should fit this setting of an action-induced enrichment is that of a *closed* monoidal fibration. Just like a monoidal closed category  $\mathcal{V}$  is one where  $\otimes : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$  has a (right) parameterized adjoint via  $- \otimes X \dashv [X, -]$  for every object  $X$ , we can consider the following notion based on Definition 3.7.

**Definition 3.12.** A monoidal fibration  $T: \mathcal{V} \rightarrow \mathbb{W}$  is (right) closed when its tensor product fibred 1-cell (11)

$$\begin{array}{ccc} \mathcal{V} \times \mathcal{V} & \xrightarrow{\otimes_{\mathcal{V}}} & \mathcal{V} \\ T \times T \downarrow & & \downarrow T \\ \mathbb{W} \times \mathbb{W} & \xrightarrow{\otimes_{\mathbb{W}}} & \mathbb{W} \end{array}$$

has a parameterized adjoint

$$\begin{array}{ccc} \mathcal{V}^{\text{op}} \times \mathcal{V} & \xrightarrow{[-,-]_{\mathcal{V}}} & \mathcal{V} \\ T^{\text{op}} \times T \downarrow & & \downarrow T \\ \mathbb{W}^{\text{op}} \times \mathbb{W} & \xrightarrow{[-,-]_{\mathbb{W}}} & \mathbb{W} \end{array}$$

Equivalently, by Theorem 3.4,  $T$  is monoidal closed when

- (i)  $\mathcal{V}$  and  $\mathbb{W}$  are monoidal closed categories,
- (ii)  $T$  is a strict closed functor,
- (iii)  $T\varepsilon = \varepsilon_T$  and  $\eta_T = T\eta$  for the respective units and counits of the adjunctions.

Notice that by Lemma 2.10, the right adjoint  $[V, -]_{\mathcal{V}}$  between the total categories automatically preserves cartesian liftings. On the other hand, for the dual notion of a *monoidal closed opfibration*, the right adjoint is not cocartesian by default.

**Remark 3.13.** In [Shu08, §13], definitions of an *internally closed* monoidal fibration over a cartesian monoidal base, as well as *externally closed* monoidal fibration over an arbitrary monoidal base are given. These are equivalent to each other under certain hypotheses; none, however, guarantee that the total category is closed on its own right. The external definition gives some, but not all, conditions in terms of the fibres and the reindexing functors for a fibred adjoint to exist, in the spirit of results such as Theorem 2.11.

Applying Theorem 3.11 we can deduce the enrichment of a monoidal closed fibration in itself, analogously to the ordinary case (Remark 2.5).

**Proposition 3.14.** *A monoidal closed fibration  $T: \mathcal{V} \rightarrow \mathbb{W}$  is  $T$ -enriched. Dually, a monoidal closed opfibration is enriched in itself.*

*Proof.* All clauses of Definition 3.3 are satisfied, since the functor  $- \otimes_{\mathcal{V}} -$  is cartesian in both variables by Definition 3.2, and also  $Ta_{XYZ}^{\mathcal{V}} = a_{TXYTZ}^{\mathbb{W}}$  and  $T\ell_X^{\mathcal{V}} = \ell_{TX}^{\mathbb{W}}$  for the respective associator and the left unitor since  $T$  is a strict monoidal functor. Therefore  $(\otimes_{\mathcal{V}}, \otimes_{\mathbb{W}})$  is indeed a  $T$ -action, just like the regular representation of a monoidal category earlier. Since this action has a parameterized adjoint, by definition of a monoidal closed fibration, the result follows.  $\square$

Finally, there is a dual version to Theorem 3.11, characterizing the enrichment of an opfibration in a monoidal opfibration.

**Theorem 3.15.** *Suppose that  $T: \mathcal{V} \rightarrow \mathbb{W}$  is a monoidal opfibration, which acts on an (ordinary) opfibration  $U: \mathcal{B} \rightarrow \mathbb{Y}$  via the opfibred 1-cell*

$$\begin{array}{ccc} \mathcal{V} \times \mathcal{B} & \xrightarrow{*} & \mathcal{B} \\ T \times U \downarrow & & \downarrow U \\ \mathbb{W} \times \mathbb{Y} & \xrightarrow{\diamond} & \mathbb{Y}. \end{array}$$

*If this action has a parameterized adjoint  $(R, S)$  as in*

$$\begin{array}{ccc} \mathcal{B}^{\text{op}} \times \mathcal{B} & \xrightarrow{R} & \mathcal{V} \\ U^{\text{op}} \times U \downarrow & & \downarrow T \\ \mathbb{Y}^{\text{op}} \times \mathbb{Y} & \xrightarrow{S} & \mathbb{W} \end{array}$$

*we can enrich the opfibration  $U$  in the monoidal opfibration  $T$ .*

**Remark 3.16.** The asymmetry between cartesian and cocartesian functors with regards to fibred and opfibred adjunctions is still apparent when comparing Theorems 3.11 and 3.15. For the former, Lemma 2.10 ensures that the right parameterized adjoint will be, at least partially as  $R(A, -)$ , cartesian; as a result, the whole parameterized adjunction lifts from  $\mathbf{Cat}^2$  to  $\mathbf{Fib}$ . On the other hand, for the latter dual theorem, the assumptions cannot ensure that the enriched hom  $R$  will be partially cocartesian. One reason for this discrepancy is that even if we change our setting from  $\mathbf{Fib}_{(\text{opl})}$  to  $\mathbf{OpFib}_{(\text{lax})}$ , the enrichment is given in both cases by the existence of a *right* adjoint (and not of a left one in the dual setting).

## 4. Applications

In this final chapter, we exhibit a few examples of the enriched fibration notion. In these cases, Theorems 3.11 and 3.15 seem to be the easiest way to deduce the enrichment, due to the fact that the enrichments on the level of bases and total categories are themselves obtained by the similarly-flavored Theorem 2.4. In what follows, we do not present all the relevant theory as it would take up many pages; instead we provide the appropriate references, in the hope that the interested reader will look for the details therein.

### 4.1 (Co)modules over (co)monoids

In the context of a locally presentable symmetric monoidal closed category  $\mathcal{V}$ , previous work [HLFV17a] establishes an enrichment of the category of monoids  $\mathbf{Mon}(\mathcal{V})$  in the symmetric monoidal category of comonoids  $\mathbf{Comon}(\mathcal{V})$ , via Theorem 2.4. The action of comonoids on monoids is induced by the internal hom of  $\mathcal{V}$ : for any coalgebra  $C$  and algebra  $B$ ,  $[C, B]$  has always the structure of an algebra via the convolution product. Its (right) parametrized adjoint  $P: \mathbf{Mon}(\mathcal{V})^{\text{op}} \times \mathbf{Mon}(\mathcal{V}) \rightarrow \mathbf{Comon}(\mathcal{V})$  which is the enriched hom-functor is called the *Sweedler hom*, since the original notion of a *measuring coalgebra*  $P(A, B)$  goes back to [Swe69].

Furthermore, in [HLFV17b] a similarly action-induced enrichment is established for the *global* category of modules in the symmetric monoidal global category of comodules, i.e. the category of all (co)modules over any (co)monoid in  $\mathcal{V}$ . The action again comes from the internal hom of the monoidal category, and its parametrized adjoint  $Q: \mathbf{Mod}^{\text{op}} \times \mathbf{Mod} \rightarrow \mathbf{Comod}$  maps an  $A$ -module  $M$  and a  $B$ -comodule  $N$  to their *measuring comodule*  $Q(M, N)$  [Bat00]. This parameterized adjunction is obtained itself using the theory of fibred adjunctions, since the functor  $U: \mathbf{Mod} \rightarrow \mathbf{Mon}(\mathcal{V})$  which gives the ‘underlying’ algebra of a module is a fibration and dually  $V: \mathbf{Comod} \rightarrow \mathbf{Comon}(\mathcal{V})$  is an opfibration. Therefore the very enrichment on the level of the total categories is accomplished via Theo-



rem 2.11, producing a general (lax) opfibred adjunction

$$\begin{array}{ccc}
 \mathbf{Mod}^{\text{op}} & \begin{array}{c} \xrightarrow{Q(-, N_B)} \\ \xleftarrow{[-, N_B]^{\text{op}}} \end{array} & \mathbf{Comod} \\
 \downarrow U^{\text{op}} & & \downarrow V \\
 \mathbf{Mon}(\mathcal{V})^{\text{op}} & \begin{array}{c} \xrightarrow{P(-, B)} \\ \xleftarrow{[-, B]^{\text{op}}} \end{array} & \mathbf{Comon}(\mathcal{V})
 \end{array} \quad (21)$$

To establish an enriched opfibration structure, we apply Theorem 3.15. First of all,  $V: \mathbf{Comod} \rightarrow \mathbf{Comon}(\mathcal{V})$  can be shown to be a monoidal opfibration, by definition of the monoidal product in  $\mathbf{Comod}$ . Moreover, since both actions on the level of total and base categories are in fact the internal hom of  $\mathcal{V}$  restricted to the appropriate subcategories, compatibility (13) also follows. Finally, the top action is cartesian [HLFV17b, (20)] hence the opfibration  $U^{\text{op}}$  is enriched in  $V$ , as in Definition 3.10.

**Proposition 4.1.** *Suppose  $\mathcal{V}$  is a locally presentable symmetric monoidal closed category. The fibration  $\mathbf{Mod} \rightarrow \mathbf{Mon}(\mathcal{V})$  is enriched in the monoidal opfibration  $\mathbf{Comod} \rightarrow \mathbf{Comon}(\mathcal{V})$ .*

An example of such a monoidal category  $\mathcal{V}$ , which also motivated this whole development, is the category of modules over a commutative ring,  $\mathbf{Mod}_R$ . Both enriching functors arise as adjoints to the linear maps space functor  $\mathbf{Mod}_R(-, -)$  restricted to the respective subcategories. In particular, for two arbitrary modules  $M$  and  $N$  over  $R$ -algebras  $A$  and  $B$ , the measuring comodule  $Q(M, N)$  which provides the enrichment of modules in comodules has its coaction over the Sweedler's measuring  $R$ -coalgebra  $P(A, B)$  which provides the enrichment of algebras in coalgebras. Similarly, the comodule composition maps  $Q(N, S) \otimes_R Q(M, N) \rightarrow Q(M, S)$  are above the coalgebra composition maps  $P(B, C) \otimes_R P(A, B) \rightarrow P(A, B)$ . This is a substantial step exhibiting the tight relations between these dual-flavored, standard (op)fibrations.

Furthermore, the forgetful  $\mathbf{Comod} \rightarrow \mathbf{Comon}(\mathcal{V})$  is in fact an example of a monoidal closed opfibration, Definition 3.12. First of all, it is the case that the category of comonoids is monoidal closed, if  $\mathcal{V}$  is locally presentable

and symmetric monoidal closed, as was already proved in [Por08, 3.2]. In [HLFV17b, 4.5], it is shown in detail how the opfibred 1-cell

$$\begin{array}{ccc} \mathbf{Comod} \times \mathbf{Comod} & \xrightarrow{(-\otimes-)} & \mathbf{Comod} \\ \downarrow & & \downarrow \\ \mathbf{Comon}(\mathcal{V}) \times \mathbf{Comon}(\mathcal{V}) & \xrightarrow{(-\otimes-)} & \mathbf{Comon}(\mathcal{V}) \end{array}$$

has a parameterized right adjoint, so the result follows by Proposition 3.14.

**Proposition 4.2.** *Suppose  $\mathcal{V}$  is a locally presentable symmetric monoidal closed category. The monoidal opfibration  $\mathbf{Comod} \rightarrow \mathbf{Comon}(\mathcal{V})$  is closed, therefore enriched in itself.*

Notice that this does not dualize for  $\mathbf{Mod} \rightarrow \mathbf{Mon}(\mathcal{V})$ , since in general the category of monoids is not monoidal closed, e.g. rings or  $R$ -algebras.

#### 4.2 Enriched (co)modules over enriched (co)categories

The above study on enrichment relations between monoids and comonoids, as well as modules and comodules, can be appropriately extended to their many-object generalizations, in the sense that a monoid can be thought of as a one-object category.

For a detailed exposition of the notions and constructions that follow, see [Vas14, §7] or from a double categorical perspective [Vas17]. Briefly, for a symmetric monoidal category with colimits preserved by  $\otimes$ , we can consider the category of  $\mathcal{V}$ -enriched categories  $\mathcal{V}\text{-Cat}$ , whose objects are monads in the bicategory of *enriched matrices*  $\mathcal{V}\text{-Mat}$  [BCSW83]. In a dual way, considering comonads therein, we can construct the category  $\mathcal{V}\text{-Cocat}$  of *enriched cocategories*, serving as a many object generalization of comonoids in  $\mathcal{V}$ . A  $\mathcal{V}$ -cocategory  $\mathcal{C}_X$  with set of objects  $X$  comes equipped with co-composition and coidentity arrows

$$\Delta_x, z: \mathcal{C}(x, z) \rightarrow \sum_{y \in X} \mathcal{C}(x, y) \otimes \mathcal{C}(y, z), \quad \epsilon_x: \mathcal{C}(x, x) \rightarrow I$$

in  $\mathcal{V}$ , satisfying coassociativity and counitality axioms. Both categories  $\mathcal{V}\text{-Cat}$  and  $\mathcal{V}\text{-Cocat}$  are in fact fibred and opfibred, respectively, over the

category of sets, via the usual forgetful functors that give the set of objects of the (co)categories. They also both live inside  $\mathcal{V}\text{-Grph}$ , the category of enriched graphs, which is bifibred over  $\mathbf{Set}$ . All these (op)fibrations are monoidal in the sense of Definition 3.2: for two  $\mathcal{V}$ -graphs (or categories, co-categories)  $G_X$  and  $H_Y$ , their tensor product is a graph  $G \otimes H$  with set of objects  $X \times Y$ , and  $(G \otimes H)((x, y), (z, w)) = G(x, z) \otimes H(y, w)$ .

When  $\mathcal{V}$  is moreover monoidal closed with products and coproducts,  $\mathcal{V}\text{-Grph}$  is monoidal closed: for two graphs  $G_X$  and  $H_Y$ , their internal hom is the graph  $\text{Hom}(G, H)$  with set of objects  $Y^X$ , given by the collection of  $\mathcal{V}$ -objects

$$\text{Hom}(G, H)(k, s) = \prod_{x', x} [G(x', x), H(kx', sx)] \text{ for } k, s \in Y^X$$

By definition of these structures, the following diagram

$$\begin{array}{ccc} \mathcal{V}\text{-Grph} & \begin{array}{c} \xrightarrow{-\otimes G_X} \\ \perp \\ \xleftarrow{\text{Hom}(G_X, -)} \end{array} & \mathcal{V}\text{-Grph} \\ \downarrow & & \downarrow \\ \mathbf{Set} & \begin{array}{c} \xrightarrow{-\times X} \\ \perp \\ \xleftarrow{(-)^X} \end{array} & \mathbf{Set} \end{array}$$

is a map of adjunctions, therefore all three clauses of Definition 3.12 are satisfied.

**Proposition 4.3.** *Suppose  $\mathcal{V}$  is a symmetric monoidal closed category with products and coproducts. The bifibration  $\mathcal{V}\text{-Grph} \rightarrow \mathbf{Set}$  mapping a  $\mathcal{V}$ -graph to its set of objects is monoidal closed, therefore it is enriched in itself.*

Similarly to how the internal hom of  $\mathcal{V}$  was lifted to an action of comonoids on monoids in Section 4.1, the internal hom of  $\mathcal{V}\text{-Grph}$  induces an action

$$\begin{aligned} K : \mathcal{V}\text{-Cocat}^{\text{op}} \times \mathcal{V}\text{-Cat} &\longrightarrow \mathcal{V}\text{-Cat} \\ (\mathcal{C}_X, \mathcal{B}_Y) &\longmapsto \text{Hom}(\mathcal{C}, \mathcal{B})_{Y^X} \end{aligned}$$

Its opposite has a parameterized adjoint, again by Theorem 2.11,

$$T : \mathcal{V}\text{-Cat}^{\text{op}} \times \mathcal{V}\text{-Cat} \rightarrow \mathcal{V}\text{-Cocat}$$

called the *generalized Sweedler hom*. We get the following (lax) opfibred parameterized adjunction

$$\begin{array}{ccc}
 \mathcal{V}\text{-Cat}^{\text{op}} & \begin{array}{c} \xrightarrow{T(-, \mathcal{B}_Y)} \\ \xleftarrow{\top} \\ \xleftarrow{K(-, \mathcal{B}_Y)^{\text{op}}} \end{array} & \mathcal{V}\text{-Cocat} \\
 \downarrow & & \downarrow \\
 \mathbf{Set}^{\text{op}} & \begin{array}{c} \xrightarrow{Y(-)} \\ \xleftarrow{\top} \\ \xleftarrow{Y(-)^{\text{op}}} \end{array} & \mathbf{Set}
 \end{array}$$

thus Theorem 3.15 applies again, see [Vas17, 4.38].

**Proposition 4.4.** *Suppose  $\mathcal{V}$  is a locally presentable, monoidal closed category. The fibration  $\mathcal{V}\text{-Cat} \rightarrow \mathbf{Set}$  is enriched in the monoidal opfibration  $\mathcal{V}\text{-Cocat} \rightarrow \mathbf{Set}$ , where both functors send the enriched structure to its set of objects.*

Finally, we can consider many object generalizations of modules and comodules, namely  $\mathcal{V}$ -modules for  $\mathcal{V}$ -categories and  $\mathcal{V}$ -comodules for  $\mathcal{V}$ -cocategories, see [Vas14, 7.6]. The former are quite standard: an  $\mathcal{A}_X$ -module  $\Psi$  can also be thought as a  $\mathcal{V}$ -profunctor  $\Psi: \mathcal{I} \rightarrow \mathcal{A}$  for  $\mathcal{I}$  the unit category. Objects are the same as  $\mathcal{A}$ , and hom-objects are  $\Psi(x) \in \mathcal{V}$  equipped with  $(\sum_{x,x'} \mathcal{A}(x, x') \otimes \Psi(x')) \rightarrow \Psi(x')$  satisfying appropriate axioms. The notion of comodules is dual, and these form global categories much like before,  $\mathcal{V}\text{-Mod}$  and  $\mathcal{V}\text{-Comod}$ . The internal hom of enriched graphs further restricts to these categories, giving an action of  $\mathcal{V}\text{-Comod}$  on  $\mathcal{V}\text{-Mod}$  via a functor

$$\begin{aligned}
 \bar{K} : \mathcal{V}\text{-Comod}^{\text{op}} \times \mathcal{V}\text{-Mod} &\longrightarrow \mathcal{V}\text{-Mod} \\
 (\Phi_C, \Psi_B) &\longmapsto \text{Hom}(\Phi, \Psi)_{\text{Hom}(C, B)}
 \end{aligned}$$

where  $\text{Hom}(\Phi, \Psi)(t) = \prod_x [\Phi(x), \Psi(tx)]$ . It has a parameterized adjoint

$$\bar{T} : \mathcal{V}\text{-Mod}^{\text{op}} \times \mathcal{V}\text{-Mod} \rightarrow \mathcal{V}\text{-Comod}$$

by Theorem 2.11 which once more heavily relies on the fact that  $\mathcal{V}\text{-Mod}$  is fibred over  $\mathcal{V}\text{-Cat}$  and  $\mathcal{V}\text{-Comod}$  is opfibred  $\mathcal{V}\text{-Cocat}$ , and there exists an

adjunction between the base categories:

$$\begin{array}{ccc}
 \mathcal{V}\text{-Mod}^{\text{op}} & \begin{array}{c} \xrightarrow{\overline{T}(-, \Psi_{\mathcal{B}})} \\ \top \\ \xleftarrow{\overline{K}(-, \Psi_{\mathcal{B}})^{\text{op}}} \end{array} & \mathcal{V}\text{-Comod} \\
 \downarrow & & \downarrow \\
 \mathcal{V}\text{-Cat}^{\text{op}} & \begin{array}{c} \xrightarrow{T(-, \mathcal{B}_Y)} \\ \top \\ \xleftarrow{K(-, \mathcal{B}_Y)^{\text{op}}} \end{array} & \mathcal{V}\text{-Cocat}
 \end{array} \tag{22}$$

**Proposition 4.5.** *If  $\mathcal{V}$  is a locally presentable symmetric monoidal closed category, the fibration  $\mathcal{V}\text{-Mod} \rightarrow \mathcal{V}\text{-Cat}$  is enriched in the opfibration  $\mathcal{V}\text{-Comod} \rightarrow \mathcal{V}\text{-Cocat}$ .*

*Proof.* Theorem 3.15 applies, to first establish the enrichment of the opfibration  $\mathcal{V}\text{-Mod}^{\text{op}} \rightarrow \mathcal{V}\text{-Cat}^{\text{op}}$ . First of all,  $\mathcal{V}\text{-Comod} \rightarrow \mathcal{V}\text{-Cocat}$  is a monoidal opfibration by definition of the respective products and cartesianness of  $\otimes_{\mathcal{V}\text{-Comod}}$ , [Vas14, 7.7.6]. The commutative square of categories and functors

$$\begin{array}{ccc}
 \mathcal{V}\text{-Comod} \times \mathcal{V}\text{-Mod}^{\text{op}} & \xrightarrow{\overline{K}^{\text{op}}} & \mathcal{V}\text{-Mod}^{\text{op}} \\
 \downarrow & & \downarrow \\
 \mathcal{V}\text{-Cocat} \times \mathcal{V}\text{-Cat}^{\text{op}} & \xrightarrow{K^{\text{op}}} & \mathcal{V}\text{-Cat}^{\text{op}}
 \end{array}$$

constitutes an opfibred action, since both  $K$  and  $\overline{K}$  are actions,  $\overline{K}$  preserves cartesian arrows by [Vas14, 7.7.3] and the action axioms are the one above each other as per their definitions. Finally, this opfibred 1-cell has an oplax opfibred parameterized adjoint by [Vas14, 7.7.5], and the proof is complete.  $\square$

### 4.3 Comparison with existing notions

The above examples were the ones that motivated the proposed enriched fibration notion – although more should be identified in future work. In this final section, we would like to discuss why other existing approaches were not applicable, due to the nature of these cases.

Recall that assuming the axiom of choice, one can construct an equivalence between fibrations  $\mathcal{A} \rightarrow \mathbb{X}$  and indexed categories, i.e. pseudofunctors  $\mathbb{X}^{\text{op}} \rightarrow \mathbf{Cat}$  via the classic *Grothendieck construction* [Gro61]. More

recently [Shu08, MV18] this correspondence has been lifted between the respective monoidal structures; we believe that a (global) enriched version of the Grothendieck construction in the future, which in a fibrewise sense appears in [BW18], will shed more light to the tight connections between our enriched fibration notion and the ones that follow. For the moment, we only sketch some of the main relevant theory and differences.

M. Bunge in [Bun13] first introduced the notion of an  $\mathbb{S}$ -indexed  $V$ -category, for  $\mathbb{S}$  an elementary topos and  $V$  an  $\mathbb{S}$ -indexed monoidal category  $V: \mathbb{S}^{\text{op}} \rightarrow \mathbf{MonCat}$ . The goal of this work was to provide a general context in order to compare as well as clarify certain misconceptions regarding different completions on 2-categories, such as the Karoubi, Grothendieck, Cauchy and Stack completion.

Independently, M. Shulman in [Shu13] also develops a theory of *enriched indexed categories* over base categories  $\mathbb{S}$  with finite products. The motivation in that paper was to capture and study ‘mixed’ fibred, indexed and internal structures in various contexts, such as Parameterized and Equivariant Homotopy Theory, abelian sheaves and many more.

Briefly, for  $\mathbb{S}$  cartesian monoidal, take  $V$  to be an  $\mathbb{S}$ -indexed monoidal category, equivalently viewed as a monoidal fibration  $\int V: \mathcal{V} \rightarrow \mathbb{S}$ . A  $V$ -enriched indexed category  $A$  is simultaneously indexed (or fibred) over the same  $\mathbb{S}$  and also ‘fibrewise’ enriched in  $\mathcal{V}$ : every category (or fiber)  $A(s)$  for  $s \in \mathbb{S}$  is  $V(s)$ -enriched, and the reindexing functors are fully faithful enriched under the appropriate change of base. Although this formulation employs the same notion of a monoidal fibration (Definition 3.2) as the base of the enrichment, there are some crucial differences resulting in two separate definitions, [Bun13, 2.4] - [Shu13, 4.1] and Definition 3.8.

First of all, Bunge’s and Shulman’s approach only concerns enrichment in fibrations over monoidal categories whose tensor product is the cartesian product. This is fundamental for the development and definitions, and not a special case of something more general; of course this was relevant to their examples at hand. On the contrary, for our examples this is evidently not the case: in (21) and (22) the base monoidal categories of the monoidal fibrations,  $\mathbf{Comon}(\mathcal{V})$  and  $\mathcal{V}\text{-Cocat}$ , are non-cartesian.

Moreover, the notion of an enriched indexed category roughly expressed in the fibred world, essentially refers to a fibration ‘enriched’ in another fi-

bration over the same base, approximately depicted as

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{\text{'fibrewise' enriched}} & \mathcal{V} \\
 & \searrow \text{fibred} & \downarrow \text{fibred} \\
 & & \mathbb{S}.
 \end{array}$$

In our examples, this fibrewise enrichment is certainly not the case: the fibre categories of our monoidal fibrations, like  $\text{Comod}_{\mathcal{V}}(C)$ , do not even have a monoidal structure themselves in order to serve as enriching bases. Furthermore, even if in [Shu13, §7] there is a short treatment of changing the indexed monoidal enriching base, and the development in [Bun13] is a special case of this via the identity functor on  $\mathbb{S}$ , in our context the enriched fibration concept involves simultaneous enrichments between both the total and the base categories of the two fibrations as essential building blocks of the structure:

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{\text{enriched}} & \mathcal{V} \\
 \text{fibred} \downarrow & & \downarrow \text{fibred} \\
 \mathbb{X} & \xrightarrow{\text{enriched}} & \mathbb{W}
 \end{array}$$

In conclusion, even if there are strong conceptual similarities between the two definitions of an enriched fibration and indexed  $\mathcal{V}$ -category, our definition does not seem to even restrict in a straightforward way to the case of fibrations over the same base, since the monoidal category  $\mathbb{W}$  is not in principle enriched over itself, nor via some sort of an identity or projection functor. As mentioned earlier, future work would aim to clarify how these two theories compare in more detail and depth. What is admittedly striking though is that several different goals and motivations have separately led to the need for a theory that combines fibred structure over a base topos or (cartesian) monoidal category, and enriched structure.

## References

- [Bat00] Marjorie Batchelor. Measuring comodules—their applications. *J. Geom. Phys.*, 36(3-4):251–269, 2000.

- [BCSW83] Renato Betti, Aurelio Carboni, Ross Street, and Robert Walters. Variation through enrichment. *J. Pure Appl. Algebra*, 29(2):109–127, 1983.
- [Bor94] Francis Borceux. *Handbook of categorical algebra. 2*, volume 51 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Categories and structures.
- [Bun13] Marta Bunge. Tightly bounded completions. *Theory Appl. Categ.*, 28:No. 8, 213–240, 2013.
- [BW18] Jonathan Beardsley and Liang Ze Wong. The enriched Grothendieck construction. arXiv:1804.03829v2 [math.CT], 2018.
- [CGR14] Eugenia Cheng, Nick Gurski, and Emily Riehl. Cyclic multicategories, multivariable adjunctions and mates. *J. K-Theory*, 13:337–396, 2014.
- [DS97] Brian Day and Ross Street. Monoidal bicategories and Hopf algebroids. *Adv. Math.*, 129(1):99–157, 1997.
- [GP97] R. Gordon and A. J. Power. Enrichment through variation. *J. Pure Appl. Algebra*, 120(2):167–185, 1997.
- [GPS95] R. Gordon, A. J. Power, and Ross Street. Coherence for tricategories. *Mem. Amer. Math. Soc.*, 117(558):vi+81, 1995.
- [Gro61] Alexander Grothendieck. *Catégories fibrées et descente*, 1961. *Seminaire de géométrie algébrique de l’Institut des Hautes Études Scientifiques (SGA 1)*, Paris.
- [Her94] Claudio Hermida. On fibred adjunctions and completeness for fibred categories. In *Recent trends in data type specification (Caldes de Malavella, 1992)*, volume 785 of *Lecture Notes in Comput. Sci.*, pages 235–251. Springer, Berlin, 1994.
- [HLFV17a] Martin Hyland, Ignacio Lopez Franco, and Christina Vasilakopoulou. Hopf measuring comonoids and enrichment. *Proc. London Math. Soc.*, 115(3):1118–1148, 2017.



- [HLFV17b] Martin Hyland, Ignacio Lopez Franco, and Christina Vasilakopoulou. Measuring comodules and enrichment. arXiv:1703.10137 [math.CT], 2017.
- [JK02] G. Janelidze and G. M. Kelly. A note on actions of a monoidal category. *Theory Appl. Categ.*, 9:61–91, 2001/02. CT2000 Conference (Como).
- [JS93] André Joyal and Ross Street. Braided tensor categories. *Adv. Math.*, 102(1):20–78, 1993.
- [Kel05] G. M. Kelly. Basic concepts of enriched category theory. *Repr. Theory Appl. Categ.*, (10):vi+137, 2005. Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714].
- [Lac00] Stephen Lack. A coherent approach to pseudomonads. *Adv. Math.*, 152(2):179–202, 2000.
- [Mar97] F. Marmolejo. Doctrines whose structure forms a fully faithful adjoint string. *Theory Appl. Categ.*, 3:No. 2, 24–44, 1997.
- [McC00] Paddy McCrudden. Categories of representations of coalgebroids. *Adv. Math.*, 154(2):299–332, 2000.
- [ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.
- [MV18] Joe Moeller and Christina Vasilakopoulou. Monoidal Grothendieck construction. To Appear, 2018.
- [Por08] Hans-E. Porst. On categories of monoids, comonoids, and bimonoids. *Quaest. Math.*, 31(2):127–139, 2008.
- [Shu08] Michael Shulman. Framed bicategories and monoidal fibrations. *Theory Appl. Categ.*, 20:No. 18, 650–738, 2008.
- [Shu13] Michael Shulman. Enriched indexed categories. *Theory Appl. Categ.*, 28:No. 21, 616–696, 2013.

- [Swe69] Moss E. Sweedler. *Hopf algebras*. Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York, 1969.
- [Vas14] Christina Vasilakopoulou. *Generalization of Algebraic Operations via Enrichment*. PhD thesis, University of Cambridge, 2014. arxiv:1411.3038 [math.CT].
- [Vas17] Christina Vasilakopoulou. Enriched duality in double categories: V-categories and V-cocategories. arXiv:1704.00329 [math.CT], 2017.
- [Win90] Glynn Winskel. A compositional proof system on a category of labelled transition systems. *Inform. and Comput.*, 87(1-2):2–57, 1990.

Christina Vasilakopoulou  
Department of Mathematics  
University of California, Riverside  
900 University Avenue  
92521, USA  
vasilak@ucr.edu