Welcome to Math 009B!

Antiderivatives and Indefinite Integrals

Idea: determine a function, if you know its derivative.

An *antiderivative* of $f(x)$ is a function $F(x)$ such that $\boxed{F'(x) = f(x)}$.
The *indefinite integral* of f is the set of all antiderivatives of f, written

$$\int f(x)\, dx$$

⋆ There are infinitely many antiderivatives, and they all differ by some constant C.

- $\int$ is the integration symbol, $f(x)$ is the *integrand*, dx is the *differential* of x, write $\int f(x)\, dx = F(x) + C$ for some antiderivative $F(x)$ of f.

Standard antiderivatives and properties

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
2. $\int e^x dx = e^x + C$
3. $\int \sin(x) dx = -\cos(x) + C$
4. $\int \frac{1}{x^2+1} dx = \arctan(x) + C$
5. $\int \frac{1}{x} dx = \ln|x| + C$
6. $\int a^x dx = \frac{a^x}{\ln a} + C$
7. $\int \cos(x) dx = \sin(x) + C$
8. $\int \sec^2(x) dx = \tan(x) + C$

- $\displaystyle\int (f(x) \pm g(x))\ dx = \int f(x)\ dx + \int g(x)\ dx$
- $\displaystyle\int (k \cdot f(x))\ dx = k \cdot \int f(x)\ dx$ for a constant number k
- $\displaystyle\int (f(x) \cdot g(x))\ dx \neq \left(\int f(x)\ dx\right) \cdot \left(\int g(x)\ dx\right)$ NOT TRUE

⋆ Differentiation and antidifferentiation are "inverse operations":

$$\frac{d}{dx}\left(\int f(x) dx\right) = f(x)!$$

Initial Value Problems: use an initial condition $f(x_0) = y_0$ to determine the constant C, i.e. a specific antiderivative out of them all!

- $f(x) = \int f'(x)\,dx$, but also $f'(x) = \int f''(x)\,dx$ and so on...

Work out the following:

1. $\int 3x^5 dx = \frac{x^6}{2} + C$
2. $\int \left(\frac{5}{x} - 2\right) dx = 5\ln|x| = 2x + C$
3. $\int (6\sin(t) + 4\sqrt[3]{t})dt = -6\cos(t) + 3\sqrt[3]{t^4} + C$
4. Find $f(x)$, if $f''(x) = 5$, $f'(0) = 7$, $f(0) = 3$. $f(x) = \frac{5x^2}{2} + 7x + 3$

The Definite Integral

Idea: indefinite integrals=antiderivatives, definite integrals=*signed* areas!

If $y = f(x)$ is defined on a closed interval $[a, b]$, the *definite integral* of f on $[a, b]$ is the total signed area from $x = a$ to $x = b$ under $f(x)$:

$$\int_a^b f(x)dx = [\text{area above } x\text{-axis}] - [\text{area below } x\text{-axis}] = A_+ - A_-$$

The endpoints a and b are called the *bounds of integration*.

Properties of the Definite Integral

1. $\int_a^a f(x)dx = 0$
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
4. $\int_a^b (f(x) \pm g(x))\, dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
5. $\int_a^b f(x)dx = \int_a^{\mathbf{c}} f(x)dx + \int_{\mathbf{c}}^b f(x)dx$ for any $\mathbf{c}$

Approximating areas

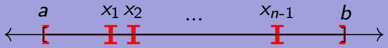
What if we cannot use geometry to evaluate areas?

$\star$ Compute an area by dividing in rectangular regions and summing.

A *partition* P of an interval $[a, b]$ is a collection of subintervals

$$[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b]$$

with $a < x_1 < x_2 < \ldots < x_{n-1} < b$.



he length of k-th subinterval $[x_{k-1}, x_k]$ is written Δx_k; if all equal, Δx.

- We can use any point of the subinterval (e.g. left endpoint, midpoint, right endpoint etc.) to form the rectangles.

Work out the following:

1. Compute $\int_{-1}^{2} (x-1)dx$ by finding the respective area. $-\frac{3}{2}$
2. Approximate $\int_{1}^{2} (x-1)dx$ using 2 subintervals and right endpoints. $\frac{3}{4}$
3. Calculate the following antiderivative $\int x(1-x^2)dx$. $\frac{x^2}{2} - \frac{x^4}{4} + C$

Riemann Sums

- Tool for summation? *Sigma notation*!

For any real numbers $a_1, a_2, \ldots, a_n$, we write

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

Properties of Sums

1. $\sum_{k=1}^{n} 1 = n$
2. $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
3. $\sum_{k=1}^{n} c \cdot a_k = c \cdot \sum_{k=1}^{n} a_k$
4. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
5. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
6. $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

$\star$ The finer the partition, i.e. the larger the number n of subintervals - the shorter their length, the better the area approximation.

What if the number of subintervals approaches infinity?

Definite Integral via Riemann sums

Let $P = [a, x_1, ..., x_{n-1}, b]$ partition of $[a, b]$, and $c_k \in [x_{k-1}, x_k]$. The *definite integral* of a function f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} \overbrace{\Delta x}^{\text{base}} \overbrace{f(c_k)}^{\text{height}}$$

The Fundamental Theorem of Calculus

Idea: definite integrals are antiderivatives!

Fundamental Theorem of Calculus (FTC), pt.1

Suppose f is continuous on $[a, b]$. Then the function defined as

$$F(x) = \int_a^x f(t)dt, \quad a \leq t \leq b$$

is differentiable on (a, b), with

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = \boxed{\text{F'(x)=f(x)}}$$

Work out the following:

1. Approximate $\int_0^3 (3x - 5)dx$ using 3 subintervals and midpoints. $-\frac{3}{2}$
2. Compute $\frac{d}{dx}\int_{-5}^x \left(\sin(t) + e^{8t}\right)\ dt$ $\sin(x) + e^{8x}$

Definite integrals and average values

FTC1 says $F(x) = \int_c^x f(t)dt$ is an antiderivative of $f(x)$: $F'(x) = f(x)$.

Fundamental Theorem of Calculus (FTC), pt.2

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

for $F(x)$ any antiderivative of $f(x)$. Equivalently, $F(b) - F(a) = \int_a^b F'(x)dx$.

$\star$ Can use this in many contexts: areas under curves, net change in values of a quantity, ...

Average value of a function

If $f(x)$ is continuous on $[a, b]$, its average value on that interval is

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)dx$$

$\star$ In fact, the value f_{avg} is the output for some input of f!

Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, then there exists some $c \in [a, b]$ such that

$$f(c)(b-a) = \int_a^b f(x)dx$$

Work out the following:

1. Evaluate $\int_0^{\pi}(\sin(x) - e^x + \cos(x))dx$. $3 - e^{\pi}$
2. What is the average velocity of an object on $[0, 3]$, if $v(t) = (t-1)^2$ ft/s? 1ft/s
3. For which time(s) is it accomplished? 0s, 2s

Substitution Rule, indefinite integrals

Idea: substitution rule is for integration what chain rule is for derivation!

Chain Rule: if $f(x)$ and $u(x)$ are differentiable, then

$$(f \circ u)'(x) = f'(u(x)) \cdot u'(x)$$

Substitution Rule for Indefinite Integrals

If $u = u(x)$, then $\displaystyle\int f'(u(x))u'(x)dx = \int f'(u)du = f(u) + C$

Work out the following: evaluate, using the substitution rule,

1. $\displaystyle\int \sin(5x+1)dx$
   $-\cos(5x+1)/5 + C$
2. $\displaystyle\int x(1-3x^2)^7 dx$
   $\dfrac{-(1-3x^2)^8}{48} + C$
3. $\displaystyle\int \frac{3x}{x+4}dx$
   $3(x+4) - 12\ln|x+4| + C$

Substitution Rule, definite integrals

Idea: for definite integrals, also substitute limits of integration!

Substitution Rule for Definite Integrals

$$\int_a^b f(u(x)) \cdot u'(x) dx = \int_{u(a)}^{u(b)} f(u) du$$

Work out the following: evaluate, using the substitution rule,

1. $\displaystyle\int_1^2 \frac{3x^2+1}{(x^3+x)^2} dx$ $\frac{2}{5}$
2. $\displaystyle\int_0^{\pi/2} \cos(x) e^{\sin(x)} dx$ $e-1$
3. $\displaystyle\int_4^{3+e} \frac{x}{3-x} dx$ $-e-2$

Integration by Parts

Idea: integration by parts is what product rule is for derivation!

<u>Recall</u>: if $u = u(x)$ and $v = v(x)$ are differentiable, then

$$(uv)' = u'v + uv'$$

Integration by parts rule (u and v are interchangable)

For $u(x)$ and $v(x)$ differentiable functions of x,

$$\int u dv = uv - \int v du$$

$\star$ Product: choose u to be the factor whose derivative is 'easier'!

Work out the following: evaluate, using integration by parts,

1. $\int x \cos(x) dx$ $x \sin x + \cos x + C$
2. $\int_1^e 9\sqrt{x} \ln(x) dx$ $2\sqrt[3]{e^2} + 4$

Integration by parts, techniques

$$\int u\,dv = uv - \int v\,du$$

- Same way for definite integrals: $\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$
- Sometimes, apply IBP repeatedly: at every step, integral becomes easier
- Multiply by 1 ($dv = 1$ so $v = x$), if derivative of integrand is easier
- New integral has 'same level of difficulty' with initial?
  Do one more time and solve for it!
- Very often, first substitute and then integrate by parts

Work out the following: evaluate, using integration by parts,

1. $\int e^x \sin(x)dx$
   $\frac{e^x(\sin(x)-\cos(x))}{2} + C$
2. $\int_0^1 e^{\sqrt{x}}dx$ 2

Trigonometric Integrals

Idea: use $\sin^2(x) + \cos^2(x) = 1$ power-reducing identities to antiderive trigonometric combinations.

Integrals of the form $\int \sin^m(x)\cos^n(x)dx$

1. if both m and n are even, use the 'power-reducing' identities

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

2. if m is odd, then $m = 2k+1$ for some integer k; rewrite

$$\sin^m(x) = \sin^{2k+1}(x) = \overbrace{\sin^{2k}(x)}^{(\sin^2(x))^k} \cdot \sin(x) = (1-\cos^2(x))^k \sin(x)$$

and substitute $u = \cos(x)$, $du = -\sin(x)dx$.

3. if n is odd, similarly rewrite $\cos^n(x) = (1-\sin^2(x))^k \cos(x)$ and substitute $u = \sin(x)$, $du = \cos(x)dx$.

- Also use $1 + \tan^2(x) = \sec^2(x)$ for $u = \tan(x)$, $du = \sec^2(x)dx$.

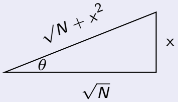
Trigonometric Substitution

Idea: need to simplify an expression $\sqrt{\pm N \pm \boxed{x}^2}$ inside an integral?
Substitute x by some trig function of an angle θ, to create perfect squares!

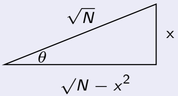
- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $1 + \tan^2(\theta) = \sec^2(\theta)$

Trigonometric Substitution

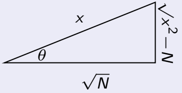
- whenever $\sqrt{N + x^2}$, substitute $x = \sqrt{N}\tan(\theta)$;
  then $\sqrt{N + N\tan^2(\theta)} = \sqrt{N}\sec(\theta)$.



- whenever $\sqrt{N - x^2}$, substitute $x = \sqrt{N}\sin(\theta)$;
  then $\sqrt{N - N\sin^2(\theta)} = \sqrt{N}\cos(\theta)$.



- whenever $\sqrt{x^2 - N}$, substitute $x = \sqrt{N}\sec(\theta)$;
  then $\sqrt{N\sec^2(\theta) - N} = \sqrt{N}\tan(\theta)$.



- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\int \sec(\theta)d\theta = \ln|\tan(\theta) + \sec(\theta)| + C$

Partial Fraction Decomposition

Idea: break up rational functions $f(x) = \frac{p(x)}{d(x)}$ into simpler fractions!

Method: to decompose $p(x)/d(x)$ where $\deg(d(x)) > \deg(p(x))$

1) factor $d(x)$ it into linear factors $(ax+b)^n$ and/or irreducible quadratic factors (ax^2+bx+c) [no real zeros]

2a) to each linear factor $(ax+b)^n$, assign the sum of partial fractions

$$\frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \ldots + \frac{C}{(ax+b)^n}$$

2b) to each quadratic factor $(ax^2+bx+c)^m$, assign the sum

$$\frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2} + \ldots + \frac{Zx+W}{(ax^2+bx+c)^n}$$

$\star$ After we clear denominators, how to determine $A, B, \ldots$?

- plug in appropriate values for x, solve for each $A, B, \ldots$ separately, or
- *polynomial equality*: if $a_n x^b + \ldots + a_1 x + a_0 = b_n x^n + \ldots b_1 x + b_0$, solve system of respective coefficient equations! $\boxed{a_k = b_k}$ $\forall k$

After PFD, new integral parts can be of various forms, indicatively:

- $\displaystyle\int \frac{1}{x+a} dx = \ln|x+a|$ or other combos with u-sub
- $\displaystyle\int \frac{1}{(x+a)^2} dx = \frac{-1}{x+a}$ or other combos with u-sub
- $\displaystyle\int \frac{2x}{x^2+b} dx = \ln|x^2+b|$ or other combos with u-sub
- $\displaystyle\int \frac{1}{x^2+1} dx = \arctan(x)$ or other combos with trig sub

Work out the following: evaluate

1. $\displaystyle\int \frac{10x}{x^2-3x-4}\, dx$ $2\ln(|x+1|) + 8\ln(|x-4|) + C$
2. $\displaystyle\int \frac{1}{x^3+x^2}\, dx$ $-\frac{1}{x} + \ln\left|\frac{x+1}{x}\right| + C$

Hyperbolic Functions

Idea: functions parameterizing hyperbolas, numerous physical applications

Hyperbolic Functions

1. $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$
2. $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

All others like $\tanh(x)$ etc. formed same way as trigonometric functions.

⋆ Very similar properties to trig counterparts, but with a twist.

Useful Properties

- $\cosh^2(x) - \sinh^2(x) = 1$
- $\tanh^2(x) + \mathrm{sech}^2(x) = 1$
- $(\sinh(x))' = \cosh(x)$
- $(\cosh(x))' = \sinh(x)$
- $(\tanh(x))' = \mathrm{sech}^2(x)$
- $(\mathrm{sech}(x))' = -\mathrm{sech}(x)\tanh(x)$

▶ Their inverse functions are in terms of logarithms:
$\mathrm{arccosh}(x) = \ln(x + \sqrt{x^2 - 1})$, $\mathrm{arcsinh}(x) = \ln(x + \sqrt{x^2 + 1})$

Improper integrals: Unbounded intervals

Idea: use limits to compute integrals whose endpoints are $\pm\infty$.

If $f(x)$ is continuous, we define the following *improper integrals*

- $\displaystyle\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$
- $\displaystyle\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx$
- $\displaystyle\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$

$\star$ The area of an unbounded region may, or may <u>not</u>, approach a number!

An improper integral *converges* when the corresponding limit exists (has a finite value). Otherwise the improper integral *diverges*.

Work out the following:

1. $\displaystyle\int \sinh^5(x)\cosh(x)dx$ $\frac{\sinh^6(x)}{6}$
2. $\displaystyle\int_0^\infty \frac{1}{(x+1)^3}\,dx$ Converges, $\frac{1}{2}$

Improper integrals: Unbounded Integrand

What about integrands being undefined on some part of interval?

Let $f(x)$ be continuous on $[a, b]$ except c where $\lim_{x \to c^{\pm}} f(x) = \pm\infty$. (VA)

- [Left Endpoint] $\int_c^b f(x)dx = \lim_{t \to c^+} \int_t^b f(x)dx$
- [Right Endpoint] $\int_a^c f(x)dx = \lim_{t \to c^-} \int_a^t f(x)dx$
- [Inbetween] $\int_a^b f(x)dx = \int_a^{c} f(x)dx + \int_{c}^b f(x)dx$

If the limit exists, the imporoper integral *converges*; otherwise it *diverges*.

Improper integrals: Comparison Theorems

Convergence and Divergence of improper integrals via Comparison

- [Converge] If $0 \leq f(x) \leq g(x)$ for any $x \in [a, \infty)$,

$$0 \leq \int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx$$

implies that if $\int_a^\infty g(x)dx$ converges, so does $\int_a^\infty f(x)dx$.

- [Diverge] If $0 \leq h(x) \leq f(x)$ for any $x \in [a, \infty)$,

$$0 \leq \int_a^\infty h(x)dx \leq \int_a^\infty f(x)dx$$

implies that if $\int_a^\infty h(x)dx$ diverges, so does $\int_a^\infty f(x)dx$.

For $f(x)$, find 'easier' function bounding it either from above or below and compute!

Work out the following: does this converge or diverge? If C, evaluate:

1. $\int_0^9 \frac{1}{\sqrt{9-x}}dx$ Converges, 6
2. $\int_0^\infty e^{5-3x}dx$ Converges, $\frac{e^5}{3}$

Areas between curves on the plane

If f and g continuous on $[a, b]$, with $f(x) \geq g(x)$ for all $x \in [a, b]$, the area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b is

$$A = \int_a^b (f(x) - g(x))dx$$

⋆ Always a positive number: the 'real' area between the graphs.

1. Find endpoints (if bounds are not given, find interestecting points)
2. Draw graphs on the plane, label endpoints, shade area
3. Compute area; may need to split in parts bounded only by 2 curves

⋆ Sometimes better compute areas wrt y-axis! Change perspective.

For a region bounded by $x = f(y)$ and $x = g(y)$, $g(y) \leq f(y)$ in $[c, d]$,

$$A = \int_c^d (f(y) - g(y))dy \qquad [f(y) \text{ is to the 'right' of } g(y)]$$

Work out the following:

1. Find the area of the region bounded by $y = x^2 - 4$ and $y = x + 2$. $\frac{125}{6}$
2. Find the area of the region bounded by $x = y^2$ and $x = 4$. $\frac{32}{3}$

The Volume of a Solid

Idea: split a solid into partitioned 'slices' and sum volumes up!

- Cross-section: intersection of solid and plane perpendicular to x-axis.

Volume of solid of cross-sectional area $A(x)$ from a to b is $\int_a^b A(x)dx$.

$\star$ When slices are cylinders, cross-sectional areas are disks!

[Disk Method] For any solid of *revolution*, obtained by rotating some curve $y = f(x)$ about some axis, if $R(x)$ is the radius of a cross-section at x,

$$V = \pi \int_a^b R^2(x)dx.$$

If cross-sectional area is between two curves, called *washer method*:

$$V_{\text{total}} = V_{\text{out}} - V_{\text{in}} = \pi \int_a^b \left(R^2(x) - r^2(x)\right) \, dx$$

Work out the following:

Volume of revolutionary solid between $y=\sqrt{x}$, $y=\frac{x}{2}$ about x-axis? $\frac{8\pi}{3}$

The Shell Method

Idea: different 'slicing' of solid gives different method for finding volume!

[Shell method] For any solid of revolution around a vertical axis, bounded by $x = a$ and $x = b$, if $r(x)$ is the radius of some shell at x and $h(x)$ its height, the volume of the solid is

$$V = 2\pi \int_a^b r(x)h(x)dx.$$

⋆ Particularly useful when we cannot solve for x or y depending on axis.

- When the axis of rotation is the y-axis, then $r(x) = x$.
- If the region is bounded above by $y = f(x)$ and below by $y = g(x)$, then $h(x) = f(x) - g(x)$.

Washer VS Shell method

Ler R be a region with x-bounds $a \leq x \leq b$ and y-bounds $c \leq y \leq d$. The volume of the revolutionary solid is given by

	Washer	Shell
Horizontal axis	$\pi \int_a^b \left(R^2(x) - r^2(x)\right) dx$	$2\pi \int_c^d r(y)h(y)dy$
Vertical axis	$\pi \int_c^d \left(R^2(y) - r^2(y)\right) dy$	$2\pi \int_a^b r(x)h(x)dx$

Work out the following: find the volume of the solid formed by revolving the region bounded by $y = \sin(x)$ and the x-axis for $0 \leq x \leq \pi$. $2\pi^2$

Arc lengths and surface areas

Idea: determine a curve's length by sums of short line segments.

If f is continuous on $[a, b]$ and differentiable on (a, b), the arc length of the curve $y = f(x)$ from a to b is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$$

The surface area of revolutionary solid about the x-axis, for $f(x) \geq 0$, is

$$SA = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx$$

whereas about the y-axis, for $a, b \geq 0$, is

$$SA = 2\pi \int_a^b x\sqrt{1 + (f'(x))^2}\,dx$$

Work out the following:

1. Find the arc length of $y=x^{3/2}$ from $x = 0$ to $x = 4$ (approx). ≈ 9.07
2. Set up integral for arc length of $y=\sqrt{1 - x^2}$ on $0 \leq x \leq 1$. $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$