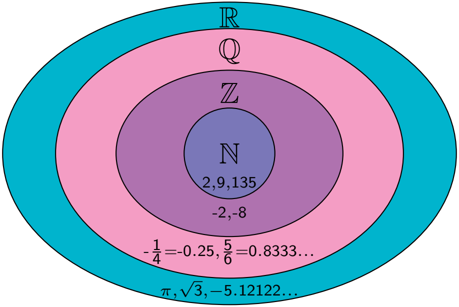
Welcome to Math 004!

The Real-Number System

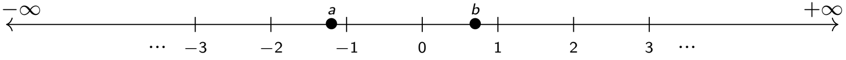
- The set of real numbers, $\mathbb{R}$, divides into the following subsets:



- $\mathbb{N}$ of natural numbers $1, 2, 3, \ldots$
- $\mathbb{Z}$ of integers $\ldots, -2, -1, 0, 1, 2, \ldots$
- $\mathbb{Q}$ of rational numbers $\frac{p}{q}$ for $p, q \in \mathbb{Z}, q \neq 0$
- irrational numbers, like $\pi = 3.141592\ldots$ or $\sqrt{2} = 1.414213\ldots$ where decimal notation neither terminates nor repeats

Interval Notation

⋆ Every real number, $a \in \mathbb{R}$, is represented by a point on the *real line*.
Then a is *less than* b, written $a < b$, if a is to the left of b!



Interval notation and Graphs

- The set $\{x|a < x < b\}$ is written (a, b), its graph 

- The set $\{x|a \leq x \leq b\}$ is written $[a, b]$, its graph 

- The set $\{x|a < x \leq b\}$ is written $(a, b]$, its graph 

- The set $\{x|a \leq x\}$ is written $[a, +\infty)$, its graph 

- All other possible combinations

▶ (a, b) is an *open* interval, $[a, b]$ a *closed* one.

Properties of the Real Numbers

For any real numbers $a, b, c \in \mathbb{R}$

$a + b = b + a$, $ab = ba$	*commutative property of + and* $*$
$a + (b + c) = (a + b) + c$, $a(bc) = (ab)c$	*associative property of + and* $*$
$a + 0 = 0 + a = a$	*additive identity property*
$-a + a = a + (-a) = 0$	*additive inverse property*
$a \cdot 1 = 1 \cdot a = a$	*multiplicative identity property*
$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$ $(a \neq 0)$	*multiplicative inverse property*
$a(b + c) = ab + ac$	*distributive property*

Absolute Value

For any $a \in \mathbb{R}$, its *absolute value* is the distance from 0 (so always positive)

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

The *distance between a and b* is $|a - b|$, equivalently $|b - a|$.

Order of Operations

Basic rules for calculations ("PEMDAS"):

- work within **P**arentheses; when nested, work from the inside out
- evaluate **E**xponentials
- do **M**ultiplications and **D**ivisions, from left to right
- do **A**dditions and **S**ubtractions, from left to right

Integer Exponents

For any positive integer $n \in \mathbb{Z}^+$, we define

$$a^n = \overbrace{a \cdot a \cdot \ldots \cdot a}^{n \text{ times}}.$$

- a is called the *base*, n is called the *exponent*.

$\boxed{a^0 \text{ is set to be } 1 \text{ (any base } a \neq 0)}$

Negative integer exponent? For $a \neq 0$, define

$$a^{-m} = \frac{1}{a^m}.$$

Consequently, for non-zero a, b and integers m, n, we always have

$$\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}.$$

Properties of Exponents

For any $a, b \in \mathbb{R}$ and $m, n \in \mathbb{Z}$, the following are true:

- $a^m \cdot a^n = a^{m+n}$ *(product rule)*
- $\dfrac{a^m}{a^n} = a^{m-n}$, $a \neq 0$ *(quotient rule)*
- $(a^m)^n = a^{m \cdot n}$ *(power rule)*
- $(a \cdot b)^m = a^m \cdot b^m$
- $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$, $b \neq 0$

Work out the following examples: (answers are now posted in green):

1. -2^3 -8
2. 7^{-2} $\frac{1}{49}$
3. $\frac{1}{3^{-2}}$ 9
4. $(2^3)^{-2}$ $\frac{1}{64}$
5. $(3xy^2)(x^3yz)$ $3x^4y^3z$
6. Simplify, using only positive exponents, $\frac{x^{-3}y}{y^2z^{-5}}$ $\frac{z^5}{x^3y}$

Polynomials

Definition

A *polynomial in one variable* is an expression of the form

$$a_n x^n + a_{n\text{-}1} x^{n\text{-}1} + \ldots + a_2 x^2 + a_1 x + a_0$$

where $n \in \mathbb{Z}^+$ and $a_n, \ldots, a_0 \in \mathbb{R}$.

- the expressions $a_i x^i$ for some $i = 0, \ldots, n$ are the *terms*
- for each term, its *degree* is the exponent of the variable
- if $a_n \neq 0$, the *degree of the polynomial* is n (the highest exponent)
- $a_n x^n$ is the *leading term*, a_0 is the *constant term*
- $a_n, \ldots, a_0$ are the *coefficients*, a_n is the *leading* coefficient

A *monomial* is a polynomial with one term, e.g. $-3z^4$.
There are also *binomials*, $y^2 - 6$, and *trinomials*, $c^6 - 7c + 1$.

$\star$ Can have polynomials in many variables! E.g. $3xy^2 + 8zxy - 2z^2$.

Operations with polynomials

Addition/Subtraction: can only preform these operations between *similar* terms, i.e. those with the same variables raised to the same powers.

Multiplication: for monomials, coefficients and each variable separately. For arbitrary polynomials, distributive property (FOIL)!

Special products of binomials

- $(a+b)^2 = a^2 + 2ab + b^2$ *square of a sum*
- $(a-b)^2 = a^2 - 2ab + b^2$ *square of a difference*
- $(a+b)(a-b) = a^2 - b^2$ *difference of squares*

$\star$ Common mistake: $(a+b)^2 \neq a^2 + b^2$! But $(a \cdot b)^2 = a^2 \cdot b^2$.

Work out the following examples:

1. which is the degree, leading coefficient and constant term of $x^2 - 8 + \frac{2}{3}x^5 - 2x$? $5, \frac{2}{3}, -8$
2. compute $(x^4 - 2x^2 + 9x) - (7x^3 + x^2 - 9x + 7)$
   $x^4 - 7x^3 - 3x^2 + 18x - 7$
3. compute $(3xy^2z^5)(-2x^6y)$ $-6x^7y^3z^5$
4. compute $(3b - 1)^2$ $9b^2 - 6b + 1$

Factoring

Idea: want to factor an expression, i.e. equivalently write as a product, as much as possible.

- Common factor: $A \cdot B + A \cdot C = A \cdot (B + C)$ for any expressions A, B, C
  - Grouping: group terms accordingly to create a common factor

Factoring trinomials $ax^2 + bx + c$

- Leading coefficient $a = 1$?
  Find two numbers s_1, s_2 with $s_1 \cdot s_2 = c$ and $s_1 + s_2 = b$; then factoring is $(x + s_1)(x + s_2)$
- Leading coefficient $a \neq 1$?
  FOIL method: find two first terms whose product is ax^2, two last terms whose product is c, and check until it's verified

$\star$ [Last time] $a^2 + 2ab + b^2$, $a^2 - 2ab + b^2$, $a^2 - b^2$ factor right away:

$$(a+b)^2 \qquad (a-b)^2 \qquad (a+b)(a-b)$$

- Always start by common factor, always finish by completely factoring!

Solving Equations

- A *linear equation* is of the form $(a, b \in \mathbb{R}, a \neq 0)$

$$ax + b = 0$$

- A *quadratic equation* is of the form $(a, b, c \in \mathbb{R}, a \neq 0)$

$$ax^2 + bx + c = 0$$

Equation-Solving Principles

- If $a = b$, then $a + c = b + c$ — *addition principle*
- If $a = b$, then $ac = bc$ — *multiplication principle*
- If $x^2 = k$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$ — *square root principle*
- If $ab = 0$, then $a = 0$ or $b = 0$ — *zero product principle*

Linear equations come down to isolating the variable
Quadratic equations come down to factoring and equating to 0

$\star$ Similarly we can solve *formulas*, i.e. equations in <u>many</u> variables that model some situation, for a chosen variable: regard all others as constants!

Work out the following examples:

1. solve the linear equation $3x + 6 = 3 + 12x$ $\quad x = \frac{1}{3}$
2. solve the quadratic equation $2a^2 - 32 = 0$ $\quad a = 4, -4$
3. solve the quadratic equation $x^2 + 5x = 0$ $\quad x = 0, -5$
4. solve $A = \frac{1}{2}bh$ for b (area of a triangle!) $\quad b = \frac{2A}{h}$
5. solve the quadratic equation $y^2 + 6y = -9$ $\quad y = -3$

Rational Expressions

Idea: work with fractions of polynomials!

A *rational expression* is the quotient of two polynomials

$$\frac{p(x)}{q(x)} = \frac{a_n x^n + \ldots + a_0}{b_m x^m + \ldots + b_0}$$

- No denominator can ever be 0: $\cancel{\frac{4}{0}}$ not defined.

So the *domain* of a rational expression are those x's for which $q(x) \neq 0$!

To simplify a rational expression, we factor both numerator and denominator, and then we 'remove factors of 1' or 'cancel'

$$\frac{a \cdot c}{b \cdot c} = \frac{a}{b} \cdot \frac{c}{c} = \frac{a}{b} \cdot 1 = \frac{a}{b} \left(= \frac{a \cdot \cancel{c}}{b \cdot \cancel{c}} \right)$$

Operations between rational expressions

⋆ To multiply two rational expressions, we multiply numerators and denominators; to divide them, we multiply with the reciprocal.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} \text{ and } \frac{x}{y} \div \frac{z}{w} = \frac{x}{y} \cdot \frac{w}{z}.$$

⋆ To add or subtract two rational expressions, use their **LCD**: a polynomial that combines all factors of the denominators involved!

$$\frac{p_1(x)}{q_1(x)} + \frac{p_2(x)}{q_2(x)} = \frac{\text{operations}\{p_1(x), p_2(x)\}}{LCD(q_1(x), q_2(x))}$$

⋆ To simplify *complex* rational expressions, perform operations in both numerator and denominator separately, and then simplify

$$\frac{\frac{p_1(x)}{q_1(x)}}{\frac{p_2(x)}{q_2(x)}} = \frac{p_1(x)}{q_1(x)} \div \frac{p_2(x)}{q_2(x)} = \frac{p_1(x) \cdot q_2(x)}{p_1(x) \cdot p_2(x)}$$

Work out the following examples:

1. what is the domain of $\dfrac{7x+5}{2x(3x+2)}$, in interval notation?

   $(-\infty, -\frac{2}{3}) \cup (-\frac{2}{3}, 0) \cup (0, \infty)$

2. multiply and simplify $\dfrac{x-2}{12x} \cdot \dfrac{3x^3+15x^2}{x^2-4x+4}$ $\qquad \dfrac{x(x+5)}{4(x-2)}$

3. $\dfrac{5}{8z} + \dfrac{3}{4z(z-3)}$ $\qquad \dfrac{5z-9}{8z(z-3)}$

Radical Expressions

Definition: nth root

A number c is said to be an *nth root* of a if $c^n = a$. It is denoted by

$$c = \sqrt[n]{a}.$$

- The sumbol $\sqrt[n]{\ }$ is called *radical*, the number n is called the *index* and what is under the radical is called the the *radicand*.

  $\star$ For n odd number, $\sqrt[n]{a}$ has the same sign $(\pm)$ as the radicand a. For n even number, $a \geq 0$ necessarily! And $\sqrt[n]{a} \geq 0$, $-\sqrt[n]{a} < 0$.

Properties of Radicals

Suppose $a, b \in \mathbb{R}$ for which the roots exist, and $m, n \in \mathbb{N}$ $(n \neq 1)$.

1. (n odd) $\sqrt[n]{a^n} = a$
2. (n even) $\sqrt[n]{a^n} = |a|$
3. $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
4. $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$
5. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$ $(b \neq 0)$

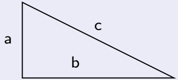
Operations Involving Radicals

⋆ We can add or subtract radicals with the same index and radicand!

$$(c \cdot \sqrt[n]{a}) \pm (d \cdot \sqrt[n]{a}) = (c \pm d)\sqrt[n]{a} \qquad \text{'similar radical terms'}$$

The Pythagorean Theorem

The sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the hypotenuse:



$$\boxed{a^2 + b^2 = c^2}$$

We can *rationalize* the denominator or numerator of a fraction, i.e. remove all roots from there, by multiplying with the root, so $\frac{\star}{\sqrt{a}} = \frac{\star \cdot \sqrt{a}}{\sqrt{a}\sqrt{a}_{=a}}$, or a *conjugate* of an expression, $(\sqrt{a} \pm b)$.

Rational Exponents

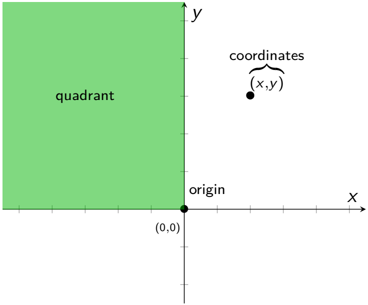
For any $a \in \mathbb{R}$ and $n \in \mathbb{N}$ for which $\sqrt[n]{a}$ exists,

$$\sqrt[n]{a} = a^{1/n} \quad \text{and} \quad \sqrt[n]{a^m} = a^{m/n}.$$

Work out the following examples:

1. simplify $\sqrt{5y^4z}\sqrt{10z}$. $5y^2z\sqrt{2}$

Graphs



- The graph of an equation represents its solutions (x, y).

To graph a line $ax + by = c$:

- solve for y
- make a table $x|y$ by plugging any two numbers for x
- label those two points (x_1, y_1) and (x_2, y_2) on a plane
- graph the (unique) line that passes through them

x-intercept and y-intercept

An x-intercept is a point $(x_0, 0)$, where the graph meets the x-axis.
A y-intercept is a point $(0, y_0)$, where the graph meets the y-axis.

$\star$ To find them, set $y = 0$ and find x and vice versa.

The distance formula

The distance d between any two points (x_1, y_1) and (x_2, y_2) on the plane is

$$d = \sqrt{(x_2 - x_2)^2 + (y_2 - y_1)^2}$$

The equation of a circle

The canonical form of the circle equation with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

Functions

Idea: when is a correspondence between two sets a function?

A *relation* is a correspondence between a set, the *domain*, and another set, the *range*, such that each element of the domain corresponds to <u>at least one</u> element of the range; related pairs are denoted by (a, b).

$\star$ A function is a particular case of a relation.

A *function* is a correspondence between a set, the *domain*, and another set, the *range*, such that each element of the domain corresponds to <u>exactly one</u> element of the range:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ a & \mapsto & f(a) \end{array}$$

Given a function $f(x)$, we can evaluate it at some *input* (element of the domain) to determine the corresponding *output* (element of the range).

Graphs and Domains of functions

- The graph of a function is the graph of the equation $y = f(x)$! Plot points are ordered pairs $(x, f(x))$.

$\star$ From the function graph, we can spot $f(a)$ for some a without knowing the formula. However, can we tell if a graph really represents a function?

The vertical-line test for a graph

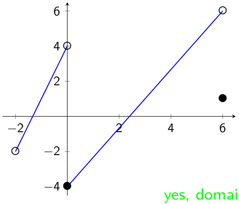
If a vertical line crosses it more than once, it is not the graph of a function.

- To identify the domain of a function formula, determine the set of all real numbers $x \in \mathbb{R}$ for which the formula is defined.

Visually, looking at a graph, domain can be found on the horizontal (x-)axis real line & range can be found on the vertical (y-)axis real line.

Work out the following examples:

1. Domain and range of relation $\{(2,10),(3,15),(4,20),(4,25)\}$? Is it a function? domain $D=\{2,3,4\}$, range $R=\{10,15,20,25\}$, no
2. Is this a graph of a function? Domain and range in interval notation?



yes, domain $(-2,6]$, range $[-4,6)$

3. For $f(x)=2x^2+3x$, find $f(1)$, $f(-2)$, $f(-x)$ and $f(2x+1)$ (expand). $5, 2, 2x^2-3x, 8x^2+14x+5$
4. Domain of $f(x)=\sqrt{x+1}$ in interval notation? $[-1,+\infty)$

Linear Functions

Idea: completely understand straight lines as graphs of functions!

Linear Functions

A function f is a *linear function* if it can be written as

$$f(x) = mx + b, \quad \text{for } m, b \in \mathbb{R} \text{ constants.}$$

- If $m = 1$, $b = 0$, the function $f(x) = x$ is the *identity* function.
- If $m = 0$, the function is a *constant* function: $f(x) = b$, a number.

Horizontal lines are given by $\boxed{y = b}$.
Vertical lines are given by $\boxed{x = c}$ - NOT graph of a function!

$\star$ For $f(x) = mx + b$, what does the coefficient 'm' *mean*?

The *slope m* of any line containing points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y\text{-change}}{x\text{-change}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

- Slope can aslo be though of as the average rate of change of a quantity!

Equations of Lines

Idea: write a line equation, given the slope and/or points of it!

- Slope-Intercept: using slope m and y-intercept $(0, b)$ or any point,
$$y = mx + b.$$
- Point-Slope: using slope m and one point (x_1, y_1), or two points,
$$y - y_1 = m(x - x_1).$$

Parallel and perpendicular lines

Parallel lines: if both vertical $(x = a)$ or same slope

$$y = mx + b_1 \text{ and } y = mx + b_2.$$

Perpendicular lines: if one vertical and other horizontal $(x = a, y = b)$ or product of their slopes is -1

$$y = m_1 x + b_1 \text{ and } y = m_2 x + b_2 \text{ with } \boxed{m_1 m_2 = -1}.$$

Work out the following examples:

1. Write an equation for the line with slope $-\dfrac{3}{8}$ and y-intercept $(0, -5)$.
   $y = -\frac{3}{8}x - 5$
2. Write an equation for the line passing through $(-3, \dfrac{1}{2})$ and $(1, \dfrac{3}{2})$.
   $y = \frac{1}{4}x + \frac{5}{4}$
3. Write an equation for the line which is perpendicular to $y = \dfrac{2}{7}x + 90$ and passes through $(0, 5)$. $y = -\frac{7}{2}x + 5$

Linear Equations Revisited

Recall: to solve a linear equation in one variable $\boxed{ax+b=0}$, use the *addition* and *multiplication* principle.

- The *solution set* of an equation is the collection of all its solutions.

Three possibilities for the linear solutions

- $ax = -b \Rightarrow x = -\frac{b}{a}$ unique solution — solution set $\{-\frac{b}{a}\}$
- no solution — solution set $\emptyset$
- infinitely many solutions — solution set $(-\infty, +\infty)$

Zeros of Functions

An input c of a function f is called a *zero* if its output is zero, $f(c) = 0$.

$\star$ Linear functions $f(x) = ax + b$ with $a \neq 0$ have exactly one zero – the first coordinate of the x-intercept of its graph!

Linear Models Applications

In order to solve a real-world problem using mathematical models,

1. familiarize with situation: draw picture, list of facts, assign variables for uknown quantities, exhibit the relevant formulas...
2. translate the word problem in mathematical language: form the equation(s) or inequality(ies) that express the problem
3. carry out mathematical manipulation: solve equation(s) or inequality(ies) for the unknown quantity(ies)
4. check whether solution verifies the problem
5. state the answer clearly and accurately, using words and units

Some notable formulas

- *Motion Formula*: $d = v \cdot t$ for d distance, v speed (velocity), t time.
- *Simple-Interest Formula*: $I = Prt$ for I simple interest, P principal of dollars, r interest rate, t years.
- *Rectangle Perimeter*: $P = 2l + 2w$ for l the length and w the width.

Work out the following examples:

1. Find the zero of $f(x) = -\frac{3}{4}x + 5$. What is the x-intercept of the line $y = f(x)$? $\frac{20}{3}$, $(\frac{20}{3}, 0)$

2. True or False?
   i) The lines $x = -1$ and $x = 1$ are perpendicular. F
   ii) The lines $y = 7 - x$ and $y = x + 3$ are parallel. F
   iii) The lines $y = 4x - 5$ and $4y = 8 - x$ are perpendicular. T

Properties of Functions

Idea: characterize functions in order to describe them more accurately.

Increasing, Decreasing, Constant Functions

- A function f is *increasing* on an open interval I if for all $a, b \in I$, $a < b \Rightarrow f(a) < f(b)$.
- A function f is *decreasing* on an open interval I if for all $a, b \in I$, $a < b \Rightarrow f(a) > f(b)$.
- A function f is *constant* on an open interval I if for all $a, b \in I$, $f(a) = f(b)$.

$\star$ What about the 'peaks' and the 'valleys' of the function's graph?

Relative Maxima and Minima

Suppose f is a function and c an element of its domain.

- $f(c)$ is a *relative maximum* if $f(c) > f(x)$ for all x in an open $I \ni c$;
- $f(c)$ is a *relative minimum* if $f(c) < f(x)$ for all x in an open $I \ni c$.

- "Piecewise" means that function changes behavior according to input!

A *piecewise function* has different output formulas for parts of its domain:

$$f(x) = \begin{cases} f_1(x), & \text{for } x < a \\ f_2(x), & \text{for } a \leq x \leq b \quad \dots \end{cases}$$

$\star$ For its graph, draw each part separately paying attention to domain, and use $\bullet$ and $\circ$ for points that are, or are not, parts of the graph.

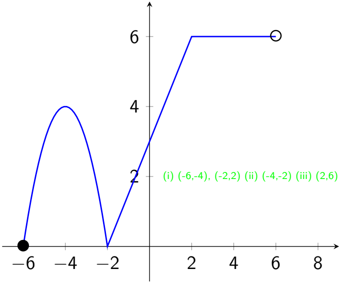
> In order to model real-world situations, we use the given facts to build up (linear, quadratic, piecewise...) functions.

Work out the following examples:

1. For the following piecewise function, find $f(-4), f(-2), f(4), f(6)$:

$$f(x) = \begin{cases} -5x - 8 & \text{for } x < -2 \\ \frac{1}{2}x + 5 & \text{for } -2 \leq x \leq 4 \\ 10 - 2x & \text{for } x > 4 \end{cases} \qquad 12, 4, 7, -2$$

2. Determine the intervals on which the function is (i) increasing, (ii) decreasing, (iii) constant.



3. For the above function, (i) domain (ii) range (iii) relative maxima (iv) relative minima. (i) [-6,6) (ii) [0,6] (iii) y=4 for x=-4 (iv) y=0 for x=-2

Operations of Functions

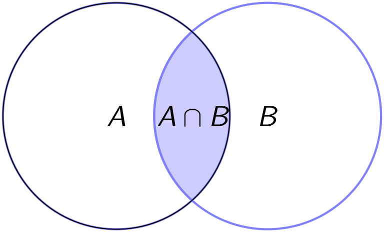
Idea: using the four basic operations, we can combine functions "pointwise" to obtain new ones!

Sum, difference, product, quotient of functions $\quad f+g, f-g, f\cdot g, f/g$

If f and g are functions, and x is in the domain of both, then

- $(f+g)(x) = f(x) + g(x)$
- $(f-g)(x) = f(x) - g(x)$
- $(f \cdot g)(x) = f(x) \cdot g(x)$
- $(f/g)(x) = f(x)/g(x)$ $(g(x) \neq 0)$

$\star$ Domain of new function? *Intersection* of old ones, and also $\cancel{\frac{A}{0}}$!



$\star$ To find these domains, we do NOT need to evaluate their formulas!
Reason only with graphs of intervals and intersections.

Domains of $f+g$, $f-g$, $f \cdot g$ and f/g

For functions f and g, with domains intervals D_f and D_g,

- $D_{f+g}, D_{f-g}, D_{f \cdot g}$ are their intersection, $D_f \cap D_g$ (common ground);
- $D_{f/g}$ is also their intersection, with an extra restriction that $g(x) \neq 0$.

The *difference quotient*, or *average rate of change*, of f is

$$\frac{f(x+h)-f(x)}{h}$$

Composing Functions

Idea: a function's input may itself be given by a different function!

Composition of functions

The *composite function* $f \circ g$ is defined as

$$(f \circ g)(x) = f(g(x)),$$

x in the domain of g, $g(x)$ in the domain of f.

Domain of $f \circ g$ $=$ intersection of domains $g(x)$ AND $f(g(x))$

- In general, $(f \circ g)(x) \neq (g \circ f)(x)$!

It is often useful to recognize a possible 'decomposition' of a function h, expressed as a composition of two functions: for which $f \& g$, $h = f \circ g$.

Work out the following examples:

1. For $f(x) = x + 1$ and $g(x) = 4x^2 - 3x - 1$, find $(g \circ f)(x)$ and its domain. $4x^2 + 5x,\ (-\infty, +\infty)$
2. For $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{2x}{x-1}$, find $(f \circ g)(x)$ and its domain.
   $\dfrac{x-1}{2x},\ (-\infty, 0) \cup (0, 1) \cup (1, +\infty)$

Symmetry

Idea: recognize the symmetries of a graph, visually and algebraically.

- For any point (x, y), flip either coordinates; is it still on graph?

A graph (not necessarily of a function) is [wrt=with respect to]

- *symmetric wrt x-axis*: $(y)\to(-y)$ gives same equation
- *symmetric wrt y-axis*: $(x)\to(-x)$ gives same equation
- *symmetric wrt origin*: both $(y)\to(-y)$, $(x)\to(-x)$ give same equation

$\star$ For functions, we have special characterizations for such properties.

Even and Odd functions graph points $(x, f(x))$

If $f(-x) = f(x)$, f is an *even function*; its graph is symmetric wrt y-axis.
If $f(-x) = -f(x)$, f is an *odd function*; its graph is symmetric wrt origin.

- Can only be one of the two. [No function can be symmetric wrt x-axis: vertical line test!]

Graph Transformations

Idea: from known basic graphs, create new ones by shifting or stretching!

Transformations for $y = f(x)$

- *vertical translation*: for $b > 0$, $y = f(x) \pm b$ shifts **up** or **down** b units

- *horizontal translation*: for $d > 0$, $y = f(x \pm d)$ shifts **left** or **right** d units

- *reflections*: $y = -f(x)$ across the x-axis, $y = f(-x)$ across the y-axis

- *vertical stretching/shrinking*: $y = af(x)$ **stretches** if $|a| > 1$, **shrinks** if $0 < |a| < 1$, moreover **reflects** across x-axis for $a < 0$.

- *horizontal stretching/shrinking*: $y = f(cx)$ **shrinks** if $|c| > 1$, **stretches** if $0 < |c| < 1$, moreover **reflects** across y-axis for $c < 0$.

- Always evaluate on a few inputs in order to determine and illustrate the change on the graph.

Complex Numbers

Idea: sometimes we need to talk about non-real solutions of equations.

The number i

The number i is defined in such a way that $\sqrt{-1} = i$, equivalently $i^2 = -1$.

Complex Numbers $\mathbb{C}$

A *complex number* is a number of the form $\boxed{a + bi}$, where $a, b \in \mathbb{R}$. The *real part* is a and the *imaginary part* is bi.

$\star$ Since (either a or) b can be 0, $\mathbb{R} \subseteq \mathbb{C}$.

The *conjugate* of a complex number $a + bi$ is $a - bi$.

Quadratic Equations and Functions

- *Quadratic equation*: $\boxed{ax^2 + bx + c = 0}$, where $a, b, c \in \mathbb{R}$ & $a \neq 0$.

Recall: to solve, use the square root principle, $x^2 = k \Rightarrow x = \pm\sqrt{k}$, and the zero product principle, $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$.

- *Quadratic function*: $\boxed{f(x) = ax^2 + bx + c}$, where $a, b, c \in \mathbb{R}$ & $a \neq 0$.

  $\star$ "Find zeros of (quadratic) function $f(x)$" is the same as "solve associated (quadratic) equation $f(x) = 0$"

Universal methods for solving quadratic equations

- *Complete the square*: create perfect squares by $d^2 \pm 2de + e^2 = (d \pm e)^2$
- *Quadratic Formula*: the solutions, for $\Delta = b^2 - 4ac$ discriminant, are

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$\Delta > 0$: 2 real solutions
$\Delta = 0$: 1 real solution
$\Delta < 0$: 2 conj. complex solutions

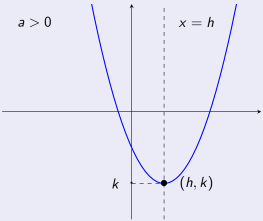
Graphs of quadratic functions

Idea: want to be able to accurately graph quadratic functions; *parabolas*!

Graph of $f(x) = a(x-h)^2 + k$

The graph of $f(x)$ is a parabola that

- opens up if $a > 0$, down if $a < 0$
- has (h, k) as its *vertex*
- has $x = h$ as axis of symmetry
- has k as minimum output if $a > 0$
- has k as maximum output if $a < 0$



$\star$ Already adequate information to identify its plot!
First cup or hat shape (positive or negative a), then vertex (h, k).

What about an arbitrary quadratic functions? Can either 'complete the square' to bring to previous, or straight away.

Graph of $f(x) = ax^2 + bx + c$

The graph of $f(x)$ is a parabola that opens up or down ($a \gtrless 0$) with vertex

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

Also $f(-\frac{b}{2a})$ is the maximum or minimum (depending on a).

Combining the above information with the x-intercepts if they exist (i.e. if $f(x) = 0$ has real solutions), and the y-intercept, we can graph the plot of ANY quadratic equation accurately!

⋆ Can also easily determine the range, as well as the increasing/decreasing parts of each parabola.

Rational Equations

Idea: solve equations that contain rational expressions, i.e. *rational equations*: e.g. $\frac{p(x)}{q(x)} + ... = \frac{s(x)}{t(x)}$ for polynomials in x.

Methodology: multiply both sides with the Least Common Denominator (LCD) of all fractions involved, in order to clear the equation of fractions.

Do not forget to exclude possible solutions, when they make some denominator 0 to begin with.

Caution: The above *doesn't* work when adding/subtracting rational expressions! Then, we again find the LCD, but use it to make all fractions have same denominator; we cannot 'clear' it unless it is an equation.

Radical Equations

Idea: solve equations in which variables appear in one or more radicands, i.e. *radical equations*: e.g. $\sqrt{p(x)} + \sqrt{q(x)} = a$.

The Principle of Powers

If $a = b$ is true, then $a^n = b^n$ is true (for any positive integer n).

Caution: The inverse isn't true in general (e.g. $(-3)^2 = 3^2$ but $-3 \neq 3$)! This means we have to always check the solutions.

Methodology: isolate one radical at one side, and then apply the principle of powers. If we have two radicals, first isolate and apply for one, and then repeat if we still have radical terms!

Work out the following examples:

1. Solve the rational equation:

$$\frac{6}{x+1} + \frac{2}{x^2+x} = \frac{8}{x} \qquad x = -3$$

2. Solve the radical equation $5 + \sqrt{x+7} = x$. $\quad x = 9$

Linear Inequalities

Idea: find those x for which an expression $f(x) \lessgtr g(x)$ is true.

Principles for Solving Inequalities (same with $\leq, \geq$)

For any real numbers a, b and c,

- if $a < b$, then $a + c < b + c$ *addition principle*
- if $a < b$ and $c > 0$, then $ac < ab$ *multiplication principle pt.1*
- if $a < b$ and $c < 0$, then $ac > bc$ *multiplication principle pt.2*

Multiplying with a negative number changes the inequality sign!

- The solution set is now an interval; we can graph it on the real line.

Compound Inequalities are two inequalities that either need to **both** hold, or **either of the two** to hold. Their solution set is either the **intersection** of the two solution sets, or their **union**.

Absolute value equalities and inequalities

Recall that the *absolute value* $|x|$ is the distance from 0 - always positive.

For any algebraic expession $f(x)$, we have that

$$|f(x)| = a \quad \Leftrightarrow \quad f(x) = a \text{ or } f(x) = -a$$

$\star$ To solve any equation that includes absolute value, first isolate the absolute value on one side and then break into *two* equations!

For any algebraic expression $f(x)$, we have that (same with $\leq, \geq$)

$$|f(x)| < a \quad \Leftrightarrow \quad -a < f(x) < a$$
$$|f(x)| > a \quad \Leftrightarrow \quad f(x) > a \text{ or } f(x) < -a$$

$\star$ Solving an absolute value inequality always comes down to solving a compound inequality!

Polynomial Functions

Idea: manipulate polynomials with degree higher than 2!

A *polynomial function* is

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where the exponents $n \in \mathbb{N}$ and the coefficients $a_i \in \mathbb{R}$. The degree is n, the leading term is $a_n x^n$, the constant term is a_0.

⋆ We can have constant ($n = 0$), linear ($n = 1$), quadratic ($n = 2$), cubic ($n = 3$), quartic ($n = 4$) and higher polynomials.

All polynomial functions, of any degree, are *continuous*.

- The *zeros* are those x_i with $p(x_i) = 0$. The *multiplicity* of a zero is 'how many times' it occurs as a zero. The x-intercepts are $(x_i, 0)$.

Every polynomial of degree $n \geq 1$ has at least 1 zero and at most n zeros.

⋆ How to identify zeros, if $p(x)$ is not factored?

The Rational Zeros Theorem

Let $f(x) = a_n x^n + a_{n\text{-}1} x^{n\text{-}1} + \ldots + a_1 x + a_0$ with integer coefficients, $a_i \in \mathbb{Z}$.
If $\dfrac{p}{q}$ is a zero of $f(x)$, then p is a factor of a_0 and q is a factor of a_n.

⋆ To find zeros of $f(x)$, first consider fractions $\dfrac{\text{factor of constant term } a_0}{\text{factor of leading coeff. } a_n}$!

Work out the following:

1. Which are the zeros of $f(x) = x(x-1)^2(x+3)$? Their multiplicity?
   $0, 1, -3$ of multiplicities $1, 2, 1$
2. Which are the x-intercepts of $f(x)$ above? $(0,0), (1,0), (-3,0)$
3. Find all possible rational zeros of $g(x) = x^3 + 3x^2 - x - 3$. $\pm 1, \pm 3$
4. Which ones are actually zeros of $g(x)$? $1, -1, -3$

Division of Polynomials

Idea: like for division of numbers, for polynomials we have

$$p(x) = d(x) \cdot q(x) + r(x) \quad \text{where}$$

$p(x)$ the *dividend*, $d(x)$ the *divisor*, $q(x)$ the *quotient*, $r(x)$ the *remainder*.

The Remainder Theorem

If c is substituted for x in some polynomial $p(x)$, then $p(c)$ is the remainder of dividing $p(x)$ by $(x - c)$: $\boxed{p(x) = (x - c) \cdot q(x) + p(c)}$

$\star$ We can use *synthetic* division when dividing with some $(x - c)$.
Also, for easier finding $p(c)$! $\rightsquigarrow$ If it is 0, then c is a *zero* of $p(x)$.

Factoring polynomials

The Factor Theorem

For a poly $p(x)$, if c is a zero ($p(c) = 0$), then $(x - c)$ is a *factor* of $p(x)$:

$$p(x) = (x - c)q(x)$$

$\star$ Use Rational Poly Thm to determine possible zeros, then synthetic division to factor. If reach quadratic, almost there!

- Morale: identify one zero at a time to completely factor, then obtain all zeros from factored form!

Work out the following examples:

1. $h(-2)$, for $h(x){=}x^4{+}2x^3{+}3x^2{-}5x{-}1$ (synthetic div)? $h(-2){=}21$
2. Possible rational zeros of $f(x) = x^3 + 2x^2 - 11x - 12$?
   $1, -1, 2, -2, 3, -3, 4, -4, 6, -6, 12, -12$
3. Completely factor $f(x)$. Which are its zeros?
   $f(x) = (x + 1)(x - 3)(x + 4)$, zeros are $x = -1, 3, -4$

Zeros of Polynomial Functions

Idea: find all zeros of any polynomial, non-rational ones included (e.g. $a + bi, c + \sqrt{d}$), and completely factor!

The Fundamental Theorem of Algebra

Every polynomial function $f(x) = a_n x^n + \ldots + a_1 x + a_0$ of degree n can be factored in n linear factors (not necessarily unique):

$$f(x) = a_n(x - c_1)(x - c_2)\ldots(x - c_n) \qquad \boxed{c_i's \text{ are the zeros of } f}$$

- Quadratics $f(x) = ax^2 + bx + c$ factor as $\boxed{f(x) = a(x - x_1)(x - x_2)}$

where $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ by the quadratic formula.

Non-real zeros: $a + bi$ and $a - bi$

If a complex $a + bi$ $(b \neq 0)$ is a zero of a polynomial function $f(x)$ with <u>real</u> coefficients, then its conjugate $a - bi$ is also a zero.

Irrational zeros: $a + b\sqrt{c}$, $a - b\sqrt{c}$

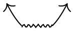
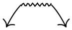
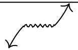
If $a + b\sqrt{c}$ (c not a perfect square) is a zero of a polynomial function $f(x)$ witn <u>rational</u> coefficients, then its conjugate $a - b\sqrt{c}$ is also a zero.

Complex and radical zeros occur in conjugate pairs!

Graphing Polynomial Functions

Idea: accurately graph cubic and quartic polynomial functions!

Leading Term Test: leading coefficient & degree give end behavior

	$a_n > 0$	$a_n < 0$
n even		
n odd		

Methodology for graphing $p(x) = a_n x^n + \ldots + a_1 x + a_0$

- determine end behavior (Leading Term Test)
- factor $p(x)$ completely (rational zeros, synthetic division, quadratic)
- (at most n) zeros ($p(x_i) = 0$) and x-intercepts $(x_i, 0)$ (for *real* zeros); odd multiplicity says graph crosses x-axis, even says graph bounces
- y-intercept $(0, f(0))$
- graph using all above information

Asymptotes of rational functions

Idea: graph rational functions $\frac{p(x)}{q(x)}$;
beware of their domain consisting of all inputs x for which $q(x) \neq 0$.

Horizontal Asymptotes

Suppose we have the rational function $f(x) = \dfrac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0}$.

- If $n > m$, there is no horizontal asymptote *larger num*
- If $n < m$, then $\boxed{y = 0}$ is a horizontal asymptote *larger denom*
- If $n = m$, then $\boxed{y = \frac{a_n}{b_m}}$ is a horizontal asymptote *same degree*

Vertical Asymptotes

For any simplified $f(x) = \frac{p(x)}{q(x)}$, if 'c' is a zero of the denominator then the line $\boxed{x = c}$ is a *vertical asymptote* of the graph of $f(x)$.

Graphing rational functions

Graph $f(x) = \frac{p(x)}{q(x)}$ [asymptotes & intercepts may or may not exist]

- Find the horizontal asymptote.
- Find the domain (need to only factor denominator).
- Simplify (need to also factor numerator); find the vertical asymptotes.
- Solve $f(x) = 0$ to find its zeros x_i; x-intercepts are $(x_i, 0)$.
- y-intercept is $(0, f(0))$.
- [Sign Table] use zeros & domain to divide real line into intervals; determine sign in each by plugging test inputs in the simplified form.
- Graph! <u>Does not</u> cross vertical asymptotes, nor horizontal ones usually.

Polynomial and Rational inequalities

Idea: solve inequalities that involve polynomials and rational expressions.

Methodology: after we bring everything on the same side

$$f(x) = a_n x^n + \ldots + a_1 x + a_0 \lesseqgtr 0,$$

solve *related equation* $f(x) = 0$, then use zeros to form *sign table*.

- The sign of any input in an interval determines whether $f(x) > 0$ (+) or $f(x) < 0$ (-) at that interval.

$\star$ Also graph-relevant! Plot parts above (+) or below (-) the x-axis.

- For $g(x) = \dfrac{a_n x^n + \ldots a_0}{b_m x^m + \ldots b_0} \lesseqgtr 0$, use **critical values** (zeros AND domain) for sign table.

Inverse Functions

Idea: for certain $f(x)$, can find their *inverse*! New function, $f^{-1}(x)$.

One-to-one functions

A function f is *one-to-one* if different inputs have different outputs:

$$a \neq b \Rightarrow f(a) \neq f(b); \quad \text{equivalently,} \quad f(a) = f(b) \Rightarrow a = b$$

Two outputs are the same ONLY if the inputs were the same to begin with!

Horizontal Line Test: if any horizontal line intersects the graph of a function only once, then the function is one-to-one.

- Any entirely increasing or decreasing f is one-to-one!

Properties of inverse functions [symbol $f^{-1} \neq \frac{1}{f}$!]

- If f is one-to-one, its inverse function f^{-1} exists.
- Domain of f=range of f^{-1}, range of f=domain of f^{-1}.

Formula for inverse functions

Suppose f is a one-to-one function.

1. Replace $f(x)$ with y.
2. Interchange x and y.
3. Solve for y.
4. Replace y with $f^{-1}(x)$.

- The graphs of f and f^{-1} are symmetric with respect to $y = x$!

If f is one-to-one, f^{-1} is the unique function such that

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$$
$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x$$

for all x's in the domain of f and f^{-1} respectively.

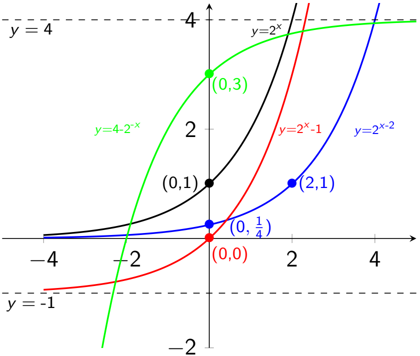
$\star$ Also useful for checking that we found the 'right' inverse!

Exponential Fuctions and Graphs

Idea: how to graph functions of the form a^x, for some number a?

Exponential Function

The function $f(x) = a^x$, where x is a real number and $a > 0, a \neq 1$, is called the *exponential function* with *base a*.



Applications and the Euler number

Compound Interest Formula

A principal amount P will grow, after t years and at interest rate r, compounded n times per year, to an amount given by the formula

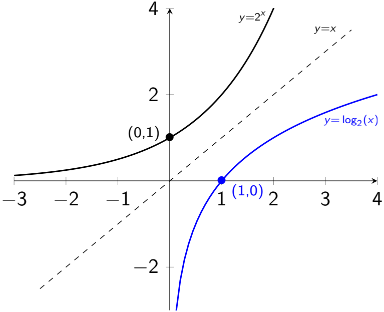
$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$\star$ P=r=t=1, get $A = (1 + \frac{1}{n})^n$. When $n \to \infty$, $A \to e$, the *Euler number*

$\boxed{\text{e=2.7182818284...}}$ irrational number!

Logarithmic Functions and Graphs

Idea: use and graph functions of the form $\log_a(x)$, i.e. inverse of a^x!



Logarithmic function of base a *[logs are exponents!]*

The function $y = \log_a(x)$ gives, for each x, the *power* to which we need to raise a in order to get x, i.e. $\boxed{y = \log_a(x) \Leftrightarrow x = a^y}$.

- $\log_a(a) = 1$ and $\log_a(1) = 0$, for any base a.

We denote the folllowing logarithms

- $\log_{10}(x)$ as $\boxed{\log(x)}$ - the *common logarithm*
- $\log_e(x)$ as $\boxed{\ln(x)}$ - the *natural logarithm*

Change-Of-Base Formula

For any logarithmic bases a and b, and any positive number M,

$$\log_b(M) = \frac{\log_a(M)}{\log_a(b)}$$

Work out the following:

1. $\log_2(16) =?$ 4
2. $\log(100) - \log(10) =?$ 2
3. $\log_5(1) =?$ 0
4. $\ln(e^3) =?$ 3
5. Which function do we obtain if we take $f(x) = \ln(x)$, reflect it along the y-axis, then shift down by 1? $\ln(-x) - 1$

Properties of Logarithmic Functions

Idea: we know properties of exponential, a^x.
What about properties of its inverse function, $\log_a(x)$?

The Product Rule

For any positive numbers M, N and any logarithmic base a

$$\log_a(MN) = \log_a(M) + \log_a(N)$$

"The logarithm of the product is the sum of the logarithms of the factors"

The Power Rule

For any positive number M, any logarithmic base a, any real number p,

$$\log_a(M^p) = p \cdot \log_a(M)$$

"The logarithm of a power of M is the exponent times the logarithm of M"

The Quotient Rule

For any positive numbers M and N, and any logarithmic base a

$$\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N)$$

"The logarithm of a quotient is the logarithm of the numerator minus the logarithm of the denominator"

Things NOT to do!

- $\log_a MN \neq (\log_a M)(\log_a N)$
- $\log_a(M+N) \neq \log_a M + \log_a N$
- $\log_a\left(\frac{M}{N}\right) \neq \frac{\log_a M}{\log_a N}$
- $(\log_a M)^p \neq p \cdot \log_a M$

$\star$ Last line: those do not simplify somehow!

Useful properties, following from definitions

- $\boxed{\log_a a^x = x}$ *"the log with base a of a to some power is the power"*
- $\boxed{a^{\log_a(x)} = x}$ *"any number a raised to the power $\log_a(x)$ gives x"*

$\star$ This is exactly the fact that a^x and $\log_a(x)$ are inverse functions!

Work out the following:

1. Compute $\log_3(\frac{1}{27})$. -3
2. Express in terms of sums and difference of logarithms: $\ln(5x^2)$.
   $\ln(5) + 2\ln(x)$
3. Express as a single logarithm, and if possible simplify:
   $\log_a(x) + 4\log_a(y) - 3\log_a(x)$. $\log_a \frac{y^4}{x^2}$

Solving Exponential Equations

Idea: know how to solve linear, quadratic, radical, rational equations...
What about equations including exponentials a^x?

- Try to express both sides of equation as exponentials with same base.

Base-Exponent Property *a^x is 1-1!*

For any $a > 0$, $a \neq 1$, we have that $\boxed{a^p = a^q \Leftrightarrow p = q}$

- If not possible, take the logarithm (any base!) on both sides.

Logarithmic equality property $\log_a(x)$ is *1-1*!

For any $M, N, a > 0$ and $a \neq 1$, $\boxed{\log_a M = \log_a N \Leftrightarrow M = N}$

$\star$ Usually take log or ln (base 10 or e) in exponential equations.

Solving Logarithmic Equations

What about equations that involve logarithmics $\log_a(x)$?

Using logarithmic properties, obtain a single logarithm on one or both sides of the equation; then

- [$\log_a(\overset{b}{\ldots}) =$number p] solve with exponential $a^p = b$ or
- [$\log_a(\overset{c}{\ldots}) = \log_a(\overset{d}{\ldots})$] use logarithmic equality, $c = d$.

⋆ Beware: for exponential equations, any solution is acceptable. For logarithmic equations, $\log_a(x)$ is only defined on positive inputs!

Work out the following:

1. Solve $2^x = 32$. $x = 5$
2. Solve $\log_3(2x - 1) - \log_3(x - 4) = 2$. $x = 5$
3. Solve $2\ln(x) + \ln(\frac{1}{8}) = \ln(2)$. $x = 4$ (-4 not in domain of $\ln(x)$!)

Systems of linear equations, in two variables

Idea: know how to solve linear equation $ax + b = 0$.
What about two equations, with two variables x, y?

Linear system in two variables x, y

A *system* of two linear equations in x and y $(a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R})$ is

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

- Graphically, its solution is the *intersecting point* of the two line graphs.

Methodology for solving systems

- *Elimination*: add or subtract (multiples of) the equations, if that eliminates some variable
- *Substitution*: solve one equation for either variable and substitute to the other

Partial Fraction Decomposition

Idea: useful to write a rational expression as a sum of simpler fractions!

Method: to decompose $p(x)/d(x)$

1. if $d(x)$ is of lower degree, divide the two polys so that $p(x)/d(x) = q(x) + r(x)/d(x)$ and follow steps (2)-(4) for $r(x)/d(x)$ instead
2. if $d(x)$ is of higher degree, factor it into linear factors $(ax+b)^n$
3. to each linear factor $(ax+b)^n$, assign the sum of n-partial fractions

$$\frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \ldots + \frac{C}{(ax+b)^n}$$

4. multiply with the LCD both sides of the formed equality: use test values of x to solve for $A, B, \ldots, C$
5. write clearly the resulting decomposition

Sequences and Series

Idea: consider lists of numbers in specific order, and calculate their sums.

Sequences

An *infinite sequence* is a function with domain the set of positive integers

$$a\colon \mathbb{Z}_+ = \{1, 2, 3, \ldots\} \longrightarrow \mathbb{R}$$

A *finite sequence* is a function with domain a finite set $\{1, 2, \ldots, n\}$ for some positive integer n.

- Write a_n for $a(n)$, and call $a_1, a_2, \ldots$ *terms* of the sequence, a_n *general term*. In *alternating* sequences, sign of term depends on n being even/odd.

$\star$ Predict the general term from a list of terms? Find pattern between given terms, and check suggested a_n works! Not necessarily unique choice.

Series

Given an infinite sequence $a_1, a_2, ..., a_n, ...$, the sum of all the terms

$$a_1 + a_2 + \ldots + a_n + \ldots$$

is called an *infinite series*. A *partial sum* is the sum of the first n terms, denoted $S_n = a_1 + ... + a_n$, and is also called a *finite series*.

- Sigma notation $S_n = \sum_{k=1}^{n} a_k = a_1 + \ldots + a_n$ using general term a_k.

Arithmetic Sequences $d = a_{n+1}$-a_n

A sequence is *arithmetic* if there exists a number d, called the *common difference*, such that for all consequent terms, $a_{n+1} = a_n + d$.
The general (n-th) term of an arithmetic sequence is $\boxed{a_n = a_1 + (n-1)d}$.

Compute Arithmetic Series

The sum of the first n terms of an arithmetic sequence is given by

$$S_n = \frac{n}{2}(a_1 + a_n).$$

Geometric Sequences and Series

Idea: in arithmetic sequences, each term is the preceding plus some d.
Now, each term is the preceding times some r!

Geometric Sequences $r = \frac{a_{n+1}}{a_n}$

A sequence is *geometric* if there exists a number r, called the *common ratio*, such that for all consequent terms, $a_{n+1} = a_n \cdot r$.
The general (n-th) term of a geometric sequence is $\boxed{a_n = a_1 \cdot r^{n-1}}$.

⋆ Like arithmetic sequences, the above are special kinds of sequences.
What about their respective sum of terms, i.e. series?

Compute Geometric Series

The sum of the first n terms of a geometric sequence is given by

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

⋆ In certain cases, the sum of the terms of an infinite geometric sequence approaches (gets very close to) a specific number, when n gets very large.

Infinite Geometric Series

Only when $|r| < 1$, the sum (or limit) of an infinite geometric sequence is

$$S_\infty = \frac{a_1}{1-r}.$$

Work out the following:

1. Find the common ratio of the geometric sequence: $-2, 2, -2, 2, \ldots$ -1
2. Find the general term of $25, 5, 1, \ldots$ (simplify) $a_n = 5^{3-n}$
3. Compute the infinite series, if it exists: $32 + 16 + 8 + \ldots$ $S_\infty = 64$
4. Compute the infinite series, if it exists: $\sum_{k=1}^{\infty} (12.5)^k$ S_∞ DNE

Permutations

Idea: *combinatorics* can be thought of as the theory of 'counting'.
The study of permutations is its part relative to order and arrangements.

Permutation

A *permutation* of a set of n objects is an ordered arrangement of them.

The Fundamental Counting Principle

Given a combined action where the first action can be performed in n_1 ways, the second action can be performed in n_2 ways and so on, the total number of ways in which the combined action can be performed is

$$n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k$$

The total number of permutations of n objects, denoted ${}_nP_n$, is

$${}_nP_n = n(n-1)(n-2)\ldots 3 \cdot 2 \cdot 1.$$

Factorial Notation

For any natural number, $\boxed{n! = n(n-1)(n-2)\dots 3\cdot 2\cdot 1}$. For zero, $0!=1$.

Therefore, the number of permutations of n objects is ${}_nP_n = n!$

$\star$ What if we want to choose fewer objects of our set, in an specific order?

Permutations of n objects taken k at a time

A permutation of a set of n objects *taken k at a time* is an ordered arrangements of k objects taken from the set.

The number of permutations of a set of n objects taken k at a time is

$$ {}_nP_k = \underbrace{n(n-1)(n-2)\dots[n-(k-1)]}_{k \text{ factors}} = \frac{n!}{(n-k)!} $$

⋆ If we allow repetition of objects, the counting of possibilities is easier.

The number of distinct arrangements of n objects taken k at a time, allowing repetition, is just n^k.

⋆ If some objects are nondistinguishable, many of the ordered arrangements will look the same, even if we don't allow repetition.

For a set of n objects in which n_1 are of one kind, n_2 are of another,..., and n_k are of a kth kind, the number of distinguishable permutations is

$$\frac{n!}{n_1! \cdot n_2! \ldots n_k!}.$$

Work out the following:

1. How many permutations are there of the word ADMIRE? (All letters used without repetition) ${}_6P_6 = 6! = 720$
2. How many 6-letter words can we make, allowing repetition of letters? $6^6 = 46656$

Combinations

Idea: for permutations, the order of choosing was important.
What about a selection *without regard to order*?

Combinations

A *combination* containing k objects chosen from a set $\{A_1, A_2, ..., A_n\}$ of n objects, $k \leq n$, is just any k-element subset (no order of elements). The number of all such possible combinations/subsets is denoted by ${}_nC_k$.

$\star$ Usually, to each combination correspond many permutations, by deciding all possible orders!

Combinations of n objects taken k at a time - Binomial notation

The total number of combinations of n objects taken k at a time is

$$\binom{n}{k} = {}_nC_k = \frac{{}_nP_k}{k!} = \frac{n!}{k!(n-k)!} \qquad \text{“}n \text{ choose } k\text{”}$$

Subsets of size of k and of size $n - k$

Number of subsets of size k is the same as that of subsets of size $n - k$!

$$\binom{n}{k} = \binom{n}{n-k} \quad \Leftrightarrow \quad {}_nC_k = {}_nC_{n-k}$$

Work out the following:

1. ${}_8C_5$? 56
2. In how many ways can we choose 6 students out of a class of 16 for a play? 8008
3. A U.S. postal zip code is five-digited, with possibly repeated numbers. How many zip codes are possible, if any of 0-9 digits can be used? 10^5