Enriched Fibration

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Co-monoids and co-modules in monoidal categories

For any monoidal $(\mathcal{V}, \otimes, I)$, there are categories $\mathbf{Mon}(\mathcal{V})$ and $\mathbf{Comon}(\mathcal{V})$ with objects (A, m, η) and (C, δ, ϵ) respectively.
$*$ If $\mathcal{V}$ symmetric, they inherit (symmetric) monoidal structure:

$$A \otimes B \otimes A \otimes B \xrightarrow{\cong} A \otimes A \otimes B \otimes B \xrightarrow{m \otimes m} A \otimes B, \quad I \cong I \otimes I \xrightarrow{\eta \otimes \eta} A \otimes B$$

For fixed A and C, have categories $\mathbf{Mod}_{\mathcal{V}}(A)$ and $\mathbf{Comod}_{\mathcal{V}}(C)$ of A-modules (M, μ) and C-comodules (X, δ). There exist functors

$$\mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \longrightarrow \mathbf{Cat} \qquad \& \qquad \mathbf{Comon}(\mathcal{V}) \longrightarrow \mathbf{Cat}$$

$$\begin{array}{ccc} A & \mapsto & \mathbf{Mod}_{\mathcal{V}}(A) \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle f^*} \\ B & \mapsto & \mathbf{Mod}_{\mathcal{V}}(B) \end{array} \qquad\qquad \begin{array}{ccc} C & \mapsto & \mathbf{Comod}_{\mathcal{V}}(C) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_!} \\ D & \mapsto & \mathbf{Comod}_{\mathcal{V}}(D) \end{array}$$

• The Grothendieck construction gives *global categories* of modules and comodules: $\mathbf{Mod} \xrightarrow{\textit{fibr}} \mathbf{Mon}(\mathcal{V})$, $\mathbf{Comod} \xrightarrow{\textit{opfibr}} \mathbf{Comon}(\mathcal{V})$.

Suppose $\mathcal{V}$ is a symmetric monoidal closed , locally presentable category.

Mon$(\mathcal{V})^{\mathrm{op}}$ and **Mon**$(\mathcal{V})$ are enriched in the symmetric monoidal **Comon**$(\mathcal{V})$.

Proof: Internal hom functor $[-,-] : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$ induces

$$H : \mathbf{Comon}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Mon}(\mathcal{V}) \longrightarrow \mathbf{Mon}(\mathcal{V})$$
$$(C, B) \longmapsto [C, B]$$

(local presentability lifts to **Comon**$(\mathcal{V})$) $\rightsquigarrow$ H^{op} has a parametrized adjoint

$$P : \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Mon}(\mathcal{V}) \longrightarrow \mathbf{Comon}(\mathcal{V})$$

which gives the hom-objects of the enrichment. □

$*$ $P(A, B)$ generalizes Sweedler's *universal measuring coalgebra*

$$\mathbf{Alg}_k(A, \mathrm{Hom}_k(C, B)) \cong \mathbf{Coalg}_k(C, P(A, B)).$$

Global categories of modules and comodules also inherit symmetric monoidal structure from $\mathcal{V}$:

$$X_C \otimes Y_D \xrightarrow{\delta\otimes\delta} X \otimes C \otimes Y \otimes D \xrightarrow{\cong} X \otimes Y \otimes C \otimes D.$$

Mod$^{\mathrm{op}}$ and **Mod** are enriched in the symmetric monoidal **Comod**.

Proof: Internal hom functor lifts to global categories

$$\begin{aligned} \bar{H} : \ \mathbf{Comod}^{\mathrm{op}} \times \mathbf{Mod} &\longrightarrow \mathbf{Mod} \\ (\ X_C\ ,\ N_B\ ) &\longmapsto [X, N]_{[C,B]} \end{aligned}$$

(general fibred adjunctions theory) $\rightsquigarrow$ $\bar{H}^{\mathrm{op}}$ has a parametrized adjoint

$$Q : \mathbf{Mod}^{\mathrm{op}} \times \mathbf{Mod} \longrightarrow \mathbf{Comod}$$

which is the enriched hom-functor. □

$*$ $Q(M, N)$ generalizes Batchelor's *universal measuring comodule*

$$\mathbf{Mod}_A(M, \mathrm{Hom}_k(X, N)) \cong \mathbf{Comod}_C(X, Q(M, N)).$$

An *action* of a monoidal category $\mathcal{V}$ on an ordinary category $\mathcal{D}$ is given by a functor $*: \mathcal{V} \times \mathcal{D} \to \mathcal{D}$ with coherent isomorphisms

$$\tau_{XYA} : (X \otimes Y) * A \xrightarrow{\sim} X * (Y * A), \quad \nu_A : I * A \xrightarrow{\sim} A.$$

Theorem

To give a category $\mathcal{D}$ with an action $*$ of a monoidal closed $\mathcal{V}$ with a parametrized adjoint is to give a tensored $\mathcal{V}$-enriched category:

$$\mathcal{V}\text{-}\mathbf{Rep}_{\mathrm{cl}} \simeq \mathcal{V}\text{-}\mathbf{Cat}_{\otimes}.$$

If $- * D \dashv F(D, -)$ for all $D \in \mathcal{D}$, there is a $\mathcal{V}$-category $\mathcal{A}$ with $\mathcal{A}_0 = \mathcal{D}$ and $F(D, D') \in \mathcal{V}$ the hom-objects.

$*$ E.g. H and $\bar{H}$ are actions with adjoints P and Q.

· A monoidal $\mathcal{V}$ is a pseudomonoid in the cartesian monoidal 2-category $(\mathbf{Cat}, \times, \mathbf{1})$.
· A $\mathcal{V}$-representation $(\mathcal{D}, *)$ is a $\mathcal{V}$-pseudomodule in $(\mathbf{Cat}, \times, \mathbf{1})$.

Idea: move from $(\mathbf{Cat}, \times, \mathbf{1})$ to $(\mathbf{Fib}, \times, 1_{\mathbf{1}})$!

- 0-cells are fibrations;
- 1-cells are commutative squares

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{S} & \mathcal{B} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathbb{X} & \xrightarrow{F} & \mathbb{Y} \end{array}$$

  where S preserves cartesian arrows;
- 2-cells are pairs of natural transformations

$$\begin{array}{ccc} \mathcal{A} & \overset{S}{\underset{T}{\Downarrow \alpha}} & \mathcal{B} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathbb{X} & \overset{F}{\underset{G}{\Downarrow \beta}} & \mathbb{Y} \end{array}$$

  with $Q\alpha = \beta_P$.

• (pseudomonoid) A *monoidal fibration* $T : \mathcal{V} \to \mathbb{W}$ is a strict monoidal functor between monoidal categories , such that $\otimes_{\mathcal{V}}$ preserves cartesian arrows.

• (pseudomodule) A *T-representation* is a fibration $P : \mathcal{A} \to \mathbb{X}$ where $\mathcal{V}$ acts on $\mathcal{A}$, $\mathbb{W}$ acts on $\mathbb{X}$, $*_{\mathcal{A}}$ is cartesian and P strictly preserves the action constraints.

• (parametrized adjunction) For a fibred 1-cell 'of two variables'

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{B} & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle H \times J}\downarrow & & \downarrow{\scriptstyle K} \\ \mathbb{X} \times \mathbb{Y} & \xrightarrow[G]{} & \mathbb{Z}, \end{array}$$

a *fibred parametrized adjoint* is a functor in $\mathbf{Cat}^2$

$$\begin{array}{ccc} \mathcal{B}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow{R} & \mathcal{A} \\ {\scriptstyle J^{\mathrm{op}} \times K}\downarrow & & \downarrow{\scriptstyle H} \\ \mathbb{Y}^{\mathrm{op}} \times \mathbb{Z} & \xrightarrow[S]{} & \mathbb{X} \end{array}$$

s.t. $F(\text{-}, B) \dashv R(B, \text{-})$, $G(\text{-}, Y) \dashv S(Y, \text{-})$ and $R(B, \text{-})$ cartesian.

- The fibration P is *enriched* in the monoidal fibration T when
  - $\mathcal{A}$ is $\mathcal{V}$-enriched, $\mathbb{X}$ is $\mathbb{W}$-enriched and

$$\begin{array}{ccc} \mathcal{A}^{\mathrm{op}} \times \mathcal{A} & \xrightarrow{\mathcal{A}(-,-)} & \mathcal{V} \\ {\scriptstyle P^{\mathrm{op}} \times P}\downarrow & \circlearrowleft & \downarrow{\scriptstyle T} \\ \mathbb{X}^{\mathrm{op}} \times \mathbb{X} & \xrightarrow[\mathbb{X}(-,-)]{} & \mathbb{W}; \end{array}$$

  - composition and identities of enrichments are compatible, i.e. $TM^{\mathcal{A}}_{A,B,C} = M^{\mathbb{X}}_{PA,PB,PC}$ and $Tj^{\mathcal{A}}_{A} = j^{\mathbb{X}}_{PA}$;
  - $\mathcal{A}(A,-)$ is cartesian.

$*$ When P is enriched in T, it is also a $\mathbb{W}$-enriched functor between $T_{\bullet}\mathcal{A}$ (via change of base) and $\mathbb{X}$.

Theorem

Suppose a monoidal fibration T acts on P. If the action has a fibred parametrized adjoint, then we can enrich P in T.

Back to (co)monoids and (co)modules. Consider the opfibrations

$$\begin{array}{c}\mathbf{Mod}^{\mathrm{op}}\\ \downarrow\\ \mathbf{Mon}(\mathcal{V})^{\mathrm{op}}\end{array} \quad \& \quad \begin{array}{c}\mathbf{Comod}\\ \downarrow\\ \mathbf{Comon}(\mathcal{V})\end{array} \quad :$$

- V : **Comod** $\to$ **Comon**$(\mathcal{V})$ is a monoidal opfibration;
- **Mod**$^{\mathrm{op}}$ is enriched in **Comod**, **Mon**$(\mathcal{V})^{\mathrm{op}}$ in **Comon**$(\mathcal{V})$ and

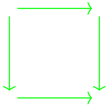
$$\begin{array}{ccc}\mathbf{Mod}^{\mathrm{op}} \times \mathbf{Mod} & \xrightarrow{\quad Q \quad} & \mathbf{Comod}\\ \downarrow & \circlearrowleft & \downarrow\\ \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Mon}(\mathcal{V}) & \xrightarrow[\quad P \quad]{} & \mathbf{Comon}(\mathcal{V});\end{array}$$

- composition laws and identities are compatible;
- $Q(-, N_B)$ preserves cartesian liftings (...)

The opfibration **Mod**$^{\mathrm{op}} \to$ **Mon**$(\mathcal{V})^{\mathrm{op}}$ is enriched in the monoidal opfibration **Comod** $\to$ **Comon**$(\mathcal{V})$.

Further steps: context of $\mathcal{V}$-categories and $\mathcal{V}$-modules, $\mathcal{V}$-operads and modules, enriched fibred theory

Thank you for your attention!



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