Welcome to Math 009A!

Department of Mathematics
University of California
Riverside, CA 92521

Mathematics 9A
Syllabus
Fall Quarter 2017

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Text: APEX Calculus Version 3.0: http://www.apexcalculus.com/

Midterm: Friday, 11/3/17, during the lecture
Final Exam: Monday, 12/11/17, 19.00-22.00

Homework will be assigned after each Wednesday lecture and will count for 20 points of the final grade. Keep good track of the deadlines!

There will be a number of quizzes in the discussion sections. The scores for the discussion quizzes will also count for 20 points of the final grade.

There will be 12 Microtutorial video quizzes posted on iLearn which will account for 10 points – 10 of the videos are required and 2 of them are for extra credit (bonus). You have only one chance to answer each question so you should watch each corresponding video in its entirety before attempting the question.

You will receive a grade for each of the midterms and the final exam. The midterm will give 60 points of your final grade, and the final exam will account for the remaining 90 points. The grades will be given according to the following scale:

A+	190 points
A	180 points
A-	170 points
B+	160 points
B	150 points
B-	140 points
C+	130 points
C	110 points
D	80 points

What is the limit of a function?

Intuitively, expresses the behavior of $y = f(x)$ near a particular value of x.

Notation "The limit of y as x approaches c is L"

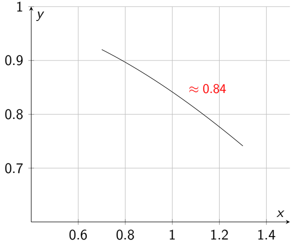
$$\lim_{x \to c} y = \lim_{x \to c} f(x) = L$$

In general, that is NOT $f(c)$; this will be true only for a class of specific cases (later). In fact, $f(c)$ may not even be defined!

Before providing the formal definition, we shall approximate limits.

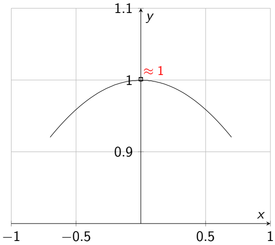
- *Graphically*: observe where $f(x)$ tends to when x tends to c.
- *Numerically*: create a table of x and $f(x)$ values for x near c.

Example: For $f(x) = \dfrac{\sin x}{x}$, approximate $\lim_{x \to 1} f(x)$ and $\lim_{x \to 0} f(x)$.



x	$\frac{\sin x}{x}$
0.99	0.844471
0.999	0.841772
1	0.841471
1.001	0.84117
1.0001	0.838447

so $\lim_{x \to 1} f(x) \approx 0.84$.



x	$\frac{\sin x}{x}$
-0.01	0.999983
-0.001	0.9999998
0	NOT defined
0.001	0.9999998
0.01	0.999983

so $\lim_{x \to 0} f(x) \approx 1$.

Non-existence of limits

- When x approaches some c, $\lim_{x \to c} f(x)$ may <u>not</u> exist.

For example, when the function $f(x)$

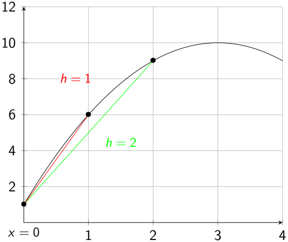
- approaches different values on either side of c;
- grows without upper/lower bound;
- oscillates.

Difference Quotients

For any function $f(x)$, the "difference quotient" formula

$$\frac{f(x+h)-f(x)}{h}$$

computes the slope of the secant line through two points on the graph.



By making h smaller, shrinking towards 0, the interval $(x, x+h)$ 'approaches the point' x! The secant line approaches the tangent line ...

Formal definition of a limit

Typically, ε and δ stand for very small positive values; they help formalise the *nearness* of a variable x to a chosen number c, expressing that their distance tends to 0, e.g. $|x - c| < \delta$.

- Think that x is near c when for small $\delta > 0$, "x is within δ units of c".

Definition: The limit of a function

Let I be an open interval containing c, and let f be a function defined on I except possibly at c. The **limit** of $f(x)$ **as** x **approaches** c is L , iff

given any $\varepsilon > 0$, there exists some $\delta > 0$ such that for all $x \neq c$, if $|x - c| < \delta$ then $|f(x) - L| < \varepsilon$.

Succinctly, $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

In practice, certain rules allow us to find limits more easily and efficiently.

Basic Limit Properties

Suppose $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = K$ for functions f, g and $c, L, K \in \mathbb{R}$.

- $\lim_{x\to c} a = a$ for any $a \in \mathbb{R}$ *(constants)*
- $\lim_{x\to c} x = c$ *(identity)*
- $\lim_{x\to c} (f(x) \pm g(x)) = L \pm K$ *(sum/difference)*
- $\lim_{x\to c} a \cdot f(x) = a \cdot L$ *(scalar multiples)*
- $\lim_{x\to c} (f(x) \cdot g(x)) = L \cdot K$ *(products)*
- $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{K}$ for $K \neq 0$ *(quotients)*
- $\lim_{x\to c} f(x)^n = L^n$ for $n \in \mathbb{Z}^+$ *(powers)*
- $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$ *(roots)*
- $\lim_{x\to c} h(f(x)) = M$ for $\lim_{x\to L} h(x) = M$ *(composition)*

Limits of Polynomials and Rational Functions

Suppose $p(x), q(x)$ are polynomials, and $c \in \mathbb{R}$.

- $\lim\limits_{x \to c} p(x) = p(c)$
- $\lim\limits_{x \to c} \dfrac{p(x)}{q(x)} = \dfrac{p(c)}{q(c)}$ when $q(c) \neq 0$

Trigonometric, Logarithmic, Power and Root functions

For $c \in \mathbb{R}$ in the domain of each function, and $n \in \mathbb{Z}^+$,

- $\lim\limits_{x \to c} \sin x = \sin c$
- $\lim\limits_{x \to c} \cos x = \cos c$
- $\lim\limits_{x \to c} \tan x = \tan c$
- $\lim\limits_{x \to c} \cot x = \cot c$
- $\lim\limits_{x \to c} \sec x = \sec c$
- $\lim\limits_{x \to c} \csc x = \csc c$
- $\lim\limits_{x \to c} a^x = a^c$
- $\lim\limits_{x \to c} \log x = \log c$
- $\lim\limits_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$

All the above theorems help "find unknown limits using known".

However, how do we handle the *indeterminate forms*, $\frac{0}{0}$?

Theorem: functions equal at all but one point

Suppose $g(x) = f(x)$ for all x in an open interval, except possibly at one value c. If $\lim_{x \to c} g(x) = L$, then $\lim_{x \to c} f(x) = L$.

As a result, if we have a rational function $\frac{p(x)}{q(x)}$ for which both $p(c)$ and $q(c)$ are 0, then $(x - c)$ is necessarily a factor of both polynomials. Then

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \lim_{x \to c} \frac{(x-c) \cdot \tilde{p}(x)}{(x-c) \cdot \tilde{q}(x)} \overset{(thm)}{=} \lim_{x \to c} \frac{\tilde{p}(x)}{\tilde{q}(x)} \overset{(rat\ fnc)}{=} \frac{\tilde{p}(c)}{\tilde{q}(c)}, \quad \text{if } \tilde{q}(c) \neq 0.$$

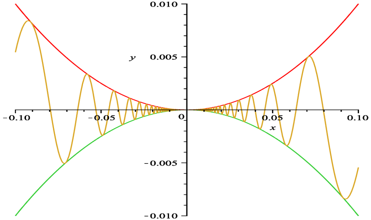
Squeeze Theorem

Suppose f, g, h are functions on some open interval $I \ni c$ such that

$$f(x) \leq g(x) \leq h(x) \text{ for all } x \in I, x \neq c.$$

If $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$, then $\lim_{x \to c} g(x) = L$.

$\star$ Useful for sin and cos functions: they can be bounded (amplitude=1)!



Some special limits to remember:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{cosx - 1}{x} = 0$$

Work out the following examples: (Answers in red)

1. If $\lim_{x\to 2} f(x) = 2$, $\lim_{x\to 2} g(x) = 3$, find -3

$$\lim_{x\to 2}\left(\frac{f(x)-2g(x)}{g(x)} - \frac{5}{3}\right)$$

2. Find 2

$$\lim_{x\to 1}\frac{x^2-1}{x-1}$$

3. If $1 \le f(x) \le x^2+2x+2$ for all x, find 1

$$\lim_{x\to(-1)} f(x)$$

4. Find 0

$$\lim_{x\to 0^+}\left(\sin x\cdot\cos(\frac{1}{x^2})\right)$$

One sided limits

Definition: Left-hand limit

Let $I \ni c$ be an open interval, and f a function defined on I except possibly at c. The *limit of $f(x)$ as x approaches c from the left* is L, denoted by

$$\lim_{x \to c^-} f(x) = L,$$

if and only if $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x < c, |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$.

Similarly ($\forall x > c$ replaces $\forall x < c$) *right-hand limits* $\lim_{x \to c^+} f(c)$ are defined.

Theorem

Let f be a function defined on an open $I \ni c$. Then

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x)$$

Continuous functions

Idea: a 'continuous' process is one that takes place gradually, without interruption or abrupt change. For functions, limits are a good indicator of where f is heading, whereas evaluating says where f actually is.

Definition: Continuous function

Left f be a function defined on an open $I \ni c$.

- f is *continuous at* c if $\lim\limits_{x \to c} f(x) = f(c)$.
- f is *continuous on* I if f is continuous for all c in I.
- f is *continuous everywhere* if f is continuous on $(-\infty, \infty)$.

$\star$ Have to first check if both $\lim\limits_{x \to c} f(x)$ and $f(c)$ exist – then if they agree.

Definition: continuity on a closed interval

If f is defined on a closed $[a,b]$, $a,b \in \mathbb{R}$, then f *is continuous on* $[a,b]$ if

1. f cntns on (a,b)
2. $\lim_{x \to a^+} f(x) = f(a)$
3. $\lim_{x \to b^-} f(x) = f(b)$

It is important to identify the intervals on which a function is continuous. Start from known functions and build up to more complicated expressions.

Theorem: Continuous functions + properties

The following functions are continuous on their domains:

$$\sin x,\ \cos x,\ \tan x,\ \cot x,\ \sec x,\ \csc x,\ \ln x,\ \sqrt[n]{x},\ a^x (a>0),\ p(x) \text{ poly}$$

If f and g are continuous on I, the following functions are also:

$$f \pm g,\ r \cdot f,\ f \cdot g,\ f/g \ {\scriptstyle (g \neq 0)},\ f^n,\ \sqrt[n]{f} \ {\scriptstyle (f \geq 0 \text{ if } n \text{ even})},\ \text{composition...}$$

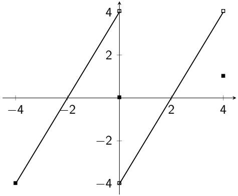
$\star$ Essentially, the previous limit theorems extended for continuity!

Work out the following examples:

1. Find $\lim_{x\to 1^+} f(x)$, $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1} f(x)$ for 8,1,DNE

$$f(x) = \begin{cases} 10 - x - x^2 & \text{if } x \le 1 \\ 2x - 1 & \text{if } x > 1 \end{cases}$$

2. Find $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0^-} f(x), f(0)$ and the continuity interval:



$-4, 4, 0, [-4, 0) \cup (0, 4)$

3. What is $\lim_{x\to 8} f(x)$? Is $f(x)$ continuous? $\frac{16}{5}$, No

$$f(x) = \begin{cases} \frac{x^2-64}{x^2-11x+24} & \text{if } x \neq 8 \\ 3 & \text{if } x = 8 \end{cases}$$

4. On which interval is $f(t) = \sqrt{5t^2 - 30}$ continuous? $(-\infty, -\sqrt{6}] \cup [\sqrt{6}, +\infty)$

Root finding and approximation

Intermediate Value Theorem

Let f be continuous on $[a, b]$ and (without loss of generality) $f(a) < f(b)$. For every value y with $f(a) < y < f(b)$, there exists a value $a < c < b$ such that $f(c) = y$.

$\star$ Particularly useful for root finding! Look for $f(c) = 0$...

Choosing an interval $[a, b]$ with $f(a) < 0, f(b) > 0$ and successively replacing an endpoint by the midpoint with the same sign is called the *Bisection Method*.

Infinity Limits

Idea: the usual ε-δ definition says that for x δ-close to c, $f(x)$ is guaranteed to be ε-close to $\lim_{x\to c} f(x)$.

Now, for some (large) M, $f(x)$ is guaranteed to be larger than M!

Limit of Infinity

We say that $\lim_{x\to c} f(x) = \infty$ if for every $M > 0$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $f(x) \geq M$. For $-\infty$, $M < 0$ and $f(x) \leq M$.

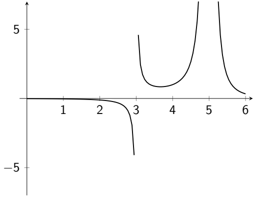
- If $\lim_{x\to c^{\pm}} f(x) = \pm\infty$, the function has a *vertical asymptote* $x = c$.

$\star$ Implicitly, the limit does not exist; it is not a numerical value L. Typical examples: those functions where $f(c) = \frac{-}{0}$, but not always.

Indeterminate forms need more work to compute the limit; symbolically they can be expressed $\frac{0}{0}, \infty \cdot 0, \infty - \infty, \frac{\infty}{\infty}, 0^0, \infty^0, 1^\infty$ after evaluation.

Work out the following examples:

1. On which interval is $g(t) = \dfrac{1}{\sqrt{1-t^2}}$ continuous? $(-1,1)$
2. Does the equation $x^8 - 3x = 5$ have a solution at $[0,2]$? At $[0,1]$?
   Yes, No
3. For $f(x) = \frac{1}{(x-3)(x-5)^2}$, find $\lim_{x\to 3^-} f(x)$, $\lim_{x\to 3^+} f(x)$, $\lim_{x\to 3} f(x)$ and $\lim_{x\to 5^-} f(x)$, $\lim_{x\to 5^+} f(x)$, $\lim_{x\to 5} f(x)$.



$-\infty, +\infty, DNE, \infty, \infty, \infty$

4. Find the vertical asymptotes of $f(x) = \dfrac{x^3}{x^2+3x-10}$.
   $x=-5, x=2$

Idea: what happens when x itself grows very large or very small?

Limits at Infinity

1. $\lim_{x\to\infty} f(x) = L \iff \forall\varepsilon > 0, \exists M > 0 \text{ s.t. } x \geq M \Rightarrow |f(x) - L| < \varepsilon.$
2. $\lim_{x\to-\infty} f(x) = L \iff \forall\varepsilon > 0, \exists M < 0 \text{ s.t. } x \leq M \Rightarrow |f(x) - L| < \varepsilon.$

- If $\lim_{x\to\pm\infty} f(x) = L$, the function has a *horizontal asymptote* $y = L$.

Limits of Rational Functions at Infinity

Suppose that $f(x)$ is a rational function, for $a_n, b_m \neq 0$,

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}.$$

- If $n = m$, $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = \frac{a_n}{b_n}$.
- If $n < m$, $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$.
- If $n > m$, $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are infinite (sign of $\frac{a_n}{b_m}$).

Work out the following examples:

1. Compute $\lim_{x \to \infty} \frac{9x - x^3}{2x^3 - x^2 + 6}$. $-1/2$
2. Find the vertical and horizontal asymptotes, if they exist, of

$$f(x) = \frac{x^2 + x - 12}{7x^3 - 14x^2 - 21x} \qquad x = 0, x = -1, y = 0$$

3. Find the vertical and horizontal asymptotes, if they exist, of

$$f(x) = \frac{x^2 - 9}{9x + 27} \qquad \text{no vertical, no horizontal}$$

4. Find $\lim_{x \to 0} f(x)$, $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$ and interval of continuity of

$$f(x) = \begin{cases} x^3 - x, & \text{if } x < 1 \\ x - 2, & \text{if } x \geq 1 \end{cases}$$

$0, 0, -1, (-\infty, 1) \cup (1, +\infty)$

The Derivative

⋆ Recall the 'difference quotient' formula for any function, that computes the slope of the secant line on its graph = average rate of change.

Definition: Derivative at a Point

Let f be continuous on an open $I \ni c$. The *derivative of f at c* is

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

if the limit exists. Then we say that f is *differentiable at c*.

If f is differentiable at every point in I, we say that f is *differentiable on I*.

⋆ Think of the derivative at a point as the instantaneous rate of change = a 'special slope' of a line approximating the graph at that point!

Definition: Tangent Line

Let f be coninuous on an open $I \ni c$ and differentiable at c. The line

$$\ell(x) = f'(c)(x - c) + f(c)$$

is the *tangent line* to the graph of f at c.

- Tangent-line approximation of function graph is useful (...not for lines).

$\star$ Let's allow the value c to vary: in this way, we can make the process "taking the derivative at" into a function itself!

Definition: Derivative Function

Let f be differentiable on an open I. The *derivative of f* is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

- Can be denoted as $f'(x) = \dfrac{df}{dx} = \dfrac{d}{dx}(f) = y' = \dfrac{dy}{dx} = \dfrac{d}{dx}(y)$.

Work out the following examples:

1. Find the derivative function of $f(x) = 6$, and of $g(x) = \frac{3}{2}x$.

   $f'(x) = 0, \; g'(x) = 3/2$

2. What is the tangent line of $h(x) = x^2$ at $x = -1$?

   $\ell(x) = -2x - 1$

3. What is the tangent line of $f(x) = \sin x$ at $x = 0$? $\ell(x) = x$

Instantaneous Rate of Change

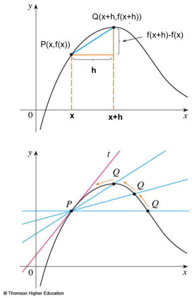
Idea: for a function f, its derivative $f'(x)$ becomes a new function of its own. It answers: 'when x changes, at what rate does f change?'

The Derivative and Motion

- If $s(t)$ is the position function of an object, its derivative function $s'(t)$ is the velocity function of the object.
- If $v(t)$ is the velocity function of an object, its derivative function $v'(t)$ is the acceleration function of the object.

- The units of the derivative $\dfrac{dy}{dx}$ correspond to $\dfrac{\text{units of } y}{\text{units of } x}$ (e.g. ft/s).

Slope of the Tangent Line



The slope of the secant line through P and Q is $\dfrac{f(x+h)-f(x)}{x+h-x}$.

As h shrinks to 0, i.e. taking the limit $h \to 0$, we find $f'(x)$ as the slope of the tangent line.

The tangent line at a point shows how *fast* the function is growing at that instance: the steeper the tangent line, the bigger the change.

- Use the tangent lines to approximate the functions: for values a close to c, where the tangent is $\ell_c(x)$, we have that $f(a) \approx \ell_c(a)$.

Differentiation Rules

Idea: abstractions which capture large classes of functions at one go.

Derivatives of Common Functions

1. *(Constant Rule)* $\frac{d}{dx}(c) = 0$
2. *(Power Rule)* $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$
3. $\frac{d}{dx}(\sin x) = \cos x$
4. $\frac{d}{dx}(\cos x) = -\sin x$
5. $\frac{d}{dx}(e^x) = e^x$
6. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

$\star$ What about combinations of the above functions?

Properties of the Derivative

Let f, g be differentiable on open I and $r \in \mathbb{R}$.

$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{df}{dx} \pm \frac{dg}{dx} = f'(x) \pm g'(x) \qquad \boxed{\text{Sum/Difference Rule}}$$

$$\frac{d}{dx}(r \cdot f(x)) = r \cdot \frac{df}{dx} = r \cdot f'(x) \qquad \boxed{\text{Scalar Multiple Rule}}$$

Higher Order Derivatives

$\star$ The derivative of a function became a new function on its own.
What is the latter's derivative?

Let $y = f(x)$ be a differentiable function on an open I.

- $f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y''$: *second derivative* of f
- $f'''(x) = \frac{d}{dx}(f''(x)) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = y'''$: *third derivative* of f
- $f^{(n)}(x) = \frac{d}{dx}(f^{(n\text{-}1)}(x)) = \frac{d}{dx}\left(\frac{d^{n\text{-}1}y}{dx^{(n\text{-}1)}}\right) = \frac{d^ny}{dx^n} = y^{(n)}$: *n^{th} derivative*

Note: the above only when each new function is differentiable!

▶ The second derivative is 'the rate of change of the rate of change' of f. E.g. for a position function f, f' is the velocity and f'' is the acceleration!

Work out the following examples:

1. After computing the first and second derivative of

$$f(t) = 9\cos t - t^6 + 2e^t + \ln(17)$$

find $f'(0)$ and $f''(\pi)$. $f'(0) = 2,\ f''(\pi) = 9 - 30\pi^4 + 2e^\pi$

2. Find the tangent line to $y = 3x^2 - x^3$ at $(1, 2)$. $\ell(x) = 3x - 1$

3. Approximate $e^{0.1}$ using the tangent line to $f(x) = e^x$ at $x = 0$.
   $e^{0.1} = f(0.1) \approx \ell_0(0.1) = 1.1$

4. Find the tangent line to $y = 2x + 3$ at $x = 375.2$. $y = 2x + 3$
   the same!

The Product Rule for Derivation

Idea: sum/difference of functions works nicely for derivatives – product?

Product Rule

Let f and g be differentiable functions on an open I. Then $f \cdot g$ is differentiable on I, with

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

$\star$ So beware: $(f(x)g(x))' \neq f'(x)g'(x)$!

- This rule (as well as the others) is in fact a Theorem. To prove it, we use the derivative *definition* and we algebraically manipulate it:

$$\frac{d}{dx}((f \cdot g)(x)) := \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \dots$$
$$= f'(x)g(x) + f(x)g'(x)$$

The Quotient Rule for Derivation

Quotient Rule

Let f and g be differentiable functions on an open I, where $g(x) \neq 0$. Then f/g is differentiable on I, and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

$\star$ Useful in many examples; in particular, trigonometric functions.

Derivatives of Trigonometric Functions

- $(\sin(x))' = \cos(x)$
- $(\cos(x))' = -\sin(x)$
- $(\tan(x))' = \sec^2(x)$
- $(\cot(x))' = -\csc^2(x)$
- $(\sec(x))' = \sec(x)\tan(x)$
- $(\csc(x))' = -\csc(x)\cot(x)$

Work out the following examples:

1. Find the derivative of

$$y = (x^2 + 5)(x^3 + 9)$$

(either by the product rule or by expanding). $x(5x^3 + 15x + 18)$

2. Find the derivative of

$$f(x) = \frac{x^2 + 3}{x} \qquad f'(x) = 1 - \frac{3}{x^2}$$

3. Find $s'(\pi)$, for $s(t) = \frac{1}{\pi^4} t^5(\cos t + e^t)$. $(5 + \pi)e^\pi - 5$

4. Find the derivative of $h(x) = e^2(\sin(\pi/4) - 1)$. 0, constant rule!

Power Rule revisited

⋆ Power Rule earlier was for $n \in \mathbb{Z}^+$; it now extends to negative integers!

Power Rule with Integer Exponents

Let $f(x) = x^n$ where $x \neq 0$ is an integer. Then

$$f'(x) = n \cdot x^{n-1}$$

There might be different choices of rules that apply to compute a derivative. There is no 'right' way! Each one leads to the one and only result$\rightarrow$the function's derivative.

⋆ Looking for the point(s) where the graph has a horizontal tangent line?

Need to solve $f'(x) = 0$.

Work out the following examples:

1. Find the derivative of $f(x) = 7e^x \sin(x)$. $7e^x(\sin x + \cos x)$
2. Find the tangent line of $y = x\cos(x)$ at the point $(\pi, -\pi)$.
   $y = -x$
3. Find the derivative of $\dfrac{x+7}{x-5}$. $\frac{-12}{(x-5)^2}$
4. Find where the graph of $f(x) = \dfrac{x^2}{x+1}$ has a horizontal tangent line.
   At $x = -2$ and $x = 0$

Chain Rule

Idea: so far, have seen derivative rules relative to basic combinations of functions (addition, multiplication etc). What about *composition*?

The Chain Rule

If f and g are differentiable functions, then $f \circ g$ is differentiable with

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

$\star$ Viewing a function as a composition of two others is *not* uniquely determined; however, in practice there is usually a natural choice.

Generalized Power Rule

Let $g(x)$ be a differentiable function, $n \neq 0$ an integer. Then

$$\frac{d}{dx}(g^n(x)) = n \cdot g^{n-1}(x) \cdot g'(x).$$

Work out the following examples:

1. Differentiate $y = \sin(e^x)$. $e^x \cos(e^x)$
2. Differentiate $y = (2 - x^2)^{10}$. $-20x(2 - x^2)^9$
3. Find an equation of the tangent line to $f(x) = 6\cos(x) + \sin^2(x)$ at $(\frac{\pi}{2}, 1)$. $-6x + 3\pi + 1$
4. Use the above tangent line, in order to approximate the value $6\cos(1.5) + \sin^2(1.5)$, i.e. $f(1.5)$. (Hint: $1.5 = \frac{3}{2}$ is very close to $\frac{\pi}{2}$!)
   $3\pi - 8$

⋆ Chain rule is very powerful; finds also new derivatives!

Derivatives of Exponential Functions

Let $f(x) = a^x$, $a > 0$, $a \neq 1$. The f is differentiable, with

$$f'(x) = \ln a \cdot a^x.$$

Idea: 'pattern' recognition for taking derivatives makes us faster.

If $u = g(x)$ denotes an arbitrary function of x, we have that

- $(u^n)' = n \cdot u^{n-1} \cdot u'$
- $(\ln(u))' = \dfrac{u'}{u}$, $(e^u)' = e^u \cdot u'$
- $(\sin(u))' = \cos(u) \cdot u'$
- $(\cos(u))' = -\sin(u) \cdot u'$
- $(\tan(u))' = \sec^2(u) \cdot u'$
- $(a^u)' = \ln a \cdot a^u \cdot u'$

- Notation: for $y = f(u)$ a function of u and $u = g(x)$ a function of x,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad [\text{equivalent to } f'(g(x)) \cdot g'(x)]$$

Work out the following examples:

1. Differentiate $y = \ln(\cos(x))$. $\quad -\tan(x)$
2. Differentiate $y = 5^{cos(t)}$. $\quad -\ln 5 \sin(t) 5^{\cos(t)}$
3. Differentiate $x^2 \cdot e^{\frac{1}{x}}$. $\quad e^{\frac{1}{x}}(2x-1)$
4. Differentiate $\sin^4(2t)$. $\quad 8\sin^3(2t)\cos(2t)$

Implicit Differentiation

Idea: so far, we know how to find the derivative of any $y = f(x)$, which *explicitly* expresses y in terms of x.
What about when the relationship between x and y is *implicit*?

Methodology for Implicit Differentiation: for equation involving x & y

- take the derivative of each term of the equation; use chain rule approach for the derivative of the y-terms
- separate y' terms on one side, and remaining terms on the other
- solve for y'; the result might very well be in terms of both x and y

⋆ Contrary to explicit functions, for implicit we can usually only check if a *pair* (x_0, y_0) satisfies the equation, rather than find some $y_0 = f(x_0)$!

- Implicit differentiation works nicely for finding tangent lines at given points (x_0, y_0).

Work out the following examples:

1. Differentiate $x^2 + y^2 = 16$. $\qquad \frac{-x}{y}$

2. Use implicit differentiation to find an equation of the tangent line to the curve at $(1, 1)$

$$x^2 + xy + y^2 = 3. \qquad \ell_1(t) = -t + 2$$

3. Differentiate $\sin(x) + \cos(y) = \sin(x)\cos(y)$. $\qquad \frac{\cos(x)(\cos(y) - 1)}{\sin(y)(\sin(x) - 1)}$

Power Rule for Differentiation

Let $f(x) = x^n$, where $n \neq 0$ is a real number. Then f is differentiable, with

$$f'(x) = n \cdot x^{n-1}.$$

- To obtain higher derivatives via implicit differentiation, just take one more derivative $\frac{d}{dx}(y')$ - in practice, uses derivation rules and algebra.

Logarithmic Differentiation

In order differentiate expressions like $y = f(x)^{g(x)}$, first apply the natural logarithm and then differentiate implicitly:

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(g(x)\ln(f(x)))$$

Work out the following examples:

1. Find the derivative of $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}} + \sqrt{\pi}$. Express it only in terms of powers of x (no radicals). $\frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}$

2. Find $\frac{dy^2}{dx^2}$ using implicit differentiation:

$$\frac{x}{y} = 10 \qquad y'' = 0$$

3. Find the tangent line at $(1, 1)$ for $y = (x)^{x^2}$, using logarithmic differentiation. $\ell_1(x) = x$

Inverse Functions

Idea: any *one to one* function f has an *inverse* function f^{-1}.
How to compute its derivative?

- $\boxed{f(f^{-1}(x)) = f^{-1}(f(x)) = x}$ says that if (a, b) is on f-graph, then (b, a) is on f^{-1}-graph! So their graphs are symmetric with respect to $y = x$.

Derivatives of Inverse Functions

Suppose: f differentiable, 1-1 on open I, $f'(x) \neq 0$, J its range, $g = f^{-1}$ and $f(a) = b$. Then g is differentiable on J, with

1. $\left(f^{-1}\right)'(b) = g'(b) = \dfrac{1}{f'(a)}$
2. $\left(f^{-1}\right)'(x) = g'(x) = \dfrac{1}{f'(g(x))}$

Inverse Trigonometric Functions

These are differentiables on open subsets of their domains, with

1. $(\sin^{-1}(x))' = \dfrac{1}{\sqrt{1-x^2}}$
2. $(\tan^{-1}(x))' = \dfrac{1}{1+x^2}$
3. $(\cos^{-1}(x))' = \dfrac{-1}{\sqrt{1-x^2}}$
4. $(\cot^{-1}(x))' = \dfrac{-1}{1+x^2}$

Work out the following examples:

1. For $f(x) = \dfrac{1}{1+x^2}$ $x \geq 0$, with the point $(1, 1/2)$ lying on its graph, find $\left(f^{-1}\right)'(1/2)$. -2
2. Find the derivative of $f(x) = \tan^{-1}(\sin(5x))$. $\dfrac{5\cos(5x)}{1+\sin^2(5x)}$

L'Hopital Rule

Idea: a method to tackle indeterminate forms like "$\frac{0}{0}$" and "$\frac{\infty}{\infty}$" for limits.

L'Hopital's Rule (LHR)

- Let $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, for f, g differentiable functions on open $I \ni c$, and $g'(x) \neq 0$ except maybe at c. Then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

- $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$, f, g differentiable on $I \ni a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

- f, g differentiable on (c, ∞), $g'(x) \neq 0$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \qquad \text{(similarly for } -\infty)$$

$\star$ Can use these rules to handle also other indeterminate forms, like "$0 \cdot \infty$" or "$\infty - \infty$". Usually rewrite them, to bring into previous forms!

- LHR helps with even more indeterminate forms: 0^0, 1^∞, ∞^0.

$$\boxed{\text{If } \lim_{x \to c} \ln(f(x)) = L, \text{ then } \lim_{x \to c} f(x) = \lim_{x \to c} e^{\ln(f(x))} = e^L.}$$

Work out the following examples:

1. Find $\displaystyle \lim_{x \to 1} \frac{x^3 - 1}{x^8 - 1}$. $\frac{3}{8}$
2. Find $\displaystyle \lim_{x \to \pi} \frac{\sin(x)}{x - \pi}$. -1
3. Find $\displaystyle \lim_{x \to 0} \frac{sin(2x)}{x + 2}$. 0, NO LHR WAS NEEDED!
4. Find $\displaystyle \lim_{x \to \infty} \frac{e^x}{\sqrt{x}}$. $+\infty$

Extreme Values

Idea: what are the largest or smaller values (outputs) of a given function?

Extreme Values

Let f defined on $I \ni c$. Then [not necessarily open I!]

- $f(c)$ is the (*absolute*) *minimum* of f on I iff $f(c) \leq f(x)\ \forall x \in I$
- $f(c)$ is the (*absolute*) *maximum* of f on I iff $f(c) \geq f(x)\ \forall x \in I$

The Extreme Value Theorem

Let f be a continuous function on a <u>closed</u> interval I. Then f has both a maximum and a minimum value on I.

Relative Minima and Maxima

Let f defined on $I \ni c$. Then

- $f(c)$ is a *relative minimum* iff $f(c) \leq f(x)$, $\forall x \in (c - \varepsilon, c + \varepsilon)$ $(\varepsilon > 0)$
- $f(c)$ is a *relative maximum* iff $f(c) \geq f(x)\ \forall x \in (c - \varepsilon, c + \varepsilon)$

- Notice how at relative extrema, the tangent line is horizontal!

Critical points

Let f defined at c. Value c is *critical* if $\boxed{f'(c)=0}$ or $\boxed{f'(c) \text{ not defined}}$. The point $(c, f(c))$ is called a *critical point*.

Work out the following examples:

1. Find $\lim\limits_{x \to 0} \dfrac{6^x - 12^x}{x}$. $\ln(\frac{1}{2})$
2. Find $\lim\limits_{x \to \infty} x^{\frac{3}{x}}$. 1
3. Find the critical points of $f(x) = \dfrac{2}{x^2+1}$. $(0, 2)$, ONLY real numbers!

Finding Extrema

Relative Extrema and Critical Points

If f has a relative extremum at $(c, f(c))$, then that is a critical point.

⋆ This doesn't mean that ALL critical points are relative extrema!

Extrema on a closed interval: if f is continuous on a closed $[a, b]$

1. evaluate f at the endpoints of the interval
2. find the critical points of f in $[a, b]$
3. absolute maximum=largest of outputs, absolute minimum=least of outputs

The Extreme Value Theorem ensures that they always exist!

The Mean Value Theorem

Idea: is instantanous rate of change ever *same* as average rate of change?

The Mean Value Theorem of Differentiation

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b).
There exists some c with $a < c < b$ (i.e. c is between a and b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Work out the following examples:

1. Find the absolute extrema of $\dfrac{4x}{x^2+1}$ at $[-4, 0]$. maximum is 0 for $x = 0$, minimum is -2 for $x = -1$
2. Find the absolute extrema of $4cos(x) + 4sin(x)$ at $[-\frac{\pi}{2}, 0]$. maximum is 4 for $x = 0$, minimum is -4 for $x = -\frac{\pi}{2}$

Mean Value Theorem, Part 2

- MVT states that the average slope is the same as $f'(c)$ for some c.

Rolle's Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$. Then there exists some $c \in (a, b)$ such that $f'(c) = 0$.

$\star$ It looks like a special case of MVT; it is needed for its proof!

Work out the following:
Consider the function $f(x) = 4 - 2x^2$ on the interval $[-6, 7]$.

1. What is the average slope of the function on this interval? -2
2. Which is the $c \in (-6, 7)$ that satisfies the MTV? $c = 1/2$

Increasing and Decreasing Functions

Idea: so far, interested in 'special points' (e.g. extrema). What about the general shape of functions?

Increasing and Decreasing Functions

Let f function defined on interval I.

- f is *increasing* on I iff for every $a < b$, $f(a) \leq f(b)$.
- f is *decreasing* on I iff for every $a < b$, $f(a) \geq f(b)$.

A function is *strictly* increasing/decreasing when $\leq, \geq$ are $<, >$.

$\star$ [MVT] Looks like f' positive for increasing f (negative for decreasing).

Conditions for Increasing/Decreasing Functions

Let f be continuous on $[a, b]$ and differentiable on (a, b).

1. If $f'(c) > 0$ for all $c \in (a, b)$, then f is increasing on (a, b).
2. If $f'(c) < 0$ for all $c \in (a, b)$, then f is decreasing on (a, b).
3. If $f'(c) = 0$ for all $c \in (a, b)$, then f is constant on (a, b).

Finding increasing and decreasing intervals: let f be differentiable on I.

1. Find the critical inputs of f on I ($f'(c) = 0$ or $f'(c)$ not defined)
2. Divide I in subintervals using the critical inputs
3. Pick any p in each subinterval and determine the sign of f':
   - If $f'(p) > 0$, f is increasing on that subinterval
   - If $f'(p) < 0$, f is decreasing on that subinterval

Work out the following examples:

1. Let $f(x) = 6 - x - x^2$. Find the open intervals on which f is increasing or decreasing. Increasing at $(-\infty, -\frac{1}{2})$, decreasing at $(-\frac{1}{2}, \infty)$

2. Let $g(x) = x + \dfrac{4}{x}$. Find the open intervals on which f is increasing or decreasing. Increasing at $(-\infty, -2) \cup (2, +\infty)$, decreasing at $(-2, 0) \cup (0, 2)$

Relative Extrema and Concativity

First Derivative Test

Let f be differentiable on I, and $c \in I$ critical input.

1. If the sign of f' changes from $+$ to $-$ at c, then $f(c)$ is a relative max.
2. If the sign of f' changes from $-$ to $+$ at c, then $f(c)$ is a relative min.
3. If the sign of f' doesn't change at c, $f(c)$ is not relative extremum.

$\star$ The next step is to study properties of f' itself. f'' will be useful!

Concave Up and Concave Down

Let f be differentiable on an interval I.

- The graph of f is *concave up* on I if f' is increasing.
- The graph of f is *concave down* on I if f' is decreasing.
- f has *no concativity* if f' is constant.

Test for Concavity

Let f be twice differentiable on an interval I. The graph of f is concave up if $f'' > 0$ on I, and is concave down if $f'' < 0$ on I.

$\star$ Similarly to relative extrema where f changes from increasing to decreasing [look for critical points, $f' = 0$ or undefined], would like to know when f's concavity changes from up to down!

Points of Inflection

A *point of inflection* is a point on the f graph where concavity changes.

If $(c, f(c))$ point of inflection, then either $f'' = 0$ or f'' is not defined at c.

Work out the following: let $f(x) = 8x^2 - x^4$. Solution at .pdf

1. Find the open intervals on which f is increasing or decreasing.
2. Identify the relative extrema of f.
3. Find the (possible) inflection points (only the x-coordinate) of f.
4. Find the open intervals on which f is concave up or down.

Inflection points and Second Derivative Test

Idea: start processing all information at the same time!

⋆ Critical inputs ($f'(x) = 0$ or f' not defined) and possible inflection points ($f''(x) = 0$ or f'' not defined) together on the real line.

- Notice that inflection points are in fact the relative extrema of $f'(x)$!

And they expess the points where f is increasing/decreasing the most.

The Second Derivative Test

Let c be a critical input of f, where $f''(c)$ is defined.

- If $f''(c) > 0$, f has a relative minimum at $(c, f(c))$.
- If $f''(c) < 0$, f has a relative maximum at $(c, f(c))$.

E.g. if a critical x is in a concave up region, then it must be a minimum!

Work out the following: for $f(x) = 4x^3 - 12x^2 + 20$,

1. find the increasing and decreasing open intervals Increasing at $(-\infty, 0) \cup (2, \infty)$, decreasing at $(0, 2)$
2. find the relative maxima and minima Max at 20 for $x = 0$, min at 4 for $x = 2$
3. find the concave up and down open intervals concave up $(1, +\infty)$, concave down $(-\infty, 1)$
4. find the inflection point(s) $(1, 12)$

Curve Sketching

Idea: using function's core properties (monotonicity, concavity etc.), produce an accurate graph of f. Behavior is mostly captured by f', f''!

Methodology for sketching f's graph

1. Specify the domain of f.
2. Find the vertical asymptotes of f.
3. Find the critical inputs of f.
4. Find the possible points of inflection of f.
5. Consider $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$ to determine end behavior.
6. Draw the real line with all of (1),(3),(4). For each created interval, determine whether f is increasing or decreasing, concave up or down.
7. Determine the relative extrema and points of inflection, if they exist.
8. Find the y & x-intercepts ($(0, f(0))$ & $(x_i, 0)$ with $f(x_i)=0$) [if asked].
9. Plot all points (critical, possible inflection, intercepts) and asymptotes on a plane; use all above behavioral info to graph.

⋆ In general, the domain is all reals with certain restrictions, e.g. points that make denominator 0, points that make the radicand negative.

⋆ The vertical asymptotes are usually determined by the denominator, but recall to first simplify if applicable.

⋆ If any of $\lim\limits_{\pm\infty}(f(x))$ is a number, recall it gives horizontal asymptote.

⋆ The intercepts dictate when the graph of the function crosses the axes.

Moral: we can sketch the graph of functions quite precisely, not by plotting very many points $(x, f(x))$ but by understanding the function's behavior at a few key places and with respect to a few key general properties.

Related Rates

Idea: an equation may relate the value of two quantites.
Sometimes, the *rate of change* of a quantity determines the other!

- Usually, write equation including the quantities, say A and B, that change through time. Then, implicitly differentiate to relate $\dfrac{dA}{dt}$ with $\dfrac{dB}{dt}$.

Work out the following:
Water flows onto a flat surface at a rate of $5cm^3/s$ forming a circular puddle $10mm$ deep. How fast is the radius growing,

1. when the radius is $1cm$? $5/2\pi \approx 0.8cm/s$
2. when the radius is $100cm$? $5/200\pi \approx 0.008cm/s$

Related Rates, continued

⋆ It is always helpful to sketch a diagram with the known & unknown data. Then, give names to quantites (which are functions of time) and relate them via an equation; finally, implicit diff. and solve for the required rate.

A few notable formulas *(non-exhaustive list!)*

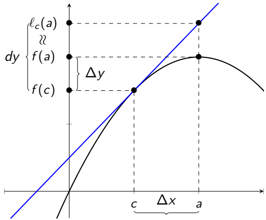
- $A = \frac{1}{2}bh = \frac{1}{2}ab\sin(\theta)$ — area of triangle
- $A = \pi r^2$ — area of a circle
- $A = \pi d^2 = 4\pi r^2$ — area of a sphere
- $V = \frac{4}{3}\pi r^2$ — volume of a sphere
- $V = \frac{1}{3}\pi r^2 h$ — volume of a cone

Work out the following:

A plane flying horizontally at an altitude of 1 mile and a speed of 500mi/h passes directly over a radar station. Find the rate at which their distance is increasing, when the plane is 2 miles away from the station. $250\sqrt{3}$, worked out at the end of notes

Differentials

- Recall: use the tangent line $\ell_c(t) = f'(c)(t-c) + f(c)$ at some input c to approximate the f-value of an input a *close* to c, $\boxed{f(a) \approx \ell_c(a).}$



Differentials of x and y

Let $y = f(x)$ be differentiable. The *differential of* x, denoted dx, is any nonzero (usually small) number. The *differential of* y, denoted dy, is

$$dy = f'(x)dx$$

$\star$ The fact that $f'(x)$ is the differentiable of y divided by that of x is **not** the alternate notation $y' = \frac{dy}{dx}$ used earlier.

Differential Notation, for $y = f(x)$ differentiable

- Δx represents a small, nonzero change in x value.
- So does dx, hence $\Delta x = dx$.
- Δy is the y-value change as x changes by Δx, $\Delta y = f(x + \Delta x) - f(x)$.
- Finally, $dy = f'(x)dx = f'(x)\Delta x \approx \Delta y$.

⋆ Not only useful for approximations, also crucial for integration (later).

Error propagation: a value is measured to be x, when it could be $x + \Delta x$.

The *propagated error* is the difference between their outputs Δy, which can be well approximated by dy.

The *relative error* is a percentage given by the ratio between dy and the output $y = f(x)$.