Abstract Systems and Sheaves

Patrick Schultz[1], David Spivak[1], **Christina Vasilakopoulou**[2]

[1]Massachusetts Institute of Technology

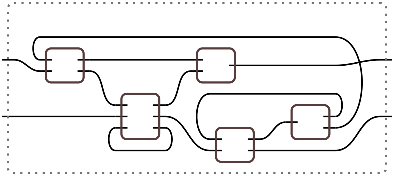
[2]Université libre de Bruxelles

DPMMS, University of Cambridge

21 February 2017

Goal: categorical framework for modeling and analysis of systems

systems as boxes
inhabitants



channels of info
flow as wires

- Coherent zoom in/out subsystems, due to *compositionality* [operad algebras]
- Appropriate notions of time for abstract systems [sheaves]

Analyse the behavior of the composite system using analysis of the particular systems components and their wired interconnection.

Outline

1. The operad of wiring diagrams
2. Interval sheaves
3. Continuous and discrete machines
4. Total and deterministic variations

From monoidal categories to operads

⋆ $\mathbf{SMonCat}_\ell$: 2-category of symmetric monoidal categories, braided lax monoidal functors, monoidal natural transformations

⋆ $\mathbf{SMonCat}_\ell(\mathcal{V}, \mathbf{Cat})$: category of $\mathcal{V}$-algebras and $\mathcal{V}$-algebra maps

A *symmetric colored operad* $\mathcal{P}$ consists of a set $\mathrm{ob}\mathcal{P}$, hom-sets of (permutation-stable) n-ary operations $\mathcal{P}(c_1, \ldots, c_n; c)$, identities $\mathrm{id}_c \in \mathcal{P}(c; c)$ and an (associative, unital) composition formula.

⋆ $\mathbf{SOpd}$: 2-category with operad functors, operad transformations

The fully faithful *underlying operad functor* $\mathbf{SMonCat}_\ell \xrightarrow{\mathcal{O}} \mathbf{SOpd}$ given by $\mathcal{O}\mathcal{V}(c_1, \ldots, c_n; c) = \mathcal{V}(c_1 \otimes \ldots \otimes c_n, c)$, induces

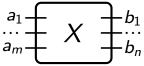
$$\mathcal{V}\text{-}\mathbf{Alg} \cong \mathbf{SOpd}(\mathcal{O}\mathcal{V}, \mathcal{O}\mathbf{Cat}) =: (\mathcal{O}\mathcal{V})\text{-}\mathbf{Alg}.$$

Monoidal category of wiring diagrams

A $\mathcal{C}$*-typed finite set* is X together with typing function $X \xrightarrow{\tau} \mathrm{ob}\mathcal{C}$; obtain an arrow category $\mathbf{TFS}_{\mathcal{C}}$, cocartesian monoidal.

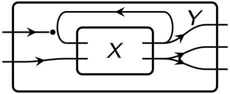
The monoidal category of wiring diagrams $\mathcal{W}_{\mathcal{C}}$:

- objects are *labeled boxes*, i.e. $X = (X^{\mathrm{in}}, X^{\mathrm{out}}) \in \mathbf{TFS}_{\mathcal{C}}^2$



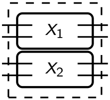
think of $X^{\mathrm{in/out}}$-elements as ports, their types as possible info values

- morphisms are $(X^{\mathrm{in}} \xrightarrow{\phi^{\mathrm{in}}} X^{\mathrm{out}} + Y^{\mathrm{in}}, Y^{\mathrm{out}} \xrightarrow{\phi^{\mathrm{out}}} X^{\mathrm{out}}) \in \mathbf{TFS}_{\mathcal{C}}^2$



think of $\phi^{\mathrm{in/out}}$ expressing which port is fed info by which

- tensor product is $X_1 \oplus X_2 = ({X_1}^{\mathrm{in}} + {X_2}^{\mathrm{in}}, {X_1}^{\mathrm{out}} + {X_2}^{\mathrm{out}})$



think of parallel placement of boxes

If $\mathcal{C}$ finitely complete, *dependent product* $\widehat{X} = \prod_x \tau(x)$ gives strong monoidal $\widehat{(-)}\colon \mathbf{TFS}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{C} \rightsquigarrow$ passage to $\mathcal{C}$-context.

Aim: model systems as algebras for $\mathcal{W}_{\mathcal{C}}$, equivalently for $\mathcal{OW}_{\mathcal{C}}$; monoidal world for formal language, operadic world for visual.

$F\colon \mathcal{W}_{\mathcal{C}} \to \mathbf{Cat}$ gives semantics to boxes, composite formula to wiring diagrams $F(X_1) \times \ldots \times F(X_n) \to F(X_1 + \ldots + X_n) \xrightarrow{F\phi} FY$.

Discrete Dynamical Systems

A DDS with input A, output B is set of states S, $A \times S \xrightarrow{f^{\mathrm{upd}}} S$, $S \xrightarrow{f^{\mathrm{rdt}}} B$; objects of category $\mathrm{DDS(A,B)}$. Model as $\mathcal{W}_{\mathbf{Set}}$-algebra:

$$\begin{array}{ccc} \mathcal{W}_{\mathbf{Set}} & \longrightarrow & \mathbf{Cat} \\ X=(X^{\mathrm{in}},X^{\mathrm{out}}) & \longmapsto & \mathrm{DDS}(\widehat{X^{\mathrm{in}}},\widehat{X^{\mathrm{out}}}) \\ \phi\downarrow & & \downarrow\mathrm{DDS}(\phi) \\ Y=(Y^{\mathrm{in}},Y^{\mathrm{out}}) & \longmapsto & \mathrm{DDS}(\widehat{Y^{\mathrm{in}}},\widehat{Y^{\mathrm{out}}}) \end{array}$$

via $(S,f^{\mathrm{upd}},f^{\mathrm{rdt}})\mapsto(S,g^{\mathrm{upd}},g^{\mathrm{rdt}})$,

$g^{\mathrm{upd}}(y,s)=f^{\mathrm{upd}}(\widehat{\phi^{\mathrm{in}}}(y,f^{\mathrm{rdt}}(s)),s)$

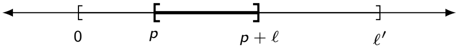
$g^{\mathrm{rdt}}(s)=\widehat{\phi^{\mathrm{out}}}(f^{\mathrm{rdt}}(s))$

Categories of intervals

$\mathbb{R}_{\geq 0}$ non-negative reals poset, $\mathrm{Tr}_p : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ translation-by-p.

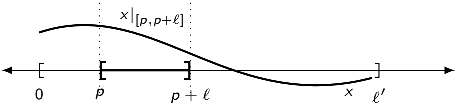
- Category **Int** of *continuous intervals* has objects $\mathbb{R}_{\geq 0}$, morphisms $\mathbf{Int}(\ell, \ell') = \{\mathrm{Tr}_p | p \in \mathbb{R}_{\geq 0} \text{ and } p \leq \ell' - \ell\}$; equivalently via image

$$[0, \ell] \overset{p}{\subseteq} [0, \ell']$$



- Category $\mathbf{Int}_N$ of *discrete intervals*, $\mathrm{ob} = \mathbb{N}$, $n \xrightarrow{\mathrm{Tr}_p} n'$ by $p \in \mathbb{N}$.

If $A \colon \mathbf{Int}^{\mathrm{op}} \to \mathbf{Set}$, view section $x \in A(\ell')$ & restriction $A(\mathrm{Tr}_p)(x)$



The *twisted arrow* category $\mathcal{C}_{tw}$ has objects $\mathcal{C}$-arrows and maps $(u, v) \colon f \to g$ commuting $\begin{array}{ccc} \cdot & \xleftarrow{u} & \cdot \\ f\downarrow & & \downarrow g \\ \cdot & \xleftarrow{v} & \cdot \end{array}$. If $\mathcal{R}$, $\mathcal{N}$ are the monoids $\mathbb{R}_{\geq 0}, \mathbb{N}$ viewed as one-object categories, $\mathbf{Int} = \mathcal{R}_{tw}$ & $\mathbf{Int}_N = \mathcal{N}_{tw}$.

Sheaves on intervals

[Johnstone site for $\mathcal{C}_{tw}$,1999] For $\ell \in \mathbf{Int}$ and $0 \leq p \leq \ell$, the pairs $[0,p] \colon p \to \ell$, $[p,\ell] \colon (\ell\text{-}p) \to \ell$ form a cover for ℓ. These generate a coverage for $\mathbf{Int}$; similarly for $\mathbf{Int}_N$.

$\star$ $\widetilde{\mathbf{Int}}$ and $\widetilde{\mathbf{Int}_N}$ are the toposes of *continuous* and *discrete interval sheaves*, i.e. $\mathbf{Int}_{(N)}$-presheaves whose compatible sections glue.

Idea: $\widetilde{\mathbf{Int}}_{(N)}$-labeled boxes have ports carrying very general time-based signals, expressed as sheaves of 'all possible behaviors'.

Examples

- $\widetilde{\mathbf{Int}_N} \simeq \mathbf{Grph}$, so every graph gives a discrete interval sheaf
- $\mathrm{L} \colon \mathbf{Set} \to \widetilde{\mathbf{Int}_\mathrm{N}}$ by $\mathrm{L(X)(n)} = \mathrm{X}^{\mathrm{n}+1}$, non-empty X-lists sheaf
- $\mathrm{F} \colon \mathbf{Set} \to \widetilde{\mathbf{Int}}$ by $\mathrm{F(X)}(\ell) = \{\mathrm{f} \colon [0,\ell] \to \mathrm{X}\}$, sheaf of functions
- $\mathrm{Ext}_\epsilon \colon \widetilde{\mathbf{Int}} \to \widetilde{\mathbf{Int}}$ by $\mathrm{Ext}_\epsilon(\mathrm{A})(\ell) = \mathrm{A}(\ell + \epsilon)$, ϵ-extension sheaf

Abstract machines

Purpose: define abstract systems in terms of **Int**-sheaves; perceive known dynamical systems as special cases; coherently interconnect arbitrary systems and study their behavior on common ground.

Characteristics of interest: for initial state and input, the machine

- uniquely evolves or 'stays idle' $\rightsquigarrow$ determinism
- always evolves $\rightsquigarrow$ totality

▶ A *continuous machine* with input & output A & $B \in \widetilde{\mathbf{Int}}$ is

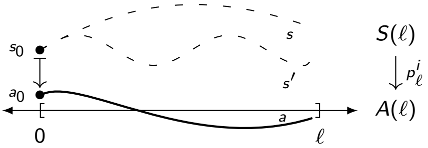
$$A \xleftarrow{p^i} S \xrightarrow{p^o} B$$

S - state sheaf
p^i - input sheaf map
p^o - output sheaf map

$\mathrm{Mch(A,B)} = \widetilde{\mathbf{Int}}/_{\mathrm{A\times B}}$ the topos of continuous (A,B)-machines.

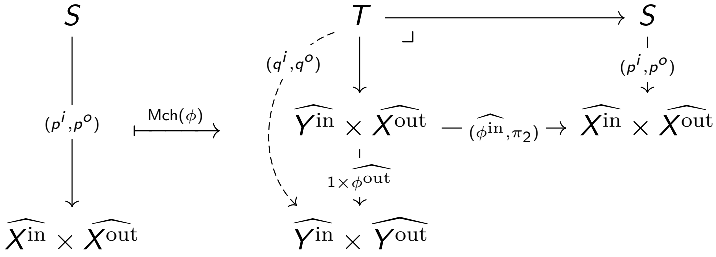
▶ For $A, B \in \widetilde{\mathbf{Int}}_N$, *discrete machines* $\mathrm{Mch_N(A,B)} = \widetilde{\mathbf{Int}}_\mathrm{N}/_{\mathrm{A\times B}}$.

Neither deterministic nor total: for input a over ℓ-interval, there may or may not be s_0-extension



Continuous machines form a $\mathcal{W}_{\widetilde{\textbf{Int}}}$-algebra

Functor $\mathrm{Mch}\colon \mathcal{W}_{\widetilde{\textbf{Int}}} \to \textbf{Cat}$ by $(X^{\mathrm{in}}, X^{\mathrm{out}}) \mapsto \mathrm{Mch}(\widehat{X^{\mathrm{in}}}, \widehat{X^{\mathrm{out}}})$ and

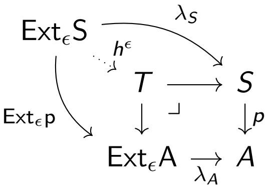


Lax monoidal by taking products of spans.

Total, deterministic and inertial maps

Left restriction $(-)|_{[0,\ell]}\colon A(\ell+\epsilon)\to A(\ell)$ gives $\lambda_A\colon \mathrm{Ext}_\epsilon A \to A$.

- An interval sheaf map $p\colon S\to A$ is *total* (resp. *deterministic*) when the induced h^ϵ has surjective (resp. injective) 0-component.

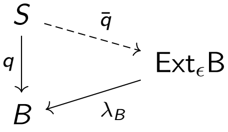


$T(0)=\{(a,s_0)\in A(\epsilon)\times S(0)\,\big|\,p(s_0)=a|_{[0,0]}\}$

If $h_0^\epsilon\colon \mathrm{Ext}_\epsilon(0)\to \mathrm{T}(0)$ is epi, for $(a,s_0)\in T(0)$ $\exists s\in S(\epsilon)$ that extends it: $s|_{[0,0]}=s_0,\ p(s)=a$

However: machines with total/det input do *not* form $\mathcal{W}$-algebra!

- An interval sheaf map $q\colon S\to B$ is ϵ-*inertial* when it factors as



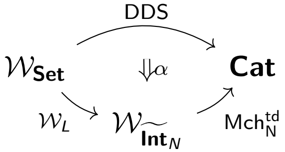
$\bar{q}_\ell\colon S(\ell)\to B(\ell+\epsilon)$

Current state determines not only current output, but ϵ-more

$\star$ $\mathsf{Mch}^{\mathsf{t}}_{(\mathsf{N})}(\mathsf{A},\mathsf{B})$ are machines with total input and inertial output, $\mathsf{Mch}^{\mathsf{d}}_{(\mathsf{N})}(\mathsf{A},\mathsf{B})$ with deterministic input and inertial output.

These form subalgebras of $\mathsf{Mch}_{(\mathsf{N})}\colon \mathcal{W}_{\widetilde{\mathbf{Int}}} \to \mathbf{Cat}$, i.e. they are closed under wiring diagrams composition.

Discrete Dynamical Systems as discrete (tot & det) machines



Realize via algebra map whose components $\mathsf{DDS}(\mathsf{X}) \xrightarrow{\alpha_{\mathsf{x}}} \mathsf{Mch}^{\mathsf{td}}_{\mathsf{N}}(\mathsf{LX})$ map $(S, f^{\mathrm{upd}}, f^{\mathrm{rdt}})$ to machine $T \to L\widehat{X^{\mathrm{in}}} \times L\widehat{X^{\mathrm{out}}}$ with $T(n) := \{(x_0, \ldots, s_n) \in (\widehat{X^{\mathrm{in}}} \times S)^n | s_{i+1} = f^{\mathrm{upd}}(x_i, s_i)\}$.

Wiring diagram algebra maps allow the translation between the various notions of systems, in a natural and flexible way.

Thank you for your attention!